Different Boundary Problems Governed by the Dynamic and Stationary Operator Nonlinear Vibration of the Plates

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Abstract: - In this paper, we propose to study some nonlinear boundary problems for the dynamically modified operator by adding a viscosity term $-\alpha\Delta u''$ to the nonlinear vibrations of the plates. The field of application for vibrating plates is extensive. To meet user needs, we have considered the geometric shape, the density of the material constituting the plate, the plate thickness, and Poisson's ratio. Once the problems have been posed, our approach then consists of transforming them into nonlinear problems of the hyperbolic type. In this work, we study six boundary value problems and we prove for each problem an existence and uniqueness theorem. Finally, we demonstrate the existence of a solution to the stationary problem using a variant of Brouwer's fixed point theorem.

Key-Words: - Airy function, Coupled problem, Elliptic-Hyperbolic, Existence and uniqueness, Faedo-Galerkin method, vibrating plate, nonlinear vibrations, Weak Solutions.

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1 Introduction

The field of application for vibrating plates is very wide. This includes, among others, the following areas:

- Individual use at home, beauty salons, well-being, relaxation, and massage.
- Sports and fitness halls, sports clubs, health, and rehabilitation professionals.
- Machines designed for soil compaction, trench back-filling, and the paving or flagging of surfaces.

In this paper, we consider a series of six boundary problems governed by the nonlinear, dynamical, and stationary modifying operator, incorporating the viscosity term $-\alpha\Delta u''$ into the nonlinear plate vibration equations.

To respond to the needs of users, we will take into account the geometric shape, the density of the material constituting the plate, the thickness of the plate, and the Poisson's ratio. Once the problems have been posed, our approach consists of transforming them into nonlinear problems of the hyperbolic type. In [1], the author studied the first problem (Dirichlet) by proving an existence and uniqueness theorem. In this work, we extend the study of [1], to five other boundary problems and prove for each problem an existence and uniqueness theorem for the dynamic case. Finally, we demonstrate the existence of a solution to the stationary problem using a variant of Brouwer's fixed-point theorem. The techniques used here are those of [1]. More precisely, the techniques of the famous Faedo-Galerkin method, used in [2], [3], [4], [5] and [6], to study nonlinear boundary value problems of the elliptic and hyperbolic type.

Although new techniques have appeared since then (such as homogenization or compensated compactness which are taught more recently) these techniques have retained their interesting properties. Remember that these techniques are currently taught in most major universities in the world; let us cite as examples [7] and [8].

The bibliography quoted here does not claim to be exhaustive, and this incompleteness must be attributed to the author's ignorance and not to the author's ill will.

The various problems being coupled, between the Airy function and the transverse displacement, the approach consists of reducing the problems governed by equations of the hyperbolic type. For this, we eliminate the Airy function from the system and prove, for each problem, by the techniques of the famous method cited above, an existence and uniqueness theorem for these modified evolution equations. These presented models play an important role in the design of artificial intelligence. In other words, mathematics makes it possible to design the basic rules of artificial intelligence. It is based on four fundamental pillars of mathematics (linear algebra, probability, statistics, and calculations).

Consider an isotropic homogeneous vibrating plate occupying an open domain $\Omega \subset \mathbb{R}^2$ on lipschitzian boundary Γ made up of two measurable and disjoint parts Γ_0 and Γ_1.

The normal and tangential derivatives are given by:

$$\frac{\partial}{\partial \eta} = \eta_1 \frac{\partial}{\partial x_1} + \eta_2 \frac{\partial}{\partial x_2} \text{ and } \frac{\partial}{\partial \tau} = -\eta_1 \frac{\partial}{\partial x_2} + \eta_2 \frac{\partial}{\partial x_1}, \tag{1}$$

where $M_i(u)$ and $N_i(u)$, i=0, denote the following differential boundary operators:

$$M(u)\,|_{\Gamma_i} = M_i(u) = \gamma_i(\sigma \Delta u + (1-\sigma)\frac{\partial^2 u}{\partial \eta_i^2}). \tag{2}$$

and

$$N(u)\,|_{\Gamma_i} = N_i(u) = -\gamma_i(\frac{\partial}{\partial \eta_i}\Delta u + (1-\sigma)\frac{\partial^3 u}{\partial \eta_i \partial \tau_i^2}). \tag{3}$$

where γ_i is the trace map on $\Gamma_i, i = 0,1$. We will denote by $a_i, i = 1$ to 3, the following positive constants

$$a_1 = \rho h,\ a_2 = \frac{Eh^3}{12(1-\sigma^2)} \text{ and } a_3 = \frac{2}{Eh}, \tag{4}$$

where E is Young's modulus, $\sigma \in]0,1[$ is the Poisson's ratio, ρ is the density of the material constituting the plate and $h$is the thickness of the plate. In what follows, we will set for u and v as two functions in Ω:

$$L(u,v) = \sigma(D_1^2 u.D_2^2 v + D_2^2 u.D_1^2 v - 2D_1 D_2 u.D_1 D_2 v). \tag{5}$$

We denote by A_σ the iterated Laplacian Δ^2 in the variable x, decomposed according to the Poisson's ratio σ , as follows, ([9]):

$$A_\sigma u = \sigma \Delta^2 u + (1-\sigma)\sum_{\alpha,\beta} \frac{\partial^4 u}{\partial x_\alpha^2 \partial x_\beta^2}. \tag{6}$$

