Investigation of the Existence of Limit Cycles in Multi Variable Nonlinear Systems with Special Attention to 3X3 Systems

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Abstract: - The proposed work addresses the dynamics of a general system and explores the existence of limit cycles (LC) in multi-variable Non-linear systems with special attention to 3x3 nonlinear systems. It presents a simple, systematic analytical procedure as well as a graphical technique that uses geometric tools and computer graphics for the prediction of limit cycling oscillations in three-dimensional systems having both explicit and implicit nonlinear functions. The developed graphical method uses the harmonic balance/harmonic linearization for simplicity of discussion which provides a clear and lucid understanding of the problem and considers all constraints, especially the simultaneous intersection of two straight lines & one circle for determination of limit cycling conditions. The method of analysis is made simpler by assuming the whole system exhibits the limit cycling oscillations predominantly at a single frequency. The discussions made either analytically/graphically are substantiated by digital simulation by a developed program as well as by the use of the SIMULINK Toolbox of MATLAB Software.

Key-Words: - Describing function, 3 x 3 nonlinear systems, limit cycles, harmonic linearization.

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1 Introduction

Limit cycles/nonlinear oscillations are the modus operandi of several physical systems and are often the basic feature of instability. Therefore, our objective is to predict the limit cycle, [1]. The importance of this problem was well felt among the researchers, [2], [3], [4], [5], [6], where the people were mostly focusing on single input single output systems. However, for the last five decades, the analysis of 2x2 nonlinear multivariable systems gained importance, particularly for the prediction of limit cycles and quite a good amount of literature is available addressing this area of research, [1], [2], [3], [4], [5], [6], [7], [8], [9], [10], [11], [12], [13], [14], [15], [16], [17], [18], [19], [20], [21], [22], [23], [24], [25], [26], [27], [28], [29], [30], [31], [32], [33], [34], [37], [38], [39], [40], [41], [42], [43], [44], [45], [46], [47], [48], [49], [50], [51]. Prediction of limit cycle and its analysis in both single as well as two dimensional nonlinear systems, a means of increasing the reliability of the describing function (DF) are well realised, [5], [6], [8], [11], [14], [22], and many others based on harmonic linearization / harmonic balance, [11], [28], [32], [36].

If there exit limit cycle oscillations, the possibility of quenching the sustained oscillations using the method of signal stabilization has been investigated, [6], [16], [23], [29], [30], in 2x2 nonlinear systems with non-memory type nonlinear elements. Prediction of limit cycling oscillations and its quenching using signal stabilization technique by a deterministic signal in memory type nonlinear elements for 2x2 multivariable systems has been discussed in [49], and the same has been addressed with Gaussian signals, [50].

Prediction and Suppression of limit cycle oscillations in 2x2 memory type nonlinear systems using arbitrary pole placement has also been discussed in some literature, [33], [45], [46], [48], and pole placement by optimal selection using Riccati Equation, [35], [51].

It has been realized that the exhibition of limit cycles in two-dimensional multivariable nonlinear systems on several occasions like Couple Core Reactor, [10], Pressurised Water Reactor (PWR) nuclear Reactor System, [19], Radar Antenna Pointing System, [9], and Interconnected Power System, [41], which can fit the structure, [24], [27], of a general two-dimensional nonlinear system.

Backlash is one of the most important nonlinearities commonly occurring in physical systems that limit the performance of speed and positions, which has been extensively discussed for two-dimensional multivariable systems, [34], [37], [41], [45], [46], [47], [48], [49], [50], [51].

The recent literature depicts some instances of multidisciplinary applications where limit cycle oscillations have been discussed. The researchers, [52], discussed three possible scenarios, namely, stable, limit cycles and chaos arise naturally in the flow and thermal dynamics of the device. The authors, [53], formulated/initialized the cell model to the limit cycle, running one-dimensional (OD) simulations of 500 stimuli at a BCL of 300ms.In [54], the dynamic behaviour of the nonlinear system switches between a stable equilibrium point and a stable limit cycle has been discussed. In [55], the stable limit cycle has been observed in autocatalytic systems through the characteristics of the Hopf bifurcation. In, [56], the exhibition of limit cycling oscillations has been observed in Biological Oscillators having both positive and negative feedbacks. The authors, [57], have observed in natural systems a closed loop as in a stable limit cycle through reviewing empirical dynamic modelling.

Scanty literature is available addressing 3 x 3 nonlinear multivariable systems in the last decade only, [27], [40], [42], [43], [44]. The researchers, [27], have speculated for investigation of limit cycles in 3x3 systems. It has been observed that the exhibition of limit cycles in three-dimensional multivariable nonlinear systems on several occasions like a boiler turbine unit is a 3 x 3 multivariable process showing nonlinear dynamics under a wide range of operating conditions, [40]. Most of the chemical process is multivariable, considering a 3x3 model of nonlinear chemical process, the limit cycling conditions have been reported, [44]. Similarly the literature, [42], [43], depicts limit cycles in 3x3 nonlinear models. However detailed analysis, well-established conclusions, and state forward techniques are still lacking in the available literature on such 3 x 3 nonlinear multivariable systems.

However a number of industrial problems with two or more higher dimensional configurations, [12], and the prediction of limit cycles via the describing function method prove to be quite essential, [5], [6], [8], [11], [12], [14], [16], [22]. Hence the exhibition of limit cycles in three-dimensional nonlinear multivariable systems which can fit the structure of general 3 x 3, [16], [27], nonlinear systems has been addressed in the present work. In the event of the existence of limit cycling oscillations, the possibility of quenching such oscillations using the signal stabilization technique by deterministic, [49], as well as Gaussian random signals, [50], may be adopted. Alternatively, suppression of limit cycles using the Pole placement technique, [51], can also be adopted. All such methods developed either by graphical or analytical approach have been substantiated by digital simulation with the developed program using MATLAB Code and also with the use of SIMULINK Toolbox of MATLAB Software.

The complexity involved in the structure, [16], [27], in particular for implicit nonlinearity or the system having the memory type nonlinearities, it would be extremely difficult to formulate and simplify the expressions even using harmonic balance, [32], [51]. Hence in the present work to get an alternative attempt has been made to develop a graphical method for the prediction of limit cycles in 3x3 non-linear systems by extension of the procedure as depicted in [1]. The developed method/procedure considers all constraints, especially the simultaneous intersection of two straight lines and one circle in three combinations. The method can be used for the prediction of limit cycles in non memory type explicit and implicit nonlinear functions and also for memory type 3x3 nonlinear systems. Such a general graphical method for 3x3 systems has never been developed before and hence claims its novelty.

The proposed paper presents the dynamics of general 3x3 nonlinear systems shown in Figure 2, Figure 3, [27], which is an equivalent representation of the general multivariable system of Figure 1, [27]. The governing equation under limit cycling condition, with reference to Figure 2 (autonomous system i.e. U=0) in frequency response form is X =-HC and C=GN(X)X: leading to X=-HGN(X)=AX, where A=-HGN(X), [27], which facilitates the determination of Eigen value of the multivariable systems (illustrated in 2.1)

$$\text{where}, X=\begin{bmatrix}X_1\\X_2\\X_3\end{bmatrix}, H=\begin{bmatrix}1&1&-1\\-1&1&1\\1&-1&1\end{bmatrix},$$

$$C=\begin{bmatrix}C_1\\C_2\\C_3\end{bmatrix}, G(\omega)=\begin{bmatrix}G_1(\omega)&0&0\\0&G_2(\omega)&0\\0&0&G_3(\omega)\end{bmatrix};$$

$$N(X)=\begin{bmatrix}N_1(X_1)&0&0\\0&N_2(X_2)&0\\0&0&N_3(X_3)\end{bmatrix}$$

X_1, X_2, X_3 and C_1, C_2, C_3 are Amplitudes of respective Sinusoids. G_1, G_2, G_3, and N_1, N_2, and N_3 are magnitudes/absolute values of respective functions. *It may be noted that for frequency response: Input is sinusoidal and outputs are steady state values considered, so that s (Laplace Operator) is replaced by $j\omega$.*

2 Analysis of Nonlinear Self-Oscillations (Limit Cycle)

The variables in Figure 1 are related by the following equations:
$Y = Y(x)$; $X = G_1 U - H G_2 Y$; $C = G_2 Y + G_3 U$

Where U, X, Y, and C are vectors of diminutions k, l, m, and n respectively and G_1, G_2, G_3, and H are linear transfer function matrices of dimensions l x k, n x m, n x k, and l x n respectively.

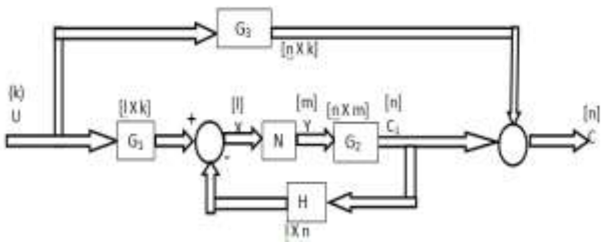


Fig. 1: Block Diagram Representation of a most General Nonlinear Multivariable System, [27]

2.1 Analysis of self-oscillations in 3x3 nonlinear systems

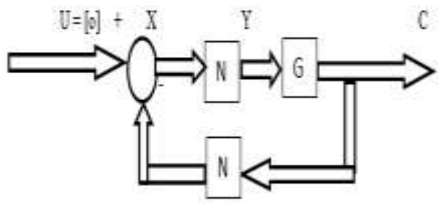


Fig. 2: Equivalent of the system of Fig. 1for U = [0]

For the autonomous state (U = 0) the system of Figure 1 can equivalently be represented as shown in Figure 2.

Considering Figure 3 and making use of the first-order harmonic linearization of the nonlinear elements (nonlinear characteristics are replaced by their respective Describing Functions: N_1, N_2, N_3), the matrix equations for the system of Figure 2 can be expressed as:

$X = -H\,C\,; C = GN\,(X)X$

Leading to $X = - H\,G\,N\,(X)\,X = A\,X$ (i)
Where, $A = -H\,G\,N\,(X)$

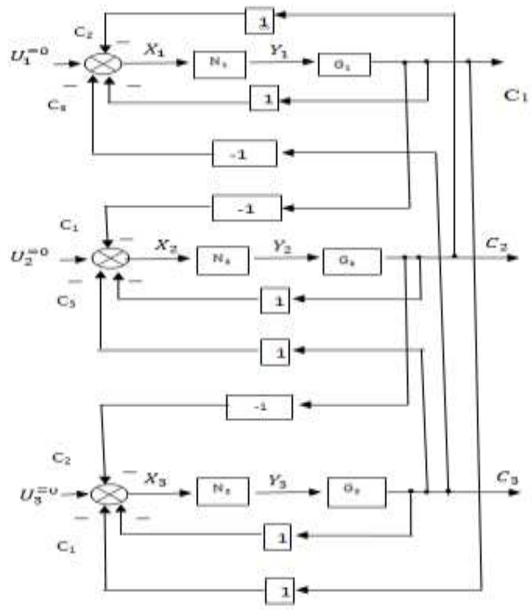


Fig. 3: An equivalent 3X3 multivariable nonlinear system of Figure 2

Visualising Eqn. (i) As a transformation of the vector X onto itself, we note that for a limit cycle to exist the following two conditions must be satisfied, [16], [27],
(I) For every nontrivial solution X, the matrix A must have an Eigen value λ, equal to unity; and
(II) The Eigen vector of A corresponding to this unity Eigen value must be coincident with X.
"Comparing the systems of Examples 3 and 4 with those of Examples 1 and 2 respectively, it is seen that if all cross feedbacks are made negative then the analysis of such system becomes trivial, [16].