2 Formulation of Problems $(\boldsymbol{P_k})$

We consider here a family of six problems governed by the dynamic equations of non-linear vibrations of the plates, that is to say for $f \in L^2(Q)$, which we are looking for a couple of functions (u, F) defined in $Q = \Omega \times]0,T[$, of boundary $\Sigma = \Gamma \times]0,T[$, solution of the problem

$$(P_k)\begin{cases} \begin{cases} a_1 u'' - \alpha \Delta u' + a_2 A_\sigma u - L(u,F) = f & \text{in } Q, \\ a_3 A_\sigma F + L(u,u) = 0 \text{ in } Q, \end{cases} \\ B^k u = \begin{cases} B_0^k u = 0 \\ B_1^k u = 0 \end{cases} \text{ on } \Sigma, \\ B^k F = \begin{cases} B_0^k F = 0 \\ B_1^k F = 0 \end{cases} \text{ on } \Sigma. \\ \begin{cases} u(x,0) = u_0(x),\ x \in \Omega \\ \partial_t u(x,0) = u_1(x),\ x \in \Omega \end{cases} \end{cases} \tag{7}$$

Where the different notations are specified as follows: for $\alpha > 0$, $-\alpha\Delta u''$ is a viscosity term, [10], for $a_i, i = 1$ to 3, see (4), for A_σ see (6) and for $L(u, F)$ see (5).

The boundary operators B_0^k and B_1^k, $k = 1$ to 6, are given by:

1) Boundary conditions on the unknown u:

$$\begin{cases} B_0^1 u = B_1^1 u = \begin{cases} u \\ \frac{\partial u}{\partial \eta} \end{cases} \text{ on } \Sigma, \quad B_0^2 u = \begin{cases} u \\ \frac{\partial u}{\partial \eta} \end{cases} \text{ on } \Sigma_0 \quad B_1^2 u = \begin{cases} u \\ M(u) \end{cases} \text{ on } \Sigma_1 \\ B_0^3 u = \begin{cases} u \\ \frac{\partial u}{\partial \eta} \end{cases} \text{ on } \Sigma_0 \quad B_1^3 u = \begin{cases} M(u) \\ N(u) \end{cases} \text{ on } \Sigma_1 \quad B_0^4 u = B_1^4 u = \begin{cases} u \\ M(u) \end{cases} \text{ on } \Sigma \\ B_0^5 u = \begin{cases} u \\ M(u) \end{cases} \text{ on } \Sigma_0 \quad B_1^5 u = \begin{cases} M(u) \\ N(u) \end{cases} \text{ on } \Sigma_1 \quad B_0^6 u = B_1^6 u = \begin{cases} M(u) \\ N(u) \end{cases} \text{ on } \Sigma \end{cases} \tag{8}$$

2) Boundary conditions on the unknown F :

$$\begin{cases} B_0^1 F = B_1^1 F = \begin{cases} F \\ \frac{\partial F}{\partial \eta} \end{cases} \text{ on } \Sigma, \quad B_0^2 F = \begin{cases} F \\ \frac{\partial F}{\partial \eta} \end{cases} \text{ on } \Sigma_0 \quad B_1^2 F = \begin{cases} F \\ M(F) \end{cases} \text{ on } \Sigma_1 \\ B_0^3 F = \begin{cases} F \\ \frac{\partial F}{\partial \eta} \end{cases} \text{ on } \Sigma_0 \quad B_1^3 F = \begin{cases} M(F) \\ N(F) \end{cases} \text{ on } \Sigma_1 \quad B_0^4 F = B_1^4 F = \begin{cases} F \\ M(F) \end{cases} \text{ on } \Sigma \\ B_0^5 F = \begin{cases} F \\ M(F) \end{cases} \text{ on } \Sigma_0 \quad B_1^5 F = \begin{cases} M(F) \\ N(F) \end{cases} \text{ on } \Sigma_1 \quad B_0^6 F = B_1^6 F = \begin{cases} M(F) \\ N(F) \end{cases} \text{ on } \Sigma \end{cases} \tag{9}$$

Physical interpretation:

f is the given volumetric force density, u is the transverse displacement, F is the normal displacement or Airy function, $M(u)$ is the bending moment and $N(u)$ is the transverse force composed of the shear force and the twisting moment. The boundary conditions (8) to (9) mean that the plate is:

- recessed at the boundary , for the first problem,
- recessed at the edge Σ_0 and simply supported at the edge Σ_1, for the second problem,
- recessed at the edge Σ_0 and free at the edge Σ_1, for the third problem,
- simply supported at the boundary Σ, for the fourth problem,
- simply supported at the boundary Σ_0 and free at the boundary Σ_1, for the fifth problem,
- free at the boundary Σ, for the sixth problem.

Remark 2.1. In (P_k) there is no initial condition on F, it depends on the fact that the system of partial differential equations does not contain a derivative in terms$t$ of F. The system (P_k) being coupled can reduce in the following way, by elimination of F. Indeed, the domain Ω is bounded and with a lipschitzian boundary, there is no particular problem in the application of the variational method to the resolution of the elliptic equation with the corresponding boundary conditions, [11].

Furthermore, the famous regularity theorems of [12] and [13], make it possible to prove that F is regular, that is to say that $F \in H^4(\Omega)$. When it comes to the problem of Neumann problem, for $k = 6$ and $\Gamma_0 = \phi$, we assume the necessary existence condition is verified:

$$\int_\Omega L(u,u)(x)dx = 0.$$

So, if G_k denotes the **Green** operator, i.e. the inverse operator of A_σ in Ω, of the problem:

$$\begin{cases} a_3 A_\sigma F + L(u,u) = 0 \text{ in } Q, \\ B^k F = \begin{cases} B_0^k F = 0 \\ B_1^k F = 0 \end{cases} \text{ on } \Sigma,\ k = 1 \text{ to } 6, \end{cases} \tag{10}$$

then

$$F = -\frac{1}{a_3} G_k(L(u,u)), \tag{11}$$

and the first equation of (7) becomes

$$a_1 u'' - \alpha \Delta u'' + a_2 A_\sigma u + \frac{1}{a_3} L(u, G_k(L(u,u))) = f \tag{12}$$

and therefore the problem (P_k), $k = 1$ to 6, becomes of the hyperbolic type, that said, we have the following:

$$(P_{A_\sigma,k}) \begin{cases} a_1 u'' - \alpha \Delta u'' + a_2 A_\sigma u + \frac{1}{a_3} L(u, G_k(L(u,u))) = f \text{ in } Q, \\ B^k u = 0 \text{ on } \Sigma, \\ \begin{cases} u(x,0) = u_0(x),\ x \in \Omega, \\ \partial_t u(x,0) = u_1(x),\ x \in \Omega. \end{cases} \end{cases} \tag{13}$$

Theorem 2.1. We assume $f, u_0,\ u_1$ given with

$$f \in L^2(Q), \tag{14}$$

$$u_0 \in V_k(\Omega) \text{ and } u_1 \in H_0^1(\Omega). \tag{15}$$

Then, there exists a unique solution (u, F) of (P_k), $k = 1$ to 6, such that

$$u \in L^\infty(0,T;V_k), \tag{16}$$

$$u' \in L^\infty(0,T;H_0^1(\Omega)), \tag{17}$$

$$F \in L^\infty(0,T;V_k), \tag{18}$$

where the V_k are the variational spaces of the problems (P_{A_σ}), $k = 1$ to 6, ([11], Chapter IV):

$$(P_{A_\sigma}) \begin{cases} -A_\sigma F = \frac{1}{a_3} L(u,u) \text{ in } Q, \\ B^k F = \begin{cases} B_0^k F = 0 \\ B_1^k F = 0 \end{cases} \text{ on } \Sigma, \end{cases} \tag{19}$$

We have

$$V_1 = H_0^2(\Omega),$$
$$V_2 = \{v \in H^2(\Omega), \gamma_0 v = \gamma_0(\frac{\partial v}{\partial \eta}) = 0 \text{ sur } \Gamma_0 \text{ et } \gamma_0 v = 0 \text{ sur } \Gamma_1\},$$
$$V_3 = \{v \in H^2(\Omega), \gamma_0 v = \gamma_0(\frac{\partial v}{\partial \eta}) = 0 \text{ sur } \Gamma_0\},$$
$$V_4 = H^2(\Omega) \cap H_0^1(\Omega),$$
$$V_5 = \{v \in H^2(\Omega), \gamma_0 v = \gamma_0(M(v)) = 0 \text{ sur } \Gamma_0\},$$
$$V_6 = H^2(\Omega).$$

The V_k, $k = 1$ to 6, are closed subspaces of $H^2(\Omega)$, hence Hilbert spaces containing $H_0^2(\Omega)$.

Remark 2.2.
It follows from (16) and (18) and from the definition (5) that $L(u,F) \in L^\infty(0,T;L^1(\Omega))$ and so the first equation of (13) implies that

$$u'' \in L^\infty(0,T;V_k'),$$

and hence the initial conditions in (P_k), $k = 1$ to 6, make sense. Indeed, to show that, it suffices to remark that $L^1(\Omega) \subset V_k' \subset H^{-2}(\Omega)$. Indeed, if $g \in L^1(\Omega)$, we have

$$|(g,v)| \le \|g\|_{L^1(\Omega)} \|v\|_{L^\infty(\Omega)} \le c\|g\|_{L^1(\Omega)} \|v\|_{V_k},\ \forall v \in V_k \tag{20}$$

Remark 2.3. In this remark, we consider the spaces $H^s(\Omega)$, where s is non-integer, developed in [14], the function F in Theorem 2.1. satisfies

$$F \in L^\infty(0,T;H^{3-\varepsilon}(\Omega)) \quad \forall \varepsilon > 0. \tag{21}$$

In fact, let $\varepsilon > 0$ arbitrarily small. So

$$L^1(\Omega) \subset H^{-1-\varepsilon}(\Omega) \subset V_k'. \tag{22}$$

In fact, if $g \in L^1(\Omega)$, we have:

$$\begin{cases} |(g,\varphi)| \le \|g\|_{L^1(\Omega)} \|\varphi\|_{L^\infty(\Omega)} \\ \quad \le c\|g\|_{L^1(\Omega)} \|\varphi\|_{H^{1+\varepsilon}(\Omega)} \end{cases},\ \forall \varphi \in H^{1+\varepsilon}(\Omega)$$

seen that $H^{1+\varepsilon}(\Omega) \subset L^\infty(\Omega)$ for $n = 2$ and $\varepsilon > 0$; [11]. Then

$$L(u,u) \in L^\infty(0,T;H^{-1-\varepsilon}(\Omega))$$

and as $F = -\frac{1}{a_3} G_k L(u,u)$, we deduce (21) using the solution of the problems (P_k), $k = 1$ to 6, and the fact that send $H^s(\Omega)$ in

$$H^{s+4}(\Omega) \cap V_k, s \geq 0.$$

3 Proof of Existence

In the proof of Theorem 2.1, we use the following lemma:

Lemma 3.1. We have

1) The application

$$V_k \times V_k \to V_k^{'},\ k = 1 \text{ to } 6$$
$$(u,v) \mapsto L(u,v)$$

is bi-linear and continuous.

2) The form $(u,v,w) \to (L(u,v),w)$ is tri-linear and continuous on V_k.

Proof: Analogous to that of [1].

i) Definition of approximate solutions.