Therefore, a standard three-dimensional nonlinear feedback system is shown in Figure 3 in the form of Figure 2" from which we get,

$$X = \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix};\ N(X) = \begin{bmatrix} N_1(X_1) & 0 & 0 \\ 0 & N_2(X_2) & 0 \\ 0 & 0 & N_3(X_3) \end{bmatrix};$$

$$G(\omega) = \begin{bmatrix} G_1 & 0 & 0 \\ 0 & G_2 & 0 \\ 0 & 0 & G_3 \end{bmatrix}; C = \begin{bmatrix} C_1 \\ C_2 \\ C_3 \end{bmatrix}$$

$$\text{And } H = \begin{bmatrix} 1 & 1 & -1 \\ -1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix}$$

and therefore $A = -H\,G\,N$

Hence

$$A = \begin{bmatrix} -N_1G_1 & -N_2G_2 & N_3G_3 \\ N_1G_1 & -N_2G_2 & -N_3G_3 \\ -N_1G_1 & N_2G_2 & -N_3G_3 \end{bmatrix}$$

The characteristic Equation is

$$|[A-\lambda I]| = 0 = \left|\begin{bmatrix} -(G_1N_1+\lambda) & -G_2N_2 & G_3N_3 \\ G_1N_1 & -(G_2N_2+\lambda) & -G_3N_3 \\ -G_1N_1 & G_2N_2 & -(G_3N_3+\lambda) \end{bmatrix}\right| \cdots \textbf{(ii)}$$

For $\lambda = 1$ (cf.I), from Eqn. (**ii**) we get,

$$\left|\begin{bmatrix} -(G_1N_1+1) & -G_2N_2 & G_3N_3 \\ G_1N_1 & -(G_2N_2+1) & -G_3N_3 \\ -G_1N_1 & G_2N_2 & -(G_3N_3+1) \end{bmatrix}\right| = 0$$

Or

$$4G_1N_1G_2N_2G_3N_3 + 2G_1N_1G_2N_2 + 2G_1N_1G_3N_3 + 2G_2N_2G_3N_3 + G_1N_1 + G_2N_2 + G_3N_3 + 1 = 0 \ldots \textbf{(Characteristic Equation).(iii)}$$

Where in general $G_1=G_1(j\omega)$, $G_2=G_2(j\omega)$, $G_3=G_3(j\omega)$, and $N_1=N_1$ $(X_1, j\omega)$, $N_2=N_2$ $(X_2, j\omega)$, $N_3=N_3$ $(X_3, j\omega)$.

Four unknowns X_1, X_2, X_3, and ω (frequency of L. C) require four independent equations for their evaluation. By separating the real and imaginary parts of the characteristic equation *(iii)* only two independent equations involving four unknown quantities can be determined. Therefore, the characteristic equation alone is not sufficient for the analysis of the limit cycle in such systems. However, by replacing the nonlinear elements with their respective DF's, ensuring harmonic balance, we can explore for possible limit cycles, the following conditions must be satisfied, [32].

(i) The Phase condition: $\theta_{c_1} + \theta_{c_2} = 180^o$

(ii) The Gain condition: $\frac{C_1}{R_1} \cdot \frac{C_2}{R_2} = 1$

(Derived from-I: Eigen Value Condition): characteristic equation

(iii) The Amplitude Ratio Condition (derived from II: Eigen Vector condition), [16], [27], [32].

This has been applied in 2x2 nonlinear systems. However, the logic can also be extended for 3x3 nonlinear systems as in the present case in the following way:

A: Non-memory type nonlinearities:

(i) The Phase condition: $\theta_{c_1} + \theta_{c_2} + \theta_{c_3} = 180^o$:

$$Where, \theta_{c_1} = \mathrm{Arg}\frac{G_1(j\omega)N_1}{1+N_1G_1(j\omega)}$$

$$\theta_{c_2} = \mathrm{Arg}\frac{G_2(j\omega)N_2}{1+N_2G_2(j\omega)}$$

$$\theta_{c_3} = \mathrm{Arg}\frac{G_3(j\omega)N_3}{1+N_3G_3(j\omega)}$$

(ii) The Gain Condition: $\frac{C_1}{R_1} \cdot \frac{C_2}{R_2} \cdot \frac{C_3}{R_3} = 1$

$$Where, \frac{C_1}{R_1} = \frac{G_1(j\omega)N_1}{1+N_1G_1(j\omega)};$$

$$\frac{C_2}{R_2} = \frac{G_2(j\omega)N_2}{1+N_2G_2(j\omega)}; \frac{C_3}{R_3} = \frac{G_3(j\omega)N_3}{1+N_3G_3(j\omega)}$$

Hence

$$\frac{G_1(j\omega)N_1G_2(j\omega)N_2G_3(j\omega)N_3}{[1+N_1G_1(j\omega)]\,[1+N_2G_2(j\omega)][1+N_3G_3(j\omega)]} = 1$$

(iii) The Amplitude Ratio Condition:

$$\frac{X_1}{X_2} = \frac{V_1}{V_2}; \frac{X_2}{X_3} = \frac{V_2}{V_3}; \frac{X_3}{X_1} = \frac{V_3}{V_1}$$

B: Memory type nonlinearities: $N_1=N(X_1, j\omega)$, $N_2 = N_2(X_2, j\omega)$; $N_3 = N_3(X_3, j\omega)$;

Accordingly the changes in derivations in *(i), (ii)* & *(iii)* are incorporated into numerical expressions

The Eigen vector V of A Matrix corresponding to λ=1, also satisfies the equation: From (II)

$$\begin{bmatrix} -(N_1G_1+1) & -N_2G_2 & G_3N_3 \\ N_1G_1 & -(N_2G_2+1) & -G_3N_3 \\ -G_1N_1 & N_2G_2 & -(N_3G_3+1) \end{bmatrix}\begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = [0]$$

Or $-(N_1G_1+1)V_1 \quad -N_2G_2V_2 \quad +N_3G_3V_3 = 0 - -\textbf{iv}(a)$

$N_1G_1V_1 \quad -(N_2G_2+1)V_2 \quad -N_3G_3V_3 = 0 - - - \textbf{iv}(b)$

$-G_1N_1V_1 \quad +N_2G_2V_2 \quad -(N_3G_3+1)V_3 = 0 - -\textbf{iv}(c)$

Addition of **iv** (a) & iv (b) yields:

$$\frac{-(2N_2G_2+1)}{1} = \frac{V_1}{V_2} \text{ or } \frac{V_1}{V_2} = |1+2N_2G_2| \cdots\cdots\cdots \textbf{(v)}$$

Addition of iv (a) & iv (c) yields:

$$|-(1+2N_1G_1)| = \frac{V_3}{V_1} \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \textbf{(vi)}$$

Similarly adding **iv** (b) and **iv** (c) we get,

$$|-(2N_3G_3+1)| = \frac{V_2}{V_3} \cdots\cdots\cdots\cdots\cdots\cdots \textbf{(vii)}$$

From Eqn. (**i**)

X = AX

$$A = \begin{bmatrix} -G_1N_1 & -G_2N_2 & G_3N_3 \\ G_1N_1 & -G_2N_2 & -G_3N_3 \\ -G_1N_1 & G_2N_2 & -G_3N_3 \end{bmatrix}$$

$$\text{Hence } X = \begin{bmatrix} -G_1N_1 & -G_2N_2 & G_3N_3 \\ G_1N_1 & -G_2N_2 & -G_3N_3 \\ -G_1N_1 & G_2N_2 & -G_3N_3 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix}$$

$$\text{or,} X_1 = -G_1N_1X_1 - G_2N_2X_2 + G_3N_3X_3, \cdots \textbf{viii}(a)$$

$$X_2 = G_1N_1X_1 - G_2N_2X_2 - G_3N_3X_3, \cdots\cdots \textbf{viii}\ (b)$$

$$X_3 = -G_1N_1X_1 + G_2N_2X_2 - G_3N_3X_3, \cdots\cdots \textbf{viii}\ (c)$$

Adding **viii** (a) and **viii** (b), we get,

$$\frac{X_1}{X_2} = |-(1 + 2\, G_2N_2)| \cdots\cdots \textbf{ix}(a)$$

Adding **viii** (a) and **viii** (c), we get,

$$\frac{X_3}{X_1} = |-(1 + 2\, G_1N_1)| \cdots\cdots\cdots \textbf{ix}(b)$$

Adding **viii** (b) and **viii** (c), we get,

$$\frac{X_2}{X_3} = |-(1 + 2\, G_3N_3)| \cdots\cdots\cdots\cdots\cdots \textbf{ix}(c)$$

Comparing (**vi**) & **ix** (b) we get

$$\frac{X_3}{X_1} = \frac{V_3}{V_1} = |-(1 + 2\, G_1N_1)| \cdots\cdots\cdots\cdots \textbf{x}(a)$$

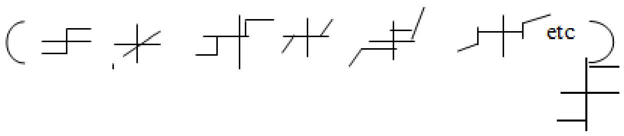
Comparing (**vii**) & **ix** (c) we get

$$\frac{X_2}{X_3} = \frac{V_2}{V_3} = |-(2\, N_3G_3 + 1)| \cdots\cdots\cdots\cdots \textbf{x}(b)$$

Comparing (**v**) & **ix** (a) we get

$$\frac{X_1}{X_2} = \frac{V_1}{V_2} = |-(1 + 2\, N_2G_2)| \cdots\cdots\cdots \textbf{x}(c)$$

2.2 Non-memory Type Nonlinear Elements



2.2.1 (A_1): Explicit nonlinear functions (DFs)

(a) $N_1 = f(X_1), N_2 = f(X_2), N_3 = f(X_3)$ *etc*

(b) G_1, G_2 and G_3: are the functions of jω in frequency response analysis.

(c) Eqn. (**iii**) can be separated into real and imaginary parts for numerical examples:

Real (**iii**) = **xi** (a) and Imaginary (**iii**) = **xi** (b)

Alternatively using the *phase* condition and *gain* condition, equations (4) and (5) are developed respectively in place of **xi** (a) and **xi** (b).

(d) Eqn. **x**: Amplitude Ratio Conditions:

$$\frac{X_3}{X_1} = \frac{V_3}{V_1} = |-(1 + 2G_1N_1)| \cdots \cdots \cdots\cdots\cdots \textbf{x}\ (a)$$

$$\frac{X_2}{X_3} = \frac{V_2}{V_3} = |-(1 + 2G_3N_3)| \cdots \cdots \cdots\cdots\cdots \textbf{x}\ (b)$$

$$\frac{X_1}{X_2} = \frac{V_1}{V_2} = |-(1 + 2G_2N_2)| \cdots \cdots \cdots\cdots\cdots \textbf{x}\ (c)$$

I. *Numerical examples:*

Example 1: Consider Figure 3 where the linear elements are $G_1(s) = \frac{2}{s(s+1)^2}$; $G_2(s) = \frac{1}{s(s+4)}$; $G_3(s) = \frac{1}{s(s+2)}$ the nonlinear elements are shown in Figure 4.

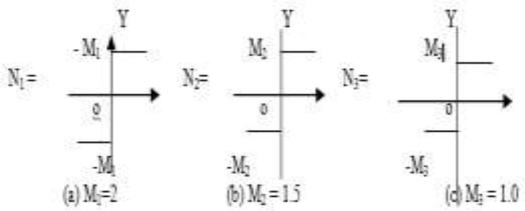


Fig. 4: All Ideal Relays

These three nonlinear elements are represented by their D.Fs as:

$N_1(X_1) = \frac{4M_1}{\pi X_1}$; $N_2(X_2) = \frac{4M_2}{\pi X_2}$; $N_3(X_3) = \frac{4M_3}{\pi X_3}$ (Explicit nonlinear functions).