The space V_k, k=1, to 6, is identified, by the application

$$v \to \left\{v, \frac{\partial v}{\partial x_1}, \ldots, \frac{\partial v}{\partial x_2}\right\},$$

to a closed subspace of

$$L^2(\Omega) \times \ldots \times L^2(\Omega)$$

which is separable and uniformly convex, so that one can project a dense countable set onto the subspace V_k. So let $\{w_1, \ldots, w_m\}$ be a basis of V_k, (for example we have $V_1 = H_0^2(\Omega)$), and let $u_m(t)$ be such that

$$u_m(t) \in [w_1, \ldots, w_m], \text{ i.e } u_m(t) = \sum_{i=1}^{m} g_{im}(t) w_i, \tag{23}$$

the g_{im} being to be determined by the conditions:

$$\begin{cases} a_1(u_m^{''}(t), w_j) + \alpha\ a(u_m(t), w_j) + a_2(A_\sigma u_m(t), w_j) \\ + \frac{1}{a_3}(L(u_m(t), G_k(u_m(t), u_m(t)), w_j) = (f(t), w_j), 1 \leq j \leq m. \end{cases} \tag{24}$$

Where

$$a(u,v) = \sum_{i=1}^{2} \int_\Omega \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_i}\, dx$$

where we use the notations of (11) and (12), with

$$u_m(0) = u_{0m} \in [w_1, \ldots, w_m],\ u_{0m} \to u_0 \text{ in } V_k,\ k = 1 \text{ to } 6, \tag{25}$$

$$u_m'(0) = u_{1m} \in [w_1, \ldots, w_m],\ u_{1m} \to u_1 \text{ in } H_0^1(\Omega). \tag{26}$$

If we define $F_m(t)$ by

$$F_m(t) = -\frac{1}{a_3} G_k(L(u_m(t), u_m(t)) \tag{27}$$

or by

$$\begin{cases} a_3 A_\sigma F_m(t) + L(u_m(t), u_m(t)) = 0, \\ F_m(t) \in V_k,\ k = 1 \text{ to } 6. \end{cases} \tag{28}$$

then (24) reads

$$\begin{cases} a_1(u_m^{''}(t), w_j) + \alpha\ a(u_m(t), w_j) + a_2(A_\sigma u_m(t), w_j) \\ -(L(u_m(t), F_m(t)), w_j) = (f(t), w_j),\ 1 \leq j \leq m. \end{cases} \tag{29}$$

Of course $F_m(t)$ is not (in general) valued in $[w_1, \ldots, w_m]$. According to the general results on the theory of systems of differential equations, one is assured of the existence of $u_m(t)$, and therefore of $F_m(t)$, on an interval $[0, t_m]$, for some $t_m > 0$.

ii) A priori estimates

We multiply (29) by $g_{jm}^{'}(t)$ and we sum in j. He comes:

$$\begin{cases} \frac{1}{2}\frac{d}{dt}\left[a_1 \left|u_m^{'}(t)\right|^2 + \alpha\ a(u_m'(t), u_m'(t)) + a_2 \left|\sigma\Delta u_m(t) + (1-\sigma)\Delta u_m(t)\right|^2\right] \\ -\left(L(u_m(t), F_m(t)), u_m^{'}(t)\right) = (f(t), u_m^{'}(t)). \end{cases} \tag{30}$$

But according to Lemma 3.1, we have

$$\left(L(u_m(t), F_m(t)), u_m^{'}(t)\right) = -\left(L(u_m(t), u_m^{'}(t)), F_m(t)\right)$$
$$= -\frac{1}{4}\left(\frac{d}{dt} L(u_m(t), u_m(t)), F_m(t)\right),$$

and by (28) this is equal to

$$\frac{a_3}{2}(A_\sigma F_m^{'}(t), F_m(t)) = \frac{a_3}{2}\frac{d}{dt}\left|\sigma\Delta F_m(t) + (1-\sigma)\Delta F_m(t)\right|^2.$$

So (30) is written again:

$$\frac{1}{2}\frac{d}{dt}\left[a_1 \left|u_m^{'}(t)\right|^2 + \alpha\ \|u_m'(t)\|^2 + a_2 \left|\sigma\Delta u_m(t) + (1-\sigma)\Delta u_m(t)\right|^2 + \frac{a_3}{2}\left|\sigma\Delta F_m(t) + (1-\sigma)\Delta F_m(t)\right|^2\right] = (f(t), u_m^{'}(t)) \tag{31}$$

and so

$$\begin{cases} \frac{1}{2}\left[a_1 |u_m'(t)|^2 + \alpha \|u_m'\|^2 + a_2 |\sigma\Delta u_m(t) + (1-\sigma)\Delta u_m(t)|^2 \right. \\ \left. + \frac{a_3}{2}|\sigma\Delta F_m(t) + (1-\sigma)\Delta F_m(t)|^2\right] = \\ \frac{1}{2}\left[a_1 |u_{1m}|^2 + \alpha \|u_{1m}\|^2 + a_2 |\sigma\Delta u_{0m} + (1-\sigma)\Delta u_{0m}|^2 \right. \\ \left. + \frac{a_3}{2}|\sigma\Delta F_m(0) + (1-\sigma)\Delta F_m(0)|^2\right] + \int_0^t (f(s), u_m^{'}(s))ds. \end{cases} \tag{32}$$

But according to (25)

$$a_1|u_{1m}|^2+\alpha\|u_{1m}\|^2+a_2|\sigma\Delta u_{0m}+(1-\sigma)\Delta u_{0m}|^2\le \text{constant}.$$