In the autonomous state L.C (assuming a single frequency limit cycle oscillation) exists when the following conditions are satisfied:

(*i*) The Phase condition: $\theta_{C1} + \theta_{C2} + \theta_{C3} = 180^0$

$$\theta_{C1} = \text{Arg}\frac{G_1(j\omega)N_1}{1 + N_1G_1(j\omega)}$$

$$\theta_{C2} = \text{Arg}\frac{G_2(j\omega)N_2}{1 + N_2G_2(j\omega)}$$

$$\theta_{C3} = \text{Arg}\frac{G_3(j\omega)N_3}{1 + N_3G_3(j\omega)}$$

(*ii*)The Gain condition: $\frac{C_1}{R_1}\frac{C_2}{R_2}\frac{C_3}{R_3} = 1$

Where

$$\frac{C_1}{R_1} = \frac{G_1(j\omega)N_1}{1+N_1G_1(j\omega)};\ \frac{C_2}{R_2} = \frac{G_2(j\omega)N_2}{1+N_2G_2(j\omega)};$$

$$\frac{C_3}{R_3} = \frac{G_3(j\omega)N_3}{1+N_3G_3(j\omega)};$$

$$\text{or } \frac{G_1(j\omega)N_1G_2(j\omega)N_2G_3(j\omega)N_3}{[1+N_1G_1(j\omega)][1+N_2G_2(j\omega)][1+N_3G_3(j\omega)]} = 1$$

(*iii*) Amplitude ratio conditions:

$$\frac{X_1}{X_2} = |1+2N_2G_2|, \frac{X_2}{X_3} = |1+2N_3G_3|,$$

$$\frac{X_3}{X_1} = |1+2N_1G_1|$$

$$N_1 = \frac{4(2)}{\pi X_1} = \frac{8}{\pi X_1};\ N_2 = \frac{4(1.5)}{\pi X_2};$$

$$N_3 = \frac{4(1)}{\pi X_3} = \frac{4}{\pi X_3} \cdots (A)$$

$$\text{Or } X_1 = \frac{8}{\pi N_1}, X_2 = \frac{6}{\pi N_2}, X_3 = \frac{4}{\pi N_3} \cdots\cdots (A)'$$

$$\text{Or } |G_1| = \frac{2}{\omega(1+\omega^2)};\ |G_2| = \frac{1}{\omega\sqrt{16+\omega^2}}$$

$$|G_3| = \frac{1}{\omega\sqrt{\omega^2+4}}$$

From (iii): (Amplitude Ratio conditions)

$$\frac{X_1}{X_2} = 1+2N_2G_2 = 1+\frac{2\times 6}{\pi X_2}\frac{1}{\omega\sqrt{16+\omega^2}} = 1+\frac{12}{\pi X_2}\frac{1}{\omega\sqrt{16+\omega^2}} \cdots\cdots\cdots\cdots\cdots\cdots (1)$$

$$\frac{X_2}{X_3} = 1+\frac{8}{\pi X_3}\frac{1}{\omega\sqrt{\omega^2+4}} \cdots\cdots\cdots\cdots\cdots\cdots (\mathbf{2})$$

$$\frac{X_3}{X_1} = 1+\frac{32}{\pi X_1}\frac{1}{\omega(1+\omega^2)} \cdots\cdots\cdots\cdots\cdots\cdots\ (\mathbf{3})$$

From *(i):* (Phase Condition)

$$\theta c_1 = \text{Arg}\frac{G_1(j\omega)N_1}{1+N_1G_1(j\omega)} = \text{Arg}\frac{2N_1}{j\omega(1-\omega^2+2j\omega)+2N_1}$$

$$= -\tan^{-1}\frac{\omega(1-\omega^2)}{2(N_1-\omega^2)} = \tan^{-1}\frac{\omega(\omega^2-1)}{2(N_1-\omega^2)}$$

$$\text{or } \tan\theta c_1 = \frac{\omega(\omega^2-1)}{2(N_1-\omega^2)}$$

Similarly,

$$\theta c_2 = \text{Arg}\frac{N_2G_2(j\omega)}{1+N_2G_2(j\omega)} = \text{Arg}\frac{N_2}{j\omega(j\omega+4)+N_2}$$

$$= -\tan^{-1}\frac{4\omega}{(N_2-\omega^2)} \quad \text{or} \quad \tan\theta c_2 = -\frac{4\omega}{(N_2-\omega^2)}$$

$$\theta c_3 = \text{Arg}\frac{G_3(j\omega)N_3}{1+N_3G_3(j\omega)} = \text{Arg}\frac{N_3}{(N_3-\omega^2)+2j\omega}$$

$$= \tan^{-1}\frac{2\omega}{(N_3-\omega^2)} \text{ Or } \tan\theta c_3 = \frac{-2\omega}{(N_3-\omega^2)}$$

Or From *(i)*: $\theta c_1 + \theta c_2 + \theta c_3 = 180^o$:

Or $\theta c_3 = 180 - (\theta c_1 + \theta c_2)$

Hence $tan\ \theta c_3 = \tan[180 - (\theta c_1 + \theta c_2)] = -\tan(\theta c_1 + \theta c_2)$

$$= \frac{-[\tan\theta c_1 + \tan\theta c_2]}{[1-\tan\theta c_1 \tan\theta c_2]}$$

$$\text{Or } \frac{-2\omega}{(N_3-\omega^2)} = \frac{-\left[\frac{\omega(\omega^2-1)}{2(N_1-\omega^2} + \frac{-4\omega}{N_2-\omega^2}\right]}{\left[1-\frac{\omega(\omega^2-1)}{2(N_1-\omega^2)}\right]\times\frac{-4\omega}{(N_2-\omega^2)}}$$

$$\text{Or } 2[2(N_1-\omega^2)(N_2-\omega^2)+4\omega^2(\omega^2-1)]$$

$$= (N_3-\omega^2)[(\omega^2-1)(N_2-\omega^2)-8(N_1-\omega^2)]$$

$$= (N_3-\omega^2)(N_2-\omega^2)(\omega^2-1)$$

$$-8(N_1-\omega^2)(N_3-\omega^2)\cdots\cdots(4a)$$

(In terms of $N_1 N_2 N_3$)

Putting the value of $N_1, N_2 \& N_3$ from A' in the above equation, we get,

$$2\left[2\left(\frac{8}{\pi X_1}-\omega^2\right)\left(\frac{6}{\pi X_1}-\omega^2\right)+4\omega^2(\omega^2-1)\right]$$

$$= \left(-\omega^2+\frac{4}{\pi X_3}\right)\left(\frac{6}{\pi X_2}-\omega^2\right)(\omega^2-1)$$

$$-8\left(\frac{8}{\pi X_1}-\omega^2\right)\left(-\omega^2+\frac{4}{\pi X_3}\right)\cdots(\mathbf{4})$$

(In terms of X_1, X_2, X_3): (uses Phase Condition)

From (*ii*): (Gain conditions)

$$\frac{C_1}{R_1}\cdot\frac{C_2}{R_2}\cdot\frac{C_3}{R_3} = \frac{G_1G_2G_3N_1N_2N_3}{[1+N_1G_1][1+N_2G_2][1+N_3G_3]} = 1$$

$$\text{Or } 0 = 1+N_2N_3G_2G_3+N_2G_2+N_3G_3+N_1G_1+N_1N_2G_1G_2+N_1N_3G_1G_3 \cdots (5a)$$

(In terms of $N_1\ N_2\ N_3$)

$$\text{Or } 0 = 1+\frac{6}{\pi X_2}\frac{4}{\pi X_2}\frac{1}{\omega\sqrt{(\omega^2+16)}}\frac{1}{\omega\sqrt{4+\omega^2}}+\frac{6}{\pi X_2}\frac{1}{\omega\sqrt{16+\omega^2}}+\frac{4}{\pi X_3}\frac{1}{\omega\sqrt{\omega^2+4}}+\frac{8}{\pi X_1}\frac{2}{\omega(1+\omega^2)}$$

$$+\frac{8}{\pi X_1}\frac{6}{\pi X_2}\frac{2}{\omega(1+\omega^2)}\frac{1}{\omega\sqrt{16+\omega^2}}+\frac{8}{\pi X_1}\frac{4}{\pi X_3}\frac{2}{\omega(1+\omega^2)}\frac{1}{\omega\sqrt{\omega^2+4}}\cdots(\mathbf{5})$$

(In terms of X_1, X_2, X_3): (Uses Gain Condition)

Summarizing: there are 4 variables: ω, X_1, X_2, X_3: To determine these 4 unknowns, 4 independent equations are necessary: *any two of Eqns. (1), (2) & (3), and equations (4) and (5) can be used.*

Use of amplitude ratio conditions:

From (1),

$$\frac{X_1}{X_2} = 1 + \frac{12}{\pi X_2}\,\frac{1}{\omega\sqrt{16+\omega^2}}$$

$$or\ \frac{X_1}{X_2} - \frac{12}{\pi X_2}\,\frac{1}{\omega\sqrt{16+\omega^2}} = 1$$

$$Or\ \frac{1}{X_2}\left[X_1 - \frac{12}{\pi\omega\sqrt{16+\omega^2}}\right] = 1$$

$$or\ X_2 = X_1 - \frac{12}{\pi\omega\sqrt{16+\omega^2}} \cdots\cdots\cdots\cdots \quad \mathbf{(6)}$$

From (2),

$$\frac{X_2}{X_3} = 1 + \frac{8}{\pi X_3}\,\frac{1}{\omega\sqrt{\omega^2+4}}$$

$$\text{Or}\ \frac{X_2}{X_3} = -\frac{8}{\pi X_3}\,\frac{1}{\omega\sqrt{\omega^2+4}} = 1$$

$$\text{Or}\ \frac{1}{X_3}\left[X_2 - \frac{8}{\pi\omega\sqrt{\omega^2+4}}\right] = 1$$

$$\text{Or}\ X_3 = X_2 - \frac{8}{\pi\omega\sqrt{\omega^2+4}} \cdots\cdots\cdots\cdots\cdots\cdots \mathbf{(7)}$$

From (3),

$$\frac{X_3}{X_1} = 1 + \frac{32}{\pi X_1\omega(1+\omega^2)}$$

$$\text{Or}\ \frac{X_3}{X_1} = -\frac{32}{\pi X_1} - \frac{1}{\pi X_1\omega(1+\omega^2)} = 1$$

$$\text{Or}\ \frac{1}{X_1}\left[X_3 - \frac{32}{\pi\omega(1+\omega^2)}\right] = 1$$

$$\text{Or}\ X_1 = X_3 - \frac{32}{\pi\omega(1+\omega^2)} \cdots\cdots\cdots\cdots\cdots\cdots \mathbf{(8)}$$

Putting (6) in (4) we get, 0=1+

$$\frac{24}{\pi^2X_3\omega^2\sqrt{\omega^2+16}\sqrt{\omega^2+4}\left[X_1 - \frac{12}{\pi\omega\sqrt{\omega^2+16}}\right]}$$

$$+\frac{6}{\pi\omega\sqrt{\omega^2+16}\left[X_1 - \frac{12}{\pi\omega\sqrt{\omega^2+16}}\right]}$$

$$\frac{4}{\pi X_3\omega\sqrt{\omega^2+4}} + \frac{16}{\pi X_1\omega(\omega^2+1)}$$

$$+\frac{96}{\pi^2X_1\omega^2(\omega^2+1)\sqrt{\omega^2+16}\left[X_1 - \frac{12}{\pi\omega\sqrt{\omega^2+16}}\right]}$$

$$+\frac{64}{\pi^2X_1X_3\omega^2(\omega^2+1)\sqrt{\omega^2+4}}$$

Putting Eqn. (8) in the above equation we get,

$$0=1+\frac{24}{\pi^2X_3\omega^2\sqrt{\omega^2+16}\sqrt{\omega^2+4}\left[X_3 - \frac{32}{\pi\omega(1+\omega^2)} - \frac{12}{\pi\omega\sqrt{\omega^2+16}}\right]}$$

$$+\frac{6}{\pi\omega\sqrt{\omega^2+16}\left[X_3 - \frac{32}{\pi\omega(1+\omega^2)} - \frac{12}{\pi\omega\sqrt{\omega^2+16}}\right]}$$

$$+\frac{4}{\pi X_3\omega\sqrt{\omega^2+4}} + \frac{16}{\pi\omega(\omega^2+1)\left[X_3 - \frac{32}{\pi\omega(1+\omega^2)}\right]}$$

$$+\frac{96}{\pi^2\omega^2(\omega^2+1)\sqrt{\omega^2+16}\left[X_3 - \frac{32}{\pi\omega(\omega^2+1)}\right]\left[X_3 - \frac{32}{\pi\omega(1+\omega^2)} - \frac{12}{\pi\omega\sqrt{\omega^2+16}}\right]}$$

$$+\frac{64}{\pi^2X_3\omega^2(\omega^2+1)\sqrt{\omega^2+4}\left[X_3 - \frac{32}{\pi\omega(\omega^2+1)}\right]}$$

Or

$$0 = 1 + \frac{8}{X_3^2\pi^2\omega^2}\left[\frac{3(1+\omega^2) + 12\sqrt{\omega^2+4} + 8\sqrt{\omega^2+16}}{(1+\omega^2)\sqrt{\omega^2+16}\sqrt{\omega^2+4}}\right]$$

$$+\frac{2}{X_3\pi\omega}\left[\frac{-12(1+\omega^2)}{\sqrt{\omega^2+4}\left[32\sqrt{\omega^2+16}+12(1+\omega^2)\right]}\right.$$

$$+\frac{3(\omega^2+1)\sqrt{\omega^2+4} + 2(\omega^2+1)\sqrt{\omega^2+16} + 8\sqrt{\omega^2+4}\sqrt{\omega^2+16}}{(\omega^2+1)\sqrt{\omega^2+4}\sqrt{\omega^2+16}}$$

$$\left.-\frac{12}{\left[16\sqrt{\omega^2+16}+3(1+\omega^2)\right]} - \frac{1}{\sqrt{\omega^2+4}}\right]$$

$$+2\left[\frac{-3(1+\omega^2)}{32\sqrt{\omega^2+16}+12(1+\omega^2)} - \frac{1}{4} + \frac{3}{2\left(32\sqrt{\omega^2+16}+12(1+\omega^2)\right)}\right]$$