By definition (27), we have

$$F_m(0)=-\frac{1}{a_3}G_k(L(u_{0m},u_{0m})). \tag{33}$$

and since $L(u_{0m},u_{0m})$ remains in a bounded subset of $L^1(\Omega)$, hence of $V_k'\subset H^{-2}(\Omega)$, $k=1$ to 6, then $F_m(0)$ remains in a bounded subset of V_k and therefore in (32)
$|\sigma\Delta F_m(0)+(1-\sigma)\Delta F_m(0)|\le$ constant.
So (32) implies that $t_m=T$ and

$$u_m,\ F_m \text{ remains in a bounded subset of } L^\infty(0,T;V_k). \tag{34}$$

$$u_m' \text{ remains in a bounded subset of } L^\infty(0,T;L^2(\Omega)). \tag{35}$$

iii) Passage to the limit
From (34) and (35), we can extract a sequence u_μ, F_μ such that

$$\begin{cases} u_\mu\to u,\ F_\mu\to F \text{ in } L^\infty(0,T;V_k) \text{ weak star},\\ u_\mu'\to u',\ F_\mu'\to F' \text{ in } L^\infty(0,T;L^2(\Omega)) \text{ weak star},\\ u_\mu\to u \text{ in } L^2(Q) \text{ strongly}.\end{cases} \tag{36}$$

Let $\phi_j, 1\le j\le j_0$ be functions of $C^1([0,T])$, such that $\phi_j(T)=0$, and

$$\psi=\sum_{j=1}^{j_0}\varphi_j\otimes w_j. \tag{37}$$

We deduce from (29), by integration by parts of the first term, for $m=\mu>j_0$ that

$$\begin{cases} -a_1\int_0^T(u_\mu',\psi')dt+\alpha\int_0^T a(u_\mu,w_j)dt+a_2\int_0^T(\sigma\Delta u_\mu+(1-\sigma)\Delta u_\mu,\Delta\psi)dt\\ -\int_0^T(L(u_\mu,F_\mu),\psi)dt=\int_0^T(f,\psi)dt+(u_{1\mu}(0),\psi(0)).\end{cases} \tag{38}$$

But according to Lemma 3.1:

$$\int_0^T(L(u_\mu,F_\mu),\psi)dt=\int_0^T(L(\psi,F_\mu),u_\mu)dt; \tag{39}$$

$$(\psi,F_\mu)\to L(\psi,F) \text{ in } L^2(Q) \text{ weakly},$$

for example and so since $u_\mu\to u$ strongly in $L^2(Q)$, we see that

$$\int_0^T(L(u_\mu,F_\mu),\psi)dt\to\int_0^T L(\Psi,F),u)dt=\int_0^T(L(u,F),\Psi)dt,$$

and

$$\int_0^T a(u_\mu,w_j)dt\to\int_0^T a(u,\Psi)dt\,.$$

Therefore, (38) implies by passage to the limit:

$$\begin{cases} -a_1\int_0^T(u',\psi')dt+\alpha\int_0^T a(u,\psi)dt+a_2\int_0^T(\sigma\Delta u+(1-\sigma)\Delta u,\Delta\psi)dt\\ -\int_0^T(L(u,F),\psi)dt=\int_0^T(f,\psi)dt+(u_1,\psi(0)),\end{cases} \tag{40}$$

and this is $\forall\psi$ of the form (37).

By passing to the limit we deduce that (40) still holds for all $\psi\in L^2(0,T;V_k), k=1\text{to}6$, such as $\psi'\in L^2(0,T;L^2(\Omega))$ and $\Psi(T)=0$.

This shows that u and F are related by the first equation of $(P_k), k=1$ to 6, and $u'(0)=u_1$. It remains only to show the second equation of $(P_k), k=1$ to 6. We can go directly to the limit on (28) (for $m=\mu$ noting that $L(u_\mu,u_\mu)\to L(u,u)$ in $D'(Q)$ for example; indeed we have

$$\int_0^T(L(u_\mu,u_\mu),\varphi)dt=\int_0^T(L(u_\mu,\varphi),u_\mu)dt,\ \forall\varphi\in D(Q)$$

and we pass to the limit as above.

4 Proof of Uniqueness

Let (u,F) and (u^*,F^*) be two solutions, let us say:

$$v_1=u-u^*,\ v_2=F-F^*. \tag{41}$$

Then we have the algebraic relations

$$a_1v_1''-\alpha\Delta v_1''+a_2A_\sigma v_1=L(u,v_2)+L(v_1,F^*), \tag{42}$$

$$a_3A_\sigma v_2=L(v_1,v_1)-2L(u,v_1), \tag{43}$$

obviously with

$$v_1(0)=0,\ v_1'(0)=0 \tag{44}$$

a) Estimates for v_2

We use remark 2.3 here. From (22) and (43) we have:

$$\|v_2\|_{H^{3-\varepsilon}(\Omega)} \leq C_1 \|L(u,v_1)\|_{L^1(\Omega)} + C_1 \|L(v_1,v_1)\|_{L^1(\Omega)}. \quad (45)$$

However,

$$\|L(u,v_1)\|_{L^1(\Omega)} \leq C_2 \|u\|_{V_k} \|v_1\|_{V_k},$$

And since

$$(u,v_1) \in L^\infty(0,T;V_k) \subset L^\infty\big(0,T;H^2(\Omega)\big),$$

we have

$$\|L(u,v_1)\|_{L^1(\Omega)} \leq C_2 \|u\|_{H^2(\Omega)} \|v_1\|_{H^2(\Omega)}$$

$$\|v_2\|_{H^{3-\varepsilon}(\Omega)} \leq C_3 \|v_1\|_{H^2(\Omega)}. \quad (46)$$

We remark that $|\Delta v_1|$ is a norm equivalent to $\|v_1\|_{H^2(\Omega)}$ over $V_k, k=1$ to 6, so that (46) defines a norm and is equivalent to

$$\|v_2\|_{H^{3-\varepsilon}(\Omega)} \leq C_4 |\Delta v_1|. \quad (47)$$

For $D^2 = D_i^2$ we have

$$D^2 v_2 \in L^\infty\big(0,T;H^{1-\varepsilon}(\Omega)\big)$$

(This holds obviously for $D_i D_j$) and, [15]

$$H^{1-\varepsilon}(\Omega) \subset L^{\frac{2}{\varepsilon}}(\Omega),$$

So

$$D^2 v_2,\ D^2 F \in L^\infty(0,T;L^{\frac{2}{\varepsilon}}(\Omega)) \quad (48)$$

and according to (47)

$$\left\|D^2 v_2\right\|_{L^{\frac{2}{\varepsilon}}(\Omega)} \leq C_5 |\Delta v_1|. \quad (49)$$

In (42), let

$$K = L(u,v_2) + L(v_1,F^*). \quad (50)$$

let us show that

$$K \in L^\infty(0,T;H^{-1}(\Omega)) \text{ and } \|K\|_{H^{-1}(\Omega)} \leq C_6 |\Delta v_1| \quad (51)$$

Indeed let $\phi \in H_0^1(\Omega)$. Then, still according to the fractional Sobolev Theorem, [15], we have, for fixed $\varepsilon > 0$:

$$\varphi \in L^{\frac{2}{1-\varepsilon}}(\Omega),\ \|\varphi\|_{L^{\frac{2}{1-\varepsilon}}(\Omega)} \leq C_7 \|\varphi\|_{H_0^1(\Omega)}. \quad (52)$$

But

$$\begin{cases} |(L(u,v_2),\varphi)| \leq C_8 \|u\|_{V_k} \left(\sum_{i,j=1}^2 \|D_i D_j v_2\|_{L^{\frac{2}{\varepsilon}}(\Omega)}\right) \|\varphi\|, \\ \leq C_8 \|u\|_{H^2(\Omega)} \left(\sum_{i,j=1}^2 \|D_i D_j v_2\|_{L^{\frac{2}{\varepsilon}}(\Omega)}\right) \|\varphi\|_{L^{\frac{2}{1-\varepsilon}}(\Omega)}, \\ \leq C_9 |\Delta v_1| \|\varphi\|_{H_0^1(\Omega)}. \end{cases} \quad (53)$$

This is true according to (49), (52) and $u \in L^\infty(0,T;V_k), k=1$ to 6. Then

$$\begin{aligned} |(L(v_1,F^*),\varphi)| &\leq C_8 \|v_1\|_{V_k} \left(\sum_{i,j=1}^2 \|D_i D_j F^*\|_{L^{\frac{2}{\varepsilon}}(\Omega)}\right) \|\varphi\|_{L^{\frac{2}{1-\varepsilon}}(\Omega)}, \\ &\leq C_8 \|v_1\|_{H^2(\Omega)} \left(\sum_{i,j=1}^2 \|D_i D_j F^*\|_{L^{\frac{2}{\varepsilon}}(\Omega)}\right) \|\varphi\|_{L^{\frac{2}{1-\varepsilon}}(\Omega)}, \\ &\leq C_{10} |\Delta v_1| \|\varphi\|_{H_0^1(\Omega)}, \end{aligned} \quad (54)$$

whence (51).

b) Equality of energy

From (42), (50) and (51), we deduce by setting

$$\begin{cases} \frac{1}{2}(a_1 |v_1'(t)|^2 + \alpha \|v_1'(t)\|^2 + a_2 |\sigma \Delta v_1(t) + (1-\sigma)\Delta v_1(t)|^2) \\ = \int_0^t (K,v_1')ds, \text{ almost everywhere} \end{cases} \quad (55)$$

Indeed, the method of Theorem 1.6. of [1] leads to

$$\frac{1}{2}(a_1|v_1'(t)|^2 + \alpha\|v_1'(t)\|^2 + a_2|\sigma\Delta v_1(t) + (1-\sigma)\Delta v_1(t)|^2)$$
$$= \int_0^\tau (K,v_1')ds + \int_\tau^t (K,v_1')ds, \text{ almost everywhere.}$$

But

$$\frac{1}{2}(a_1|v_1'(\tau)|^2 + \alpha\|v_1'(\tau)\|^2 + a_2|\sigma\Delta v_1(\tau) + (1-\sigma)\Delta v_1(\tau)|^2)$$
$$= \int_0^\tau (K,v_1')ds,$$

$$\begin{cases} \frac{1}{2}(a_1 |v_1'(t)|^2 + \alpha \|v_1'(t)\|^2 + a_2 |\sigma \Delta v_1(t) + (1-\sigma)\Delta v_1(t)|^2) = \\ \frac{1}{2}(a_1 |v_1'(\tau)|^2 + \alpha \|v_1'(\tau)\|^2 + a_2 |\sigma \Delta v_1(\tau) + (1-\sigma)\Delta v_1(\tau)|^2) \\ + \int_\tau^t (K,v_1)ds, \end{cases} \quad (56)$$

for almost all τ and t.
According to (44), we can extend v_1 by 0 for $t < 0$ and K being also extended $t < 0$. In (56), we do the same thing for $\tau < 0$. Therefore, we obtains (55).

c) Uniqueness

We complete the proof easily from (55) and (51); In fact

$$\begin{cases} \left|\int_0^t (K,v_1')ds\right| \leq C_6 \int_0^t |\sigma\Delta v_1(s) + (1-\sigma)\Delta v_1(s)| \|v_1'(s)\|_{H_0^1(\Omega)} ds \\ \leq C_{11} \int_0^t |\sigma\Delta v_1(s) + (1-\sigma)\Delta v_1(s)| \|v_1'(s)\| ds, \end{cases} \quad (57)$$

and then, we deduce from (55) that

$$\begin{cases} \|v_1'(t)\|^2 + |\sigma\Delta v_1(t) + (1-\sigma)\Delta v_1(t)|^2 \\ \leq C_{12} \int_0^t (\|v_1'(s)\|^2 + |\sigma\Delta v_1(s) + (1-\sigma)\Delta v_1(s)|^2) ds. \end{cases} \quad (58)$$

Thanks to Gronwall's inequality, we obtain $v_1 = 0$.
Consequently, from (47) we deduce that $v_2 = 0$.