Or multiplying both sided by X_3^2 in the above we get,

$$0 = X_3^2 + \frac{8}{\pi^2\omega^2}\left[\frac{3(1+\omega^2)+12\sqrt{\omega^2+4}+8\sqrt{\omega^2+16}}{(1+\omega^2)\sqrt{\omega^2+4}\sqrt{\omega^2+16}}\right]$$

$$+\frac{2X_3}{\pi\omega}\left[\frac{-12(1+\omega^2)}{\sqrt{\omega^2+4}\left[32\sqrt{\omega^2+16}+12(1+\omega^2)\right]} - \frac{1}{\sqrt{\omega^2+4}} - \frac{12}{\left[16\sqrt{\omega^2+16}+3(1+\omega^2)\right]}\right.$$

$$\left.+\frac{3(\omega^2+1)\sqrt{\omega^2+4}+2(\omega^2+1)\sqrt{\omega^2+16}+8\sqrt{\omega^2+4}\sqrt{\omega^2+16}}{(\omega^2+1)\sqrt{\omega^2+4}\sqrt{\omega^2+16}}\right]$$

$$+2X_3^2\left[\frac{-12(1+\omega^2) - 32\sqrt{\omega^2+16} - 12(1+\omega^2) + 6}{4\left(32\sqrt{\omega^2+16}+12(1+\omega^2)\right)}\right]$$

$$Or\ 0 = X_3^2\left[\frac{\left(32\sqrt{\omega^2+16}\right)+6}{2\left(32\sqrt{\omega^2+16}+12(1+\omega^2)\right)}\right]$$

$$+\frac{2X_3}{\pi\omega}\left[\frac{+12(1+\omega^2)^2\sqrt{\omega^2+16}+192(\omega^2+1)\sqrt{\omega^2+4}\sqrt{\omega^2+16}+36(\omega^2+1)^2\sqrt{\omega^2+4}+64(\omega^2+1)(\omega^2+16)}{(1+\omega^2)\sqrt{\omega^2+16}\sqrt{\omega^2+4}\left[32\sqrt{\omega^2+16}+12(1+\omega^2)\right]}+\right.$$

$$\left.\frac{+256(\omega^2+16)\sqrt{\omega^2+4}-32(\omega^2+16)-12(1+\omega^2)\sqrt{\omega^2+16}}{(1+\omega^2)\sqrt{\omega^2+16}\sqrt{\omega^2+4}\left[32\sqrt{\omega^2+16}+12(1+\omega^2)\right]} - \frac{12}{\left[16\sqrt{\omega^2+16}+3(1+\omega^2)\right]}\right]$$

$$\frac{+8}{\pi^2\omega^2}\left[\frac{3(1+\omega^2)+12\sqrt{\omega^2+4}+8\sqrt{\omega^2+16}}{(1+\omega^2)\sqrt{\omega^2+4}\sqrt{\omega^2+16}}\right] \cdots\cdots\cdots\cdots \mathbf{(S)}$$

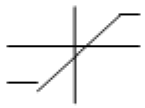
which can also be used for the determination of X_3 for a fixed value of ω.

Eqn. (4) (Uses phase condition) can also be further simplified as:

$$\text{Or } \frac{192}{\pi^2 X_1 X_2} - \frac{24(\omega^2-1)}{\pi^2 X_2 X_3} + \frac{256}{\pi^2 X_1 X_3} + \frac{6\omega^2(\omega^2-5)}{\pi X_2}$$
$$-\frac{96\omega^2}{\pi X_1} + \frac{4\omega^2}{\pi X_3}(\omega^2-9) = \omega^6 - 19\omega^4 + 8\omega^2 \cdots (9)$$

I. Equations (4-uses Phase Condition) or (9), (5-uses Gain Condition) and any two of (6), (7), (8) (use Amplitude Ratio Condition) can be used to solve for ω, X_1, X_2 & X_3: for explicit non-memory type nonlinearities.

2.2.1 (A_2): Implicit nonlinear functions (DFs)



(a) $N_1 = f(X_1), N_2 = f(X_2), N_3 = f(X_3)$

(b) G_1, G_2 and G_3 : Are the functions of j ω in frequency response analysis.

(c) Eqn. **(iii)** can be separated into real and imaginary parts for numerical examples:

Real **(iii)** = **xi** (a); Imaginary **(iii)** = **xi** (b)

Alternatively using the *phase* condition and *gain* condition, equations (4) and (5) are developed respectively in place of **xi** (a) and **xi** (b).

(d) Eqn. **x**: Amplitude Ratio Conditions:

$$\frac{X_3}{X_1} = \frac{V_3}{V_1} = |-(1+2G_1N_1)| \cdots\cdots\cdots \cdots \mathbf{x}\,(a)$$

$$\frac{X_2}{X_3} = \frac{V_2}{V_3} = |-(1+2G_3N_3)| \cdots\cdots\cdots \cdots \mathbf{x}\,(b)$$

$$\frac{X_1}{X_2} = \frac{V_1}{V_2} = |-(1+2G_2N_2)| \cdots\cdots\cdots \cdots \mathbf{x}\,(c)$$

Under autonomous state to determine limit cycle in 3 X 3 nonmemory type system with implicit nonlinear function there are four unknowns: ω (Frequency of LC), N_1 or X_1, N_2 or X_2, N_3 or X_3: There are five Eqns. **xi**(a), **xi**(b) and **x**(a), **x** (b) and **x** (c) can be used. To determine the above four unknowns Eqns. (4), (5) and any two of equations **x**(a), **x**(b) & **x**(c) can be used.

II. *Numerical problems:*

Example 2: Consider Figure 3 where the linear elements are $G_1(s) = \frac{2}{s(s+1)^2}$; $G_2(s) = \frac{1}{s(s+4)}$;

$G_3(s) = \frac{1}{s(s+2)}$; And the nonlinear elements are shown in Figure 5.

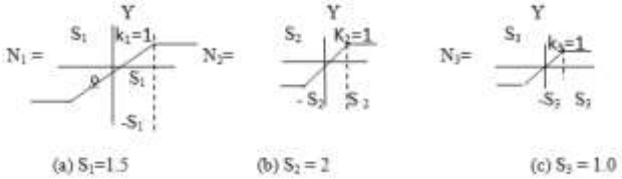


Fig. 5: All Ideal Saturation type nonlinear elements (with slopes k_1, k_2 & k_3)

These nonlinear elements are represented by their D.Fs as shown in Eqn. (10).

$$\left.\begin{aligned} N_1(X_1) &= \frac{2}{\pi}\left[\sin^{-1}\frac{1.5}{X_1} + \frac{1.5}{X_1}\sqrt{1-\left(\frac{1.5}{X_1}\right)^2}\right] \\ N_2(X_2) &= \frac{2}{\pi}\left[\sin^{-1}\frac{2}{X_2} + \frac{2}{X_2}\sqrt{1-\left(\frac{2}{X_2}\right)^2}\right] \\ N_3(X_3) &= \frac{2}{\pi}\left[\sin^{-1}\frac{1}{X_3} + \frac{1}{X_3}\sqrt{1-\left(\frac{1}{X_3}\right)^2}\right] \end{aligned}\right\} \quad (10)$$

In the autonomous state L.C (assuming a single frequency limit cycle oscillation) exists when the following conditions are satisfied:

(a) The Phase condition: $\theta_{C1} + \theta_{C2} + \theta_{C3} = 180^0$.

$$Loop1: \theta c_1 = Arg\,\frac{G_1N_1}{1+G_1N_1};$$

$$Loop\ 2;\ \theta c_2 = Arg\,\frac{G_2N_2}{1+G_2N_2};$$

$$Loop\ 3: \theta c_3 = Arg\,\frac{G_3N_3}{1+G_3N_3}$$

$$\text{Or } \tan\theta c_1 = \tan\frac{G_1N_1}{1+G_1N_1};$$

$$\tan\theta c_2 = \tan\frac{G_2N_2}{1+G_2N_2};$$

$$\tan\theta c_3 = \tan\frac{G_3N_3}{1+G_3N_3}$$

$$\text{Or } \theta c_3 = 180 - (\theta c_1 + \theta c_2)$$

$$\text{or} \tan \theta c_3 = \tan[180 - (\theta c_1 + \theta c_2)]$$

$$= -\tan(\theta c_1 + \theta c_2)$$

Hence

$$\tan \theta c_3 = -\tan(\theta c_1 + \theta c_2) = -\frac{[\tan \theta c_1 + \tan \theta c_2]}{[1 - \tan \theta c_1 \tan \theta c_2]}$$

$$\text{Or } \frac{-2\omega}{(N_3 - \omega^2)} = -\frac{[\tan \theta c_1 + \tan \theta c_2]}{[1 - \tan \theta c_1 \tan \theta c_2]}$$

$$\text{Or } 2 \times [2(N_1 - \omega^2)(N_2 - \omega^2) + 4\omega^2(\omega^2 - 1)]$$

$$= (N_3 - \omega^2)[(\omega^2 - 1)(N_2 - \omega^2) - 8(N_1 - \omega^2)] \cdots (4a)$$

$$\text{Or } 4N_1N_2 - N_2N_3\omega^2 + 8N_1N_3 + N_2N_3 - 12\omega^2 N_1 - 5\omega^2 N_2 - 9N_3\omega^2 + N_3\omega^4 + \omega^4 N_2$$
$$= -21\omega^4 + 8\omega^2 + \omega^6 \cdots\cdots (4)$$

(b) Gain Condition:

$$1 + N_2N_3\, G_2G_3 + N_1N_2\, G_1G_2 + N_1N_3\, G_1G_3$$
$$+ N_1G_1 + N_2G_2 + N_3G_3 = 0 \cdots (5a)$$

$$\text{Or } 1 + N_2N_3 \frac{1}{\omega\sqrt{16+\omega^2}} \frac{1}{\omega\sqrt{\omega^2+4}} + N_1N_2 \frac{2}{\omega(1+\omega^2)} \frac{1}{\omega\sqrt{16+\omega^2}} + N_1N_3 \frac{2}{\omega(1+\omega^2)} \frac{1}{\omega\sqrt{\omega^2+4}}$$
$$+ N_1 \frac{2}{\omega(1+\omega^2)} + \frac{N_2}{\omega\sqrt{16+\omega^2}} + \frac{N_3}{\omega\sqrt{\omega^2+4}} = 0 \cdots\cdots (5)$$

(c) Eqn. **x**: Amplitude Ratio Conditions:

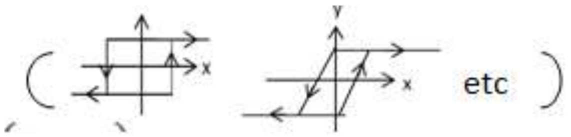
$$\frac{X_3}{X_1} = \frac{V_3}{V_1} = |-(1 + 2G_1N_1)| \cdots \cdots\cdots \mathbf{x}\,(a)$$

$$\frac{X_2}{X_3} = \frac{V_2}{V_3} = |-(1 + 2G_3N_3)| \cdots \cdots \cdots \mathbf{x}\,(b)$$

$$\frac{X_1}{X_2} = \frac{V_1}{V_2} = |-(1 + 2G_2N_2)| \cdots \cdots \cdots \mathbf{x}\,(c)$$

Under autonomous state to determine limit cycle in 3 X 3 non-memory type system, there are *four* unknowns: ω (Frequency of LC), N_1 or X_1, N_2 or X_2, N_3 or X_3: There are five Eqn. **xi**(a), **xi**(b) and **x**(a), **x** (b) and **x** (c). To determine the above four unknowns Eqns. (4), (5) and *any two of equations* **x** (a), **x** (b) & **x** (c) can be used. It may be noted that in this case X_1, X_2, and X_3 are determined from N_1, N_2, and N_3 respectively using *Newton Raphson method*.