5 Stationary Problems (S_k), $k = 1$ *to* 6

We now propose to prove an existence theorem for a solution, using a variant of Brouwer's fixed-point theorem [1], for stationary problems corresponding to problems (7). Therefore we are looking for a pair of functions u, $F \in V_k$, *k=1* to *6*, such that

$$(S_k) \begin{cases} \begin{cases} a_2 A_\sigma u - L(u, F) = f \text{ in } \Omega, \\ a_3 A_\sigma F + L(u, u) = 0 \text{ in } \Omega, \end{cases} \\ B^k u = \begin{cases} B_0^k u = 0 \\ B_1^k u = 0 \end{cases} \text{ on } \Gamma, \\ B^k F = \begin{cases} B_0^k F = 0 \\ B_1^k F = 0 \end{cases} \text{ on } \Gamma. \end{cases} \quad (59)$$

Where the different notations are those of the previous paragraphs 1 and 2.
We will need the following lemma:

Lemma 5.1. ([1]). Let $\xi \to P(\xi)$ be a continuous map from $\mathbb{R}^m$ to it self, such that, for a suitable $\rho > 0$, we have:

$$(P(\xi), \xi) \geq 0, \forall \xi \text{ such as } |\xi| = \rho. \quad (60)$$

where if $\xi = \{\xi_i\}, \eta = \{\eta_i\} \in R^m$:

$$(\xi, \eta) = \sum_{i=1}^{m} \xi_i \eta_i, |\xi| = (\xi, \xi)^{\frac{1}{2}} \quad (61)$$

Then there exists $\xi, |\xi| \leq \rho$, such that
$P(\xi) = 0$.
We use Lemma 5.1. to show the following theorem:

Theorem 5.1. Let f be in V_k', then the problem is (S_k), $k = 1$ to 6, admit a solution.
Proof.

1) **Approximate solutions.**
Let $w_1,..., w_m$ ... a « base » de $V_k, k = 1$ à 6, formed for example, by functions of $D(\Omega)$ as in the dynamic case. We are looking for $u_m \in [w_1,..., w_m]$, i.e. $u_m = \sum_{i=1}^{m} \xi_i w_i$, such that

$$a_2(A_\sigma u_m, w_i) + \frac{1}{a_3}(L(u_m, G_k(u_m, u_m), w_i) = (f, w_i), 1 \leq i \leq m, \quad (62)$$

If we define F_m by

$$F_m = -\frac{1}{a_3} G_k(L(u_m, u_m). \quad (63)$$

Then (62) is equivalent to

$$\begin{cases} (a_2 A_\sigma u_m - (L(u_m, F_m), w_i) = (f, w_i), 1 \leq i \leq m, \\ a_3 A_\sigma F_m + L(u_m, u_m) = 0 \end{cases} \quad (64)$$

We have to show that (62) admits a solution. We use Lemma 5.1 for this as follows. To $\xi = \{\xi_i\}$ we associate $u_m = \sum_{i=1}^{m} \xi_i w_i$ then

$$\eta_i = (a_2 A_\sigma u_m - L(u_m, F_m), w_i) - (f, w_i), 1 \leq i \leq m, \quad (65)$$

and we put

$$P(\xi) = \{\eta_i\} \quad (66)$$

So,

$$\begin{cases} (P(\xi), \xi) = \sum_{i=1}^{m} \eta_i \xi_i = (a_2 A_\sigma u_m - L(u_m, F_m), u_m) - (f, u_m) \\ = |\sigma\Delta u_m + (1-\sigma)\Delta u_m|^2 - L(u_m, F_m), u_m) - (f, u_m), \\ = a_2 |\Delta u_m|^2 - L(u_m, F_m), u_m) - (f, u_m). \end{cases} \quad (67)$$

But by Lemma 3.1 and by (64),

$$(L(u_m, F_m), u_m) = (L(u_m, u_m), F_m) = -a_3 |\Delta F_m|^2,$$

so that (49) gives

$$(P(\xi), \xi) = a_2 |\Delta u_m|^2 + a_3 |\Delta F_m|^2 - (f, u_m) \quad (68)$$

But

$$|(f, u_m)| \leq \|f\|_{V_k'} \|u_m\|_{V_k} c_1 |\Delta u_m|,$$

and so

$$(P(\xi), \xi) \geq a_2 |\Delta u_m|^2 + a_3 |\Delta F_m|^2 - c_1 |\Delta u_m|. \quad (69)$$

So $(P(\xi), \xi) \geq 0$ if $|\Delta u_m| \geq \frac{c1}{a2}$ (where $a_2 = \frac{Eh^3}{12(1-\sigma^2)}$, see (4)) condition fulfilled if $|\xi| = \rho$, ρ large enough. We can therefore use Lemma 5.1; there thus exists $u_m \in [w_1,..., w_m]$a solution of (62), or, which comes to the same thing, of (64). Moreover, if u_m is a solution, we have $P(\xi) = 0$ and (68), (69) give

$$(P(\xi), \xi) = a_2 |\Delta u_m|^2 + a_3 |\Delta F_m|^2 = (f, u_m) \leq c_1 |\Delta u_m|. \quad (70)$$