B: *For Memory type nonlinear elements*:



(a) $N_1 = f(X_{m_1}, \omega), N_2 = f(X_{m_2}, \omega), N_3 = f(X_{m_3}, \omega)$,

(b) G_1, G_2, G_3 are functions of j ω: (For frequency response)

where ω is the frequency of L. C oscillations.

(c) Eqn. (**iii**) can be separated by Gain and Phase conditions

(i) Gain of the loop should be 1

(ii) Phase of the Loop should be θ =180⁰ =

$$\angle G_1 + \angle G_2 + \angle G_3 + \angle N_1 + \angle N_2 + \angle N_3$$

Where phase angles of G come from j ω but phase angles of N come from the phase shift of concerned D. F.

3 Methods of Determination of Limit Cycles (L. C)

3.1 Analytical

(A1): For non-memory type nonlinear elements: Explicit nonlinear functions (DFs).

Eqn. (**iii**) (characteristic equation) is separated into real and imaginary parts.

(A2): For non-memory type nonlinear elements: Implicit nonlinear functions (DFs).

(a) $N_1 = |N_1(X_1)|,\ N_2 = |N_2(X_2)|,$

$N_3 = |N_3(X_3)|$

(b) $G_1 = |G_1(j\omega)|, G_2 = |G_2(j\omega)|, G_3 = |G_3(j\omega)|$

(c) Use eqn. x (a), x (b) & x (c)

Solve for ω = 0.1, 0.2, 0.3.................3.0 with relevant Eqns.

(B): For memory type nonlinearity.

Eqn. (**iii**) (characteristic equation) is separated into real and imaginary parts.

(a) $N_1 = f(X_{m_1}, \omega), N_2 = f(X_{m_2}, \omega),$

$N_3 = f(X_{m_3}, \omega)$

(b) Conditions Loop Gain = 1

(c) Phase Loop $= \theta = 180^0 = \angle G + \angle N = \angle G_1 + \angle G_2 + \angle G_3 + \angle N_1 + \angle N_2 + \angle N_3$

, where phase angles of G come from jω and the phase of N comes from the phase shift of DFs.

Example 4, [16]: 3 X 3 systems:

$$H=\begin{bmatrix}1 & 1 & 1\\ 1 & 1 & 1\\ 1 & 1 & 1\end{bmatrix}: \frac{V_1}{V_2}=\frac{V_2}{V_3}=\frac{V_1}{V_3}=\frac{X_1}{X_2}=1=\frac{X_1}{X_3}=\frac{X_2}{X_3}$$

(All the cross feedbacks are made negative: the analysis of such systems becomes trivial)

Hence Example 1: 2 X 2 systems:

$$H=\begin{bmatrix}1 & 1\\ -1 & 1\end{bmatrix}: \frac{X_1}{X_2}=\frac{V_1}{V_3}=\frac{|N_2G_2|}{|1+N_1G_1|}=\frac{|1+N_2G_2|}{|N_1G_1|}=|1+2N_2G_2|$$

Example 2: 3 X 3 system:

$$H=\begin{bmatrix}1 & 1 & -1\\ -1 & 1 & 1\\ 1 & -1 & 1\end{bmatrix}:$$

$$\frac{X_1}{X_2}=\frac{V_1}{V_2}=|1+2N_2G_2|;\ \frac{X_2}{X_3}=\frac{V_2}{V_3}=|1+2N_3G_3|;$$

$$\frac{X_3}{X_1}=\frac{V_3}{V_1}=|1+2N_1G_1|;$$

Simultaneous solution of Eqns. (4-Gain Condition), (5-Phase Condition) and any two of Eqns. (6), (7), (8) (Amplitude Ratio Condition), will yield the four unknown parameters $\omega, X_1 X_2 X_3$ of possible self-oscillations in the system considered.

3.2 Graphical

When a closed loop system exhibits limit cycles, the signal at any point of the loop is transmitted around the loop to that point without any change in amplitude and phase. Thus, a system exhibits a limit cycle when the loop gain is one and the loop phase shift is $\pm 2n\pi$, [21], [51], when n is an integer.

For the proposed work the complexity involved in the structure in particular for implicit or having the memory type nonlinearities, it would be extremely difficult to formulate and simplify the expressions, even in the harmonic balance method, [32], [51]. Hence it is felt necessary to develop a graphical technique using the harmonic balance method as discussed in the references, [16], [27], [49], [51], for non-memory and memory type nonlinearities respectively.

3.2.1 Graphical Method for 2X2 system

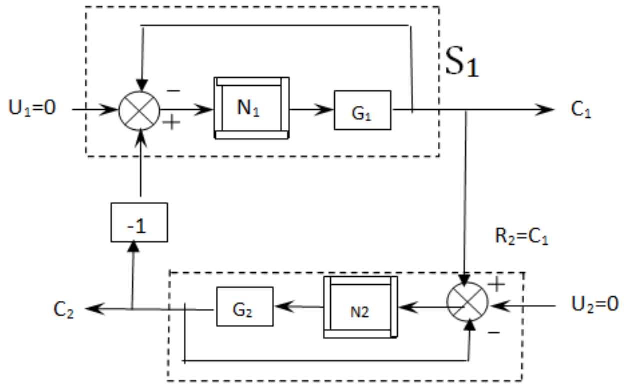


Fig. 6: A class of 2 x 2 nonlinear systems

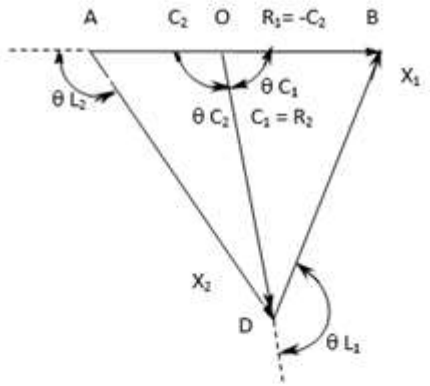


Fig. 7: Phase diagram for the system of Figure 6

Consider the system of Figure 6. The self-oscillation revealed by the system is shown in the phase diagram in Figure 7. The sides of the triangle OBD correspond to the quantities for subsystem S_1 and those of the triangle OAD correspond to subsystem S_2 of Figure 8. For a fixed frequency ω, the angles ΘL_1 (Arg. G_1 (j ω)) and ΘL_2 (Arg. G_2 (j ω)) are fixed.

As a result, the angle ODB must remain constant at (180-ΘL_1) constraining D to lie on a circle as given in Figure 9. The phase representing X_2 would lie along the straight line drawn at an angle ΘL_2 with the phase C_2 (= - R_1). The intersections of the straight line with the circle shown in Figure 9 will represent possible self-oscillations if the following conditions are satisfied:

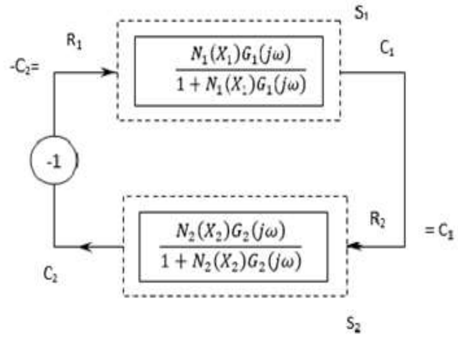


Fig. 8: Linear equivalent for the system of Figure 6

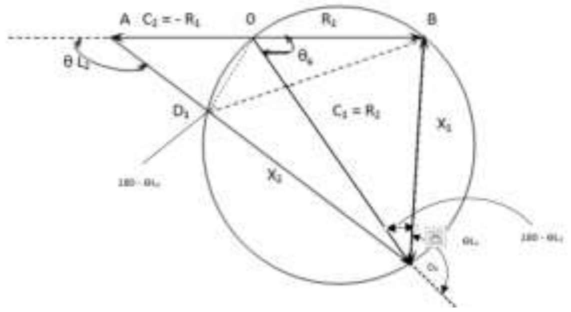


Fig. 9: Locus of D_i satisfying constant loop phase shift condition

(a) The point D_i lie on the intersection of segment O D_i B of a circle subtending an angle (180^0-ΘL_1) on OB and a straight line A D_i making an angle ΘL_2 with OA.

(b) O $D_i = C_1 = Y_1G_1$: i = 1 or 2
And B $D_i = X_1$, where $Y_1 = N_1X_1$ and

(c) OA = $C_2 = Y_2G_2$
And A $D_i = X_2$, where $Y_2 = N_2X_2$

The need to consider separate phase diagrams for various values of R_1 for checking possible self-oscillations can be eliminated if all the quantities are normalized with respect to the magnitude R_1, leading to a single phase diagram for a particular value of ω as, shown in Figure 10, wherein, OB = $R_1/R_1 = 1$; OA = $C_2/R_1 = -1.0$

O Di = $C_1/R_1 = C_1/-C_2$, i = 1 or 2
B Di = X_1/R_1; A Di = X_2/R_1

In Figure 10, 'C' is the center of the circle OC Di B of radius r = OC.
In Figure 10, Figure 11 (a), and Figure 11(b), selecting 'O' as the origin, the coordinates of the point Di (i = 1, 2) can be determined in the following manner:

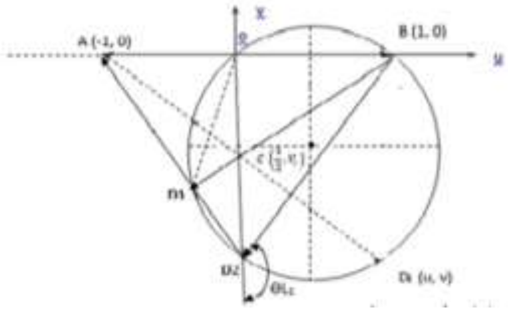


Fig. 10 Normalized Phase Diagram for the system of Figure 6

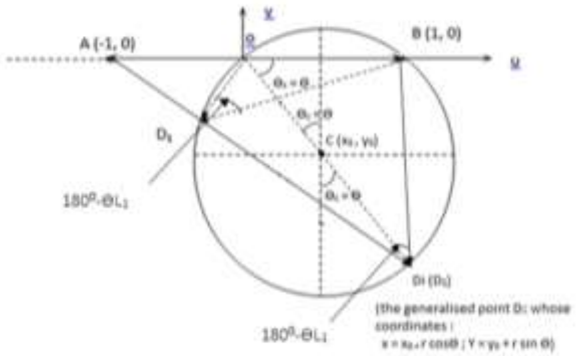


Fig. 11(a) Normalized Phase Diagram with General D_i

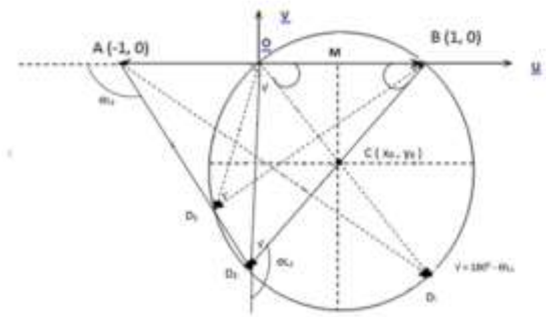


Fig. 11(b) Normalized Phase Diagram with D_1 or D_2 (D_i)

Consider the Figure 11 (b), OCD_2B is a circumscribed circle where radius = r
Fundamental relation: sine Law:

$$\frac{OB}{sinOD_2B} = \frac{1}{\sin(180-\theta L_1)}$$

$$= \frac{BD2}{sinD_2OB} = \frac{2r}{sinD_2OB} = \frac{OD_2}{sinOBD_2}$$

Explanation: OCB is an isosceles triangle:
$\angle COM = \angle CBM$

OCD_2 is an isosceles triangle: $\angle CD_2O = \angle COD_2$

Hence $\angle D_2OB = \angle CBM + \angle CD_2O$

In BOD_2 triangle: $\angle BD_2O + \angle OBD_2 + \angle BOD_2 = 180^0$

Or $\angle BD_2O + \angle OBD_2 + \angle BD_2O + \angle OBD_2 = 180^0$

$\because$ [$\angle BOD_2 = \angle CD_2O + \angle COB$]

Or $2\angle BD_2O + 2\angle OBD_2 = 180^0$

Or $\angle BOD_2 + \angle OBD_2 = 90^0$ [$BOD_2 = \angle BOD_2 + \angle OBD_2$]

$\angle BOD_2 = 90^0$ [Since $BOD_2 = \angle BD_2O + \angle OBD_2$]

Hence from sine law,

$$\frac{BD_2}{\sin 90} = \frac{2r}{\sin 90} = \frac{2r}{1} = \frac{1}{\sin(180^0 - \theta L_1)} = \frac{1}{\sin \theta L_1}$$

Or $r = \frac{1}{2 \sin \theta L_1}$ = radius of the circle

The coordinates of the center of the circle:

X coordinate of C = $x_0 = OM = \frac{1}{2}$

Y coordinate of C = $y_0 = CM = \sqrt{CO^2 - OM^2}$

$= \sqrt{r^2 - (1/2)^2}$

$$\sqrt{\frac{1}{4\sin^2 \theta L1} - 1/4} = \sqrt{\frac{1 - sin^2\theta L_1}{4sin^2\theta L_1}}$$

$$= \sqrt{\frac{\cos^2\theta L_1}{4\sin^2\theta L_1}} = \frac{1}{2tan\theta L_1}$$

If O is considered as the origin (0, 0): y_0 of C = $-\frac{1}{2tan\theta L1}$

Equation to the straight line A D_i; y = m x where, X coordinate of A = u +AO = u+1 and m = slope of the line = $\tan\Theta L_2$

Hence the Equation to straight line AD_i

$Y = v = (u+1)\tan \Theta L_2$ or $u = v \cot \Theta L_2 - 1$ (11)

The coordinates of the intersection point:

Consider the Figure 11 (a):

The x coordinate of D_i (2) = $x_0 + r\cos\theta$

Y coordinate of D_i (2) = $y_0 + r \sin \Theta_c$

The equation for a circle can be written as:

$$(u - \frac{1}{2})^2 + (v - \frac{1}{2\tan\theta L_1})^2 = (\frac{1}{2sin\theta L1})^2 \quad (12)$$

(We can determine the coordinate of D_i (the point of intersection of the straight line (A D_i) with the circle): $\boldsymbol{u_i}$ and $\boldsymbol{v_i}$ eliminating either u or v from equation (12) using eqn. (11) :

$(v\cot\Theta L_2 - 1 - \frac{1}{2})^2 + (v - \Theta\ 5 \cot\Theta L_1)^2 = (\frac{1}{2\sin\theta L_1})^2 = (0.5\ cosec\Theta L_1)^2$

Or $v^2\cot^2\Theta L_2 - 2x\frac{3}{2} v \cot\Theta L_2 + \frac{9}{4} + v^2 - 2 \times 0.5 \cot\Theta L_1 + 0.25\cot^2\theta L_1$

$= 0.25\,(1 + \cot^2\theta L_1) = 0.25 + 0.25\cot^2\theta L_1$

Or $v^2(\cot^2\Theta L_2 + 1) - v\,(3\cot\Theta L_2 + \cot\Theta L_1) + \frac{9}{4} - 0.25 = 0$

Or $v^2 cosec^2\Theta L_2 - v\,(3\cot\Theta L_2 + \cot\Theta L_1) + 2 = 0$ (13)

Hence, $v_i =$

$$\frac{(3\cot\theta L_2 + \cot\theta L_1) \pm \sqrt{(3\cot\theta L_2 + \cot\theta L_1)^2 - 8cosec^2\theta L_2}}{2cosec^2\theta L_2} \quad (14)$$

And $u_i = v_i \cot\theta L_2 - 1$ (15)

It may be noted that the coordinates of the points D_i (i = 1, 2) in the normalized phase diagram are functions only of the angles ΘL_1 and ΘL_2. For a specific system the angles ΘL_1 and ΘL_2 are known for an assumed value of ω, and, therefore, the coordinates v_i and u_i can be evaluated and subsequently we obtain:

$N_1(X_1) = \frac{Y_1}{X_1}$, (where $Y_1 = \frac{C_1}{G_1}$)

$$= \frac{C_1}{G_1 X_1} = \frac{OD_i}{G_1 BD_i} = \frac{1}{G_1}\sqrt{\frac{(u_i)^2 + (v_i)^2}{(1-u_i)^2 + (v_i)^2}} \quad (16)$$

$$N_2(X_2) = \frac{Y_2}{X_2} = \frac{C_2}{G_2 X_2} = \frac{OA}{G_2 AD_i} = \frac{1}{G_2\sqrt{(u_i+1)^2 + (v_i)^2}} \quad (17)$$

$$\frac{X_1}{X_2} = \frac{BD_i}{AD_i} = \sqrt{\frac{(1-u_i)^2 + (v_i)^2}{(1+u_i)^2 + (v_i)^2}} \quad (18)$$

Subsequently, the values of X_1 and X_2 corresponding to the values of N_1 (X_1) and N_2 (X_2) obtained in Eqns. (16) & (17) are obtained from the expressions for the DFs, and the ratio $\frac{X_1}{X_2}$ thus obtained can be compared with that obtained from Eqn. (18). The process is to be repeated for various assumed values of ω. The values of ω at which the ratio $\frac{X_1}{X_2}$ computed from Eqn. (18) and that from the D.F expressions are equal determines the frequencies of self Oscillations.

Once ω is determined the amplitudes of other variables of interest are calculated directly from the different equations developed above or directly from the normalized Phase Diagram drawn to scale corresponding to the Limit Cycling (LC) frequency.

3.2.2 Graphical Method for 3 x 3 systems

The steps depicted and illustrated in section 3.2.1 are extended for 3x3 nonlinear systems. The normalised phase diagrams are drawn with three combinations such as:

Combination 1: For subsystems S1 & S2: C1 (+ve) and C2 (-ve)

Combination 2: For subsystems S3 & S2: C2 (+ve) and C3 (-ve).

Combination 3: For subsystems S1 & S3: C3 (+ve) and C1 (-ve).

Example 1 & Example 2 are revisited:

Linear elements are represented by $G_1(s) = \frac{2}{s(s+1)^2}$; $G_2(s) = \frac{1}{s(s+4)}$; $G_3(s) = \frac{1}{s(s+2)}$ and Nonlinear elements are taken Ideal relays as shown in Figure 4 and ideal saturations as shown in Figure 5.

$$\Theta L_1 = \text{Arg.}\,(G_1(j\,\omega)) = -90 - 2\tan^{-1}(\omega):$$

$$\Theta L_2 = \text{Arg.}\,(G_2(j\,\omega)) = -90 - \tan^{-1}(\tfrac{\omega}{4}):$$

$$\Theta L_3 = \text{Arg.}\,(G_3(j\,\omega)) = -90 - \tan^{-1}(\tfrac{\omega}{2}):$$

For a fixed value of ω the Combinations of Subsystems 1, 2, and 3, Normalised Phase Diagrams are shown in Figure 12(a), (b), and (c) respectively. However, any one of these combinations can be used for the determination of limit cycling conditions and the related quantities of interest.

$$N_2 = (11-3\omega^2)\omega^2 \pm \sqrt{(11-3\omega^2)\omega^4 - 8(\omega^2+16)(1-\omega^2)^2\omega^2} \quad \cdots\cdots\cdots (1.10), [16]$$

$$N_1 = \frac{(\omega^2-1)}{8} N_2 + \frac{9\omega^2-\omega^4}{8} \quad \cdots (1.8), [16]$$

$$\frac{X_1}{X_2} = \frac{(1+\omega^2)\sqrt{[\omega^2(\omega^2+16-2N_2)+N_2^2]}}{2N_1\sqrt{\omega^2+16}} \quad \cdots (1.11), [16]$$

$$\frac{X_1}{X_2} = \frac{BD_i}{AD_i} = \sqrt{\frac{(1-u_i)^2+(u_i)^2}{(1+u_i)^2+u_i^2}} \quad \cdots (1.18), [16]$$

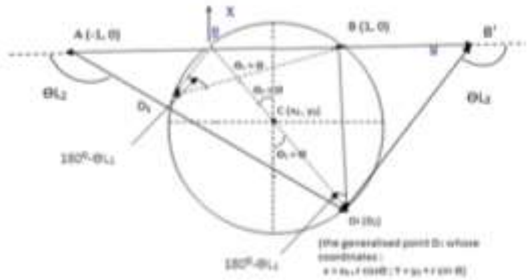


Fig. 12 (a): Normalised Phase Diagram with C_1, C_2 & C_3 for the combination 1

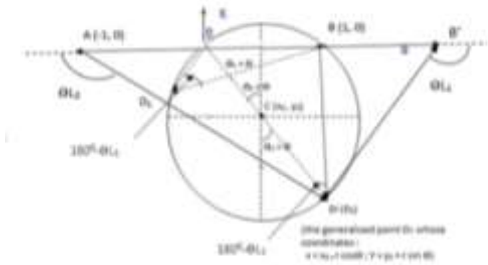


Fig. 12 (b): Normalised Phase Diagram with C_1, C_2 & C_3 for the combination 2

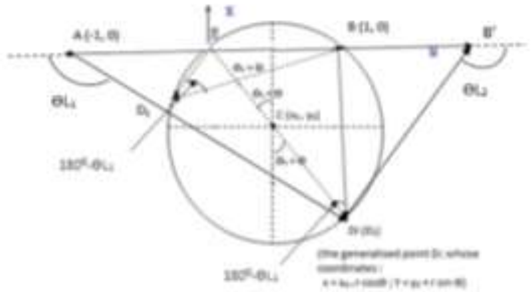
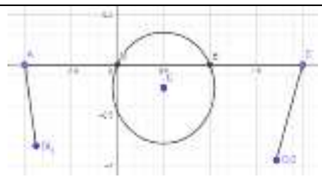
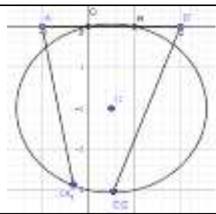
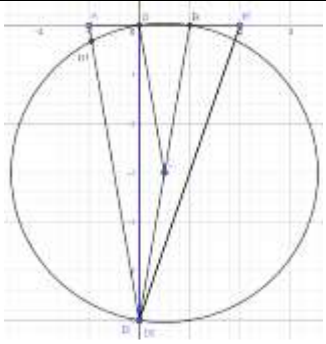
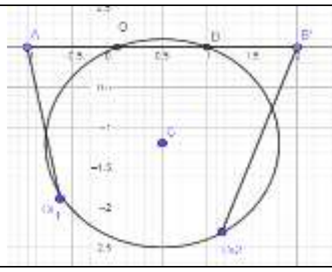


Fig. 12 (c): Normalised Phase Diagram with C_1, C_2 & C_3 for the combination 3

Table 1 shows the θL_1, θL_2, θL_3, r (radius), and the intersection points of the straight lines and circle for combination 1 corresponding to example 1. It may be noted that Table 1 contains $\frac{X_1}{X_2}$ obtained from Eqn. 1.11 and Eqn. 18 are matched at a limited cycling frequency.

Table 1(a). Shows the θ_{L1}, θ_{L2},θ_{L3}, r (radius), and the intersection points of the straight lines and circles for combination 1 corresponding to example 1

ω	**θ_{L1}**	**θ_{L2}**	**θ_{L3}**	**r**	**X_1/X_2 from eqn. 18**	**X_1/X_2 from eqn. 1.11(A1)**	**Normalized Phase Diagrams**	**Remark**
0.600	-151.93	-98.531	-106.7	-0.55257	-	-		No intersection of straight lines and circle
0.650	-156.05	-99.23	-108	0.58256	-	-		No intersection of straight lines and circle
0.700	-159.98	-99.926	-109.29	-2.128	-	-		No intersection of straight lines and circle
0.701	**-160.06**	**-99.94**	**-109.32**	**-3.1323**	**1.0**	**1.02 (matched)**		The intersection of st. lines & circle found: Confirms the occurrence of limit cycles ω=0.701, C1 = OD2 = 6 C2 = 1 C3 = 1 X1=BD2=6.08 X2=AD2=6.08 X3=B'D2=6.32
0.750	-163.74	-100.62	-110.56	-1.3583	-	-		No intersection of straight lines and circle

4 Digital Simulation

I. *Numerical problems*

Example 1 & Example 2 are revisited: A 3x3 system represented by Figure 3 has three nonlinear elements as shown in Figure 4 and Figure 5 for Ex.1 & Ex.2 respectively and three linear transfer functions are $G_1(s) = \frac{2}{s(s+1)^2}$; $G_2(s) = \frac{2}{s(s+4)}$ and $G_3(s) = \frac{1}{s(s+2)}$

Partial Fraction Expansion of $G_1(s)$, $G_2(s)$ and $G_3(s)$:

$$G_1(s) = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{(s+1)^2} = \frac{A\,(s+1)^2 + Bs(s+1)}{s(s+1)^2}$$

$$Or\ \frac{s^2(A+B) + s\,(2A+B+C) + A}{s\,(s+1)^2} = \frac{2}{s(s+1)^2}$$

Or A=2, A+B=0: B=-A =-2, 2A+B+C=0: C=-2A-B=-4+2=-2

$$\text{Hence } G_1(s) = \frac{2}{s} - \frac{2}{s+1} - \frac{2}{(s+1)^2} : \frac{2}{s}, \frac{-2}{s+1}, \frac{-2}{s+1}\left(\frac{1}{s+1}\right)$$

$$G_2(s) = \frac{1}{s(s+4)} = \frac{A}{s} + \frac{B}{(s+4)} = \frac{A(s+4)+Bs}{s(s+4)} = \frac{4A + s(A+B)}{s(s+4)}$$

$$Or\ 4A = 1 : A = \frac{1}{4}, A + B = 0 : B = -A = -\frac{1}{4}$$

$$\text{Hence } G_2(s) = \frac{0.25}{s} - \frac{0.25}{s+4}$$

$$G_3(s) = \frac{1}{s(s+2)} = \frac{A}{s} + \frac{B}{s+2} = \frac{A(s+2)+Bs}{s(s+2)} = \frac{2A + (B+A)s}{s(s+2)}$$

$$Or\ 2A = 1 : A = \frac{1}{2}, A + B = 0 : B = -A = -\frac{1}{2}$$

$$\text{Hence } G_3(s) = \frac{0.5}{s} - \frac{0.5}{s+2}$$

For a very small value of the sampling period T, TG(z) ≈ G(s). Figure 13 and Figure 14 represent the canonical equivalent and Digital equivalent to Figure 3 for Examples 1 & 2 respectively.

Z-transfer functions from Laplace functions:

$$G_1(s) : \frac{2}{s} \Rightarrow \frac{2z}{z-1}; \frac{-2}{s+1} \Rightarrow \frac{-2z}{z-e^{-T}};$$

$$\frac{-2}{(s+1)^2} \Rightarrow \frac{-2Tz\,e^{-T}}{(z-e^{-T})^2}$$

$$G_2(s) : \frac{0.25}{s} \Rightarrow \frac{0.25z}{(z-1)}; \frac{-0.25}{s+4} \Rightarrow \frac{-0.25z}{z-e^{-4T}}$$

$$G_3(s) : \frac{0.5}{s} \Rightarrow \frac{0.5z}{(z-1)}; \frac{-0.5}{s+2} \Rightarrow \frac{-0.5z}{z-e^{-2T}}$$

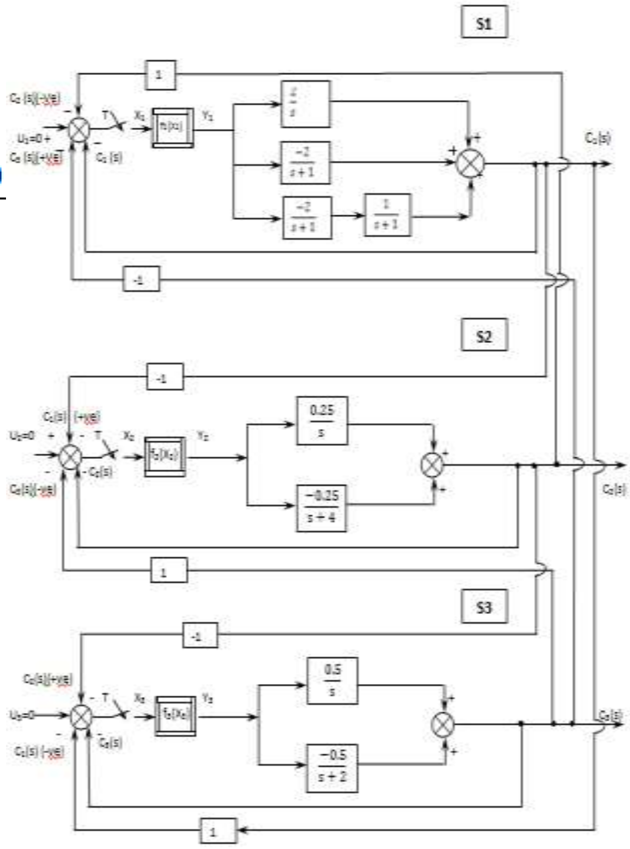


Fig. 13: Equivalent Canonical form of Fig. 3 for Ex.1 & 2

From Figure 14 the following algorithm has been derived:

$$\textbf{(5)}\quad \frac{OW1(z)}{Y_1(z)} = \frac{2z}{z-1} \Rightarrow 2Y_1(z) = \frac{z-1}{z} OW1(z) = OW1(z) - z^{-1}OW1(z)$$

Taking inverse z-transform: OW1 (n T) = 2Y_1 (n T) + OW1$(\overline{n-1T})$

$$(2)\quad \frac{OW2(z)}{Y_1(z)} = \frac{-2z}{z-e^{-T}} \Rightarrow -2Y_1(z) = \frac{z-e^{-T}}{z} OW2(z) = OW2(z) - z^{-1}e^{-T}OW2(z)$$

Taking inverse z-transform: OW2 (n T) = -2Y_1 (n T) +e^{-T} OW2$(\overline{n-1T})$

$$(3)\quad \frac{OW3(z)}{Y_1(z)} = \frac{-2Tze^{-T}}{(z-e^{-T})^2} \Rightarrow -2Tze^{-T}Y_1(z) = \frac{(z-e^{-T})^2}{z} OW3(z)$$
$$= z^* OW3(z) - 2e^{-T}\,OW3(z) + e^{-2T}z^{-1}OW3(z)$$

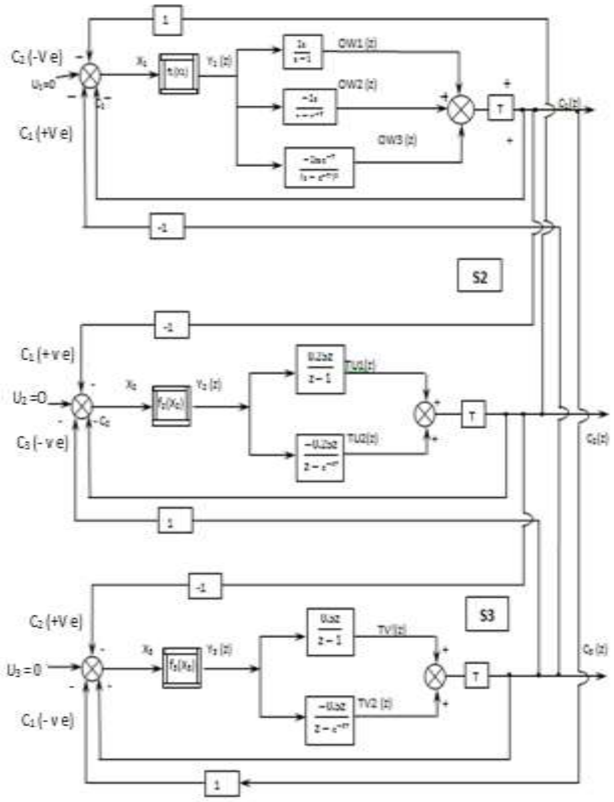


Fig. 14: The Digital representation of Fig. 3 for Ex. 1 & 2

Or $-2Te^{-T}z^{-1}Y_1(z)$=OW3 (z) $-2e^{-T}z^{-1}$OW3 (z) $+e^{-2T}z^{-2}$ OW3 (z)

Taking inverse z-transform: OW3 (n T) = $-2Te^{-T}Y_1(\overline{n-1T})+2e^{-T}$ OW3$(\overline{n-1T})-e^{-2T}$OW3 $(\overline{n-2T})$

(4) $\frac{TU1(z)}{Y_2(z)}=\frac{0.25z}{(z-1)}\Rightarrow 0.25\,Y_2(z)=\frac{z-1}{z}TU1(z)-z^{-1}TU1(z)$

Taking inverse z-transform: TU1 (n T) = 0.25Y_2 (n T) +TU1$(\overline{n-1T})$

(5)
$\frac{TU2(z)}{Y_2(z)}=\frac{-0.25z}{(z-e^{-4T})}\Rightarrow -0.25\,Y_2(z)=\frac{z-e^{-4T}}{z}TU2(z)=TU2(z)-z^{-1}e^{-4T}TU2(z)$

Taking inverse z-transform: TU2 (n T) = -0.25Y_2 (n T) + e^{-4T}TU2$(\overline{n-1T})$

(6)
$\frac{TV1(z)}{Y_3(z)}=\frac{0.5z}{(z-1)}\Rightarrow 0.5\,Y_3(z)=\frac{z-1}{z}TV1(z)=TV1(z)-z^{-1}TV1(z)$

Taking inverse z-transform: TV1 (n T) =0.5Y_3 (n T) +TV1$(\overline{n-1T})$

(7)
$\frac{TV2(z)}{Y_3(z)}=\frac{-0.5z}{(z-e^{-2T})}\Rightarrow -0.5\,Y_3(z)=\frac{z-e^{-2T}}{z}TV2(z)=TV2(z)-z^{-1}*AK2*TV2(z)$

Taking inverse z-transform: TV2 (n T) = -0.5Y_3 (n T) + AK2* TV2$(\overline{n-1T})$

Let us take$(\overline{n-1T})$ is the zeroth instant; nT is the first instant, so we can write:

OW1$(\overline{n-1T})$ = OW1Nϕ⇒OW1N;

OW1 (n T) = OW1N1

OW2$(\overline{n-1T})$ = OW2Nϕ⇒OW2N;

OW2 (n T) = OW2N1

OW3$(\overline{n-2T})$ = OW3N (-1)⇒OW3NN;

OW3 $(\overline{n-1T})$ = OW3Nϕ⇒ OW3N ;

OW3 (nT)=OW3N1

Now C_1 (nT) = OWN1 = T* [OW1N1 + OW2N1 + OW3N1]

= T* [OW1 (nT) +OW2 (nT) + Ow3 (nT)]

=T*[2Y_1 (nT) + OW1N-2Y_1 (n T) + AK1* OW2N – 2* T* AK1 * OY1N + 2* AK1* OW3N

- AK2*OW3 NN] =OWN1=C_1

Similarly,

TU1$(\overline{n-1T})$ = TU1Nϕ ⇒ TU1N; TU1 (nT) = TU1N1, TU2$(\overline{n-1T})$ = TU2Nϕ= TU2N; TU2 (nT) = TU2N1

Now C_2 (nT) = TUN1 = T* [TU1N1 + TU2N1] = T* [TU1 (nT) + TU2 (nT)] = T* [0.25 Y_2 (n T) + TU1N – 0.25Y_2 (n T) + AK3*TU2N] = TUN1 = C_2

Similarly,

TV1$(\overline{n-1T})$ = TV1Nϕ ⇒ TV1N; TV1 (nT) = TV1N1

TV2$(\overline{n-1T})$ = TV2Nϕ ⇒ TV2N; TV2 (nT) = TV2N1

Now C_3 (n T) = TVN1 = T*[TV1N1 + TV2N1] = T*[TV1 (n T) + TV2 (n T)]

= T*[0.5Y3 (n T) + TV1N-0.5Y3 (n T) +AK2* TV2N] = TVN1 = C_3

Next Run:

R_1=ORN1=$C_3 - C_2$ =TVN1 – TUN1

R_2 = TRN1 = $C_1 - C_3$ = OWN1 – TVN1

R_3 = THRN1= $C_2 - C_1$ = TUN1 – OWN1

X_1 = OXN1 = ORN1 – OWN1, OYN1 = OF (OXN1)

X_2 = TXN1 = TRN1 – TUN1, TYN1 = TF (TXN1)

X_3 = THXN1 = THRN1 – TVN1, THYN1 = THF (THXN1)

An appropriate program in MATLAB code following the above algorithm produces the results. The results/images of digital simulation along with that of using SIMULINK Toolbox are presented in Figure 16 and the numerical values obtained there are shown in Table 2.

5 Use of SIMULINK Toolbox in MATLAB

The SIMULINK Toolbox is used to determine X_1, X_2, X_3, C_1, C_2 & C_3 for both Examples 1 and 2 and the results so obtained are compared with those of graphical method and digital simulation (Figure 15). Figure 15 shows the SIMULINK representation for the prediction of Limit Cycles,

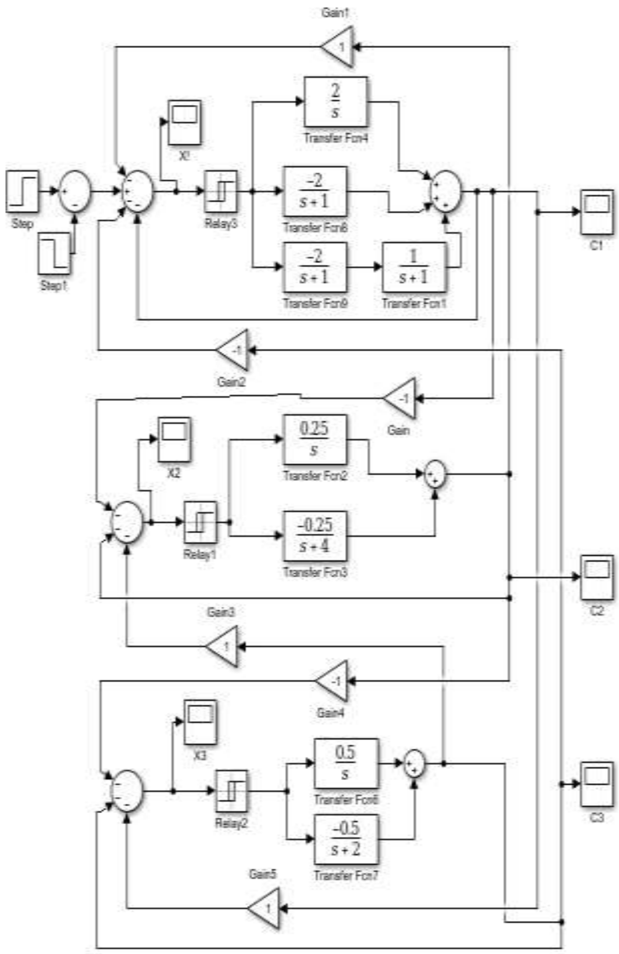


Fig. 15 (a): Diagram used in SIMULINK Toolbox for the solution of example 1.

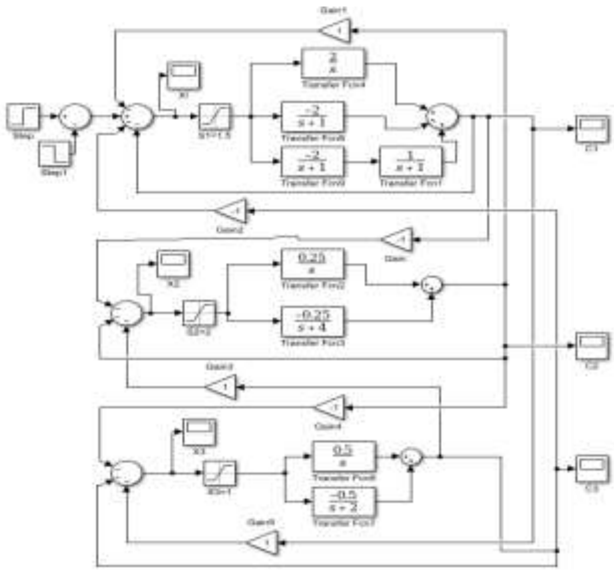


Fig. 15(b): Diagram used in SIMULINK toolbox for the solution of example 2.

6 Comparison of Results

Figure 16 and Figure 17 show the Results/Images obtained from Digital Simulation and with the use of the SIMULINK Toolbox of Example-1 and Example-2 respectively.

Tables 2(a) and 2(b) show the numerical results of Example 1 and 2 for Ideal Relay and Saturation respectively using different methods.

It has been observed that the graphical results are almost matching with that obtained by digital simulation as well as by use of SIMULINK Toolbox both in images and also in Tables with numerical values.

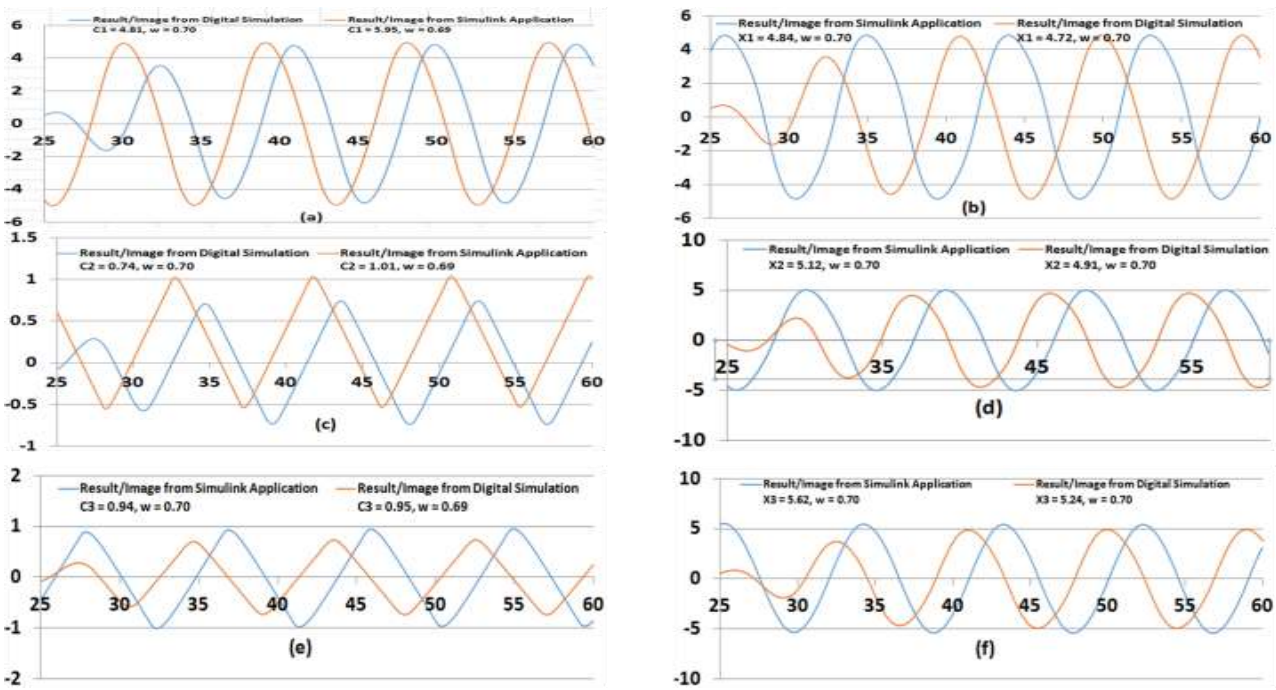


Fig. 16: Results/Images from digital simulation and SIMULINK for C1, C2, C3, X1, X2, and X3 of Example 1 (relay type nonlinearities)

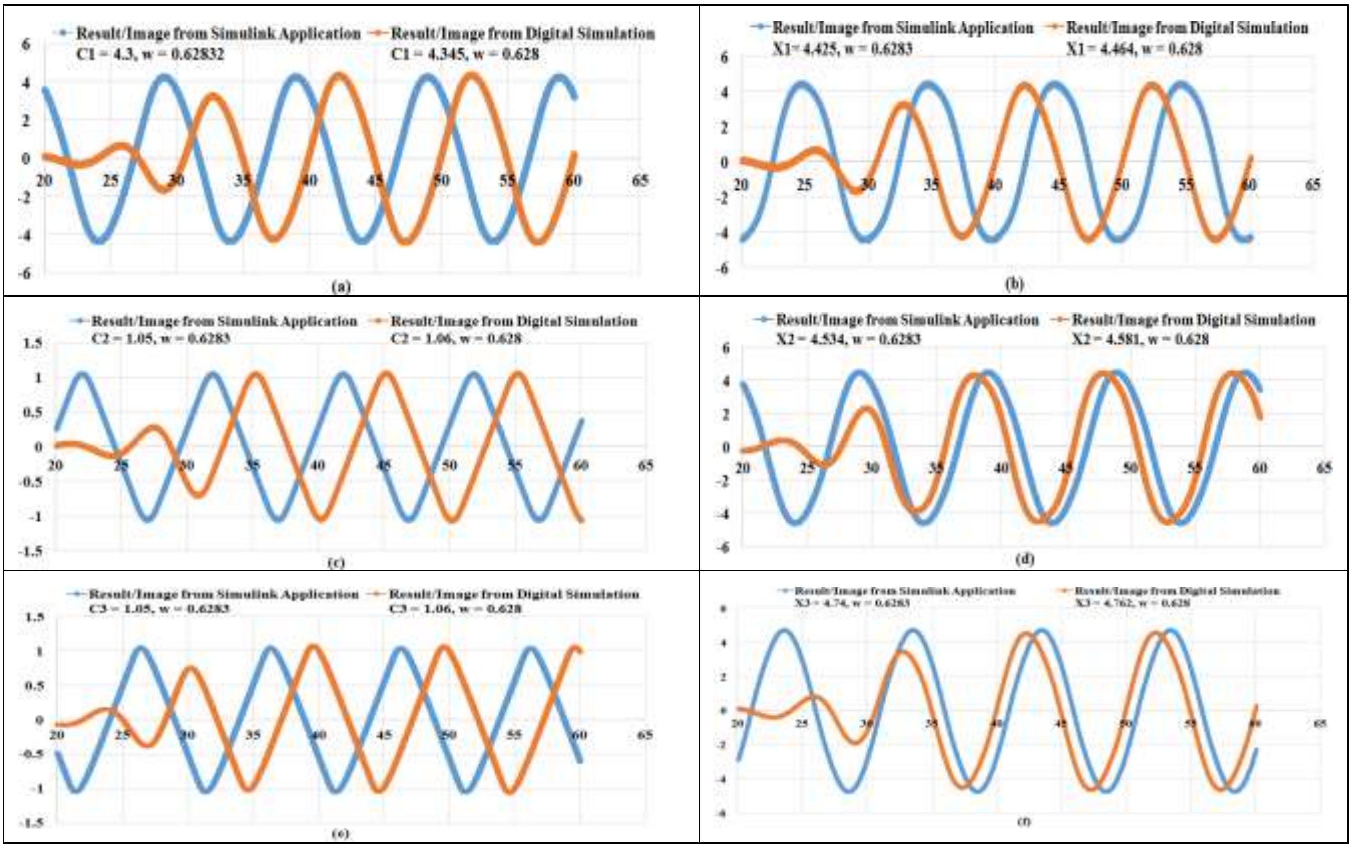


Fig. 17: Results/Images from digital simulation and SIMULINK for C1, C2, C3, X1, X2, and X3 of Example 2 (saturation type nonlinearities)

Table 2(a). Results obtained using different methods corresponding to Ideal Relay Example-1

Sl. No	Methods	C_1	C_2	C_3	X_1	X_2	X_3	ω
1	Graphical	6.0	1.0	1.0	6.08	6.08	6.32	0.701
2	Digital Simulation	4.83	0.74	0.95	4.72	4.91	5.23	0.70
3	Using SIMULINK TOOL BOX OF MATLAB	5.95	1.01	0.96	4.84	5.12	5.62	0.70

Table 2(b). Results obtained using different methods corresponding to Example2: (Saturation)

Sl. No	Methods	C_1	C_2	C_3	X_1	X_2	X_3	ω
1	Digital Simulation	4.345	1.06	1.06	4.464	4.581	4.762	0.628
2	Use of SIMULINK TOOL BOX OF MATLAB	4.30	1.05	1.05	4.425	4.534	4.74	0.6283

7 Conclusion

The work presented, claims the novelty in the following lines: It discusses multivariable (n x n) nonlinear systems and proposes a general formulation in matrix form applicable in both non-memory and memory-type nonlinearities for prediction of limit cycles. The complexity arises in the formulation, particularly for implicit non-memory type non-linearity or memory type nonlinearities, it may be extremely difficult to formulate and simplify the expressions even using the harmonic linearization method, [32], [51]. Hence an attempt has been made to develop a graphical technique using the harmonic linearization/harmonic balance method for the prediction of limit cycles in 3 x 3 nonlinear systems which is not available elsewhere. This is an opening and has the brighter future scope of adopting the techniques like signal stabilization, [49], [50], suppression of limit cycles, [51], in the event of the existence of limit cycling oscillations for 3 x 3 or higher dimensional nonlinear systems which brings in development in the design of nonlinear systems on several occasions.

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Asutosh Patnaik has made the validation of the results using the geometric tools and SIMULINK toolbox of MATLAB software.

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Conflict of Interest

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