2) Passing to the limit.
We deduce from (70) that

$$u_m, F_m \text{ remains in a bounded subset of } V_k, k = 1 \text{ to } 6. \quad (71)$$

We can therefore extract two sequences u_μ, F_μ such that

$$\begin{cases} u_\mu \longrightarrow u \text{ in } V_k \text{ weak} \\ F_\mu \longrightarrow F \text{ in } V_k \ weak \end{cases} \quad (72)$$

and, the injection of $V_k \subset H^2(\Omega) \to L^2(\Omega)$ being compact,

$$\begin{cases} u_\mu \longrightarrow u \text{ in } L^2(\Omega) \text{ strong (and even in } H^1(\Omega) \text{ strong)}, \\ F_\mu \longrightarrow F \text{ dans } L^2(\Omega) \text{ strong (and even in } H^1(\Omega) \text{ strong)}. \end{cases} \quad (73)$$

Let i be fixed, $\mu > i$; we have:

$$(a_2 A_\sigma u_\mu, w_i) - L(u_\mu, F_\mu), w_i) = (f, w_i), 1 \leq i \leq m.$$

but

$$\left(L(u_\mu, F_\mu), w_i\right) = \left(L(u_\mu, w_i), F_\mu\right)$$

and

$$L(u_\mu, w_i) \to L(u, w_i) \text{ in } L^2(\Omega) \text{ weak,}$$

which, with (73), gives

$$\left(L(u_\mu, w_i), F_\mu\right) \to (L(u, w_i), F)$$

Consequently

$$(a_2 A_\sigma u, w_i) - L(u, F), w_i) = (f, w_i), 1 \leq i \leq m.$$

We derive the first equation of (59) and extend the same procedure to the limit in the second equation of (64).

6 Conclusion and Perspectives

In the first part of this work, within the framework of solid mechanics, and more precisely in plate theory, we have established an existence theorem for various dynamic problems governed by the operator of non-linear vibration plates. Additionally, we have proven an existence and uniqueness theorem for modified evolution equations using the compactness method.

As future perspectives, it would be interesting to extend this work in cases where

- The Sobolev $W^{1,p}(\Omega)$, [16], spaces with a constant or variable exponents.
- The plates have polygonal borders.

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References:

[1] Lions, J. L., Some methods for solving nonlinear boundary problems, («Quelques méthodes de résolution des problèmes aux limites non linéaires»), *Dunod*, Paris (1969).

[2] M. Boukrouche, B. Merouani, F. Zoubai. On a nonlinear elasticity problem with friction and Sobolev spaces with variable exponents, *Fixed Point Theory Algorithms Sci.Eng.*, vol. 14, 2022.

[3] Y. Boukhatem, B. Benabderrahmane, A. Rahmoune, Faedo-Galerkin Method for a Nonlinear Boundary Problem (Méthode de Faedo-Galerkin pour un Problème aux limites non Linéaire), *Analele Universit˘at¸ii Oradea Fasc. Matematica*, Tom XVI, (2009), 167-181.

[4] F. Zoubai, B. Merouani. A nonlinear elasticity system in Sobolev spaces with variable exponents, Bull. Math. Soc. Sci. Math. Roumanie, Tome 64, (112) No. 1, 2021, 17 − 33

[5] S. Manaa, S Boulaaras, H Benseridi, M. Dilmi, S. Alodhaibi, Analysis for flow of an incompressible brinkman-type fluid in thin medium with friction, *Journal of Function Spaces*, 2021, 1-8.

[6] Jian Xue, Lihua Chen, Yue Sun & Wei Zhang, Free Vibration of Stiffened Plate with Cracked Stiffeners, *Adv. Appl. Math. Mech.*, 16 (2024), pp. 253-278.

[7] Hervé Le Dret, Nonlinear elliptic partial differential equations, (« Équations aux dérivées partielles elliptiques non linéaires»), Springer 2013, Laboratoire Jacques-Louis Lions Université Pierre et Marie Curie Paris France.

[8] T. Gallouët, R. Herbin, Master 2 in Mathematics, Partial differential equations , («Master 2 de Mathématiques, Equations aux dérivées partielles»), Université Aix Marseille, 2022.

[9] J. Necas, Direct methods in Theory of elliptic equations, («Les méthodes directes en Théorie des équations elliptiques»), Masson, Paris, 1967.

[10] N. F. Morozov, Nonlinear vibrations of thin plates with allowance for rotational inertia.

Dokl. Akad. Nauk, 176 (1967). *Traduit dans Soviet Math.*, 8 (1967), 1137-1141.

[11] F. Bouzeghaya, Science Doctorate Thesis, Analysis of Some Problems in Continuum Mechanics, («Thèse de Doctorat Sciences. Analyse de Quelques Problèmes en Mécanique des milieux continus»), soutenue en Nov. 2013, à l'Université F. ABBAS de Sétif, Algérie. Thèse de doctorat F. Bouzeghaya, Chapter IV, [Online]. http://dspace.univ-setif.dz:8888/jspui/handle/123456789/1955 (Accessed Date: January 25, 2024).

[12] S. Agmon, A. Douglis, L. Nirenberg. Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions I, *Communications on Pure & Applied Maths*, 12, 1959, p.623-727.

[13] S. Agmon, A. Douglis, L. Nirenberg. Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions II, *Communications on Pure & Applied Maths*, 17, 1964, p. 35-92.

[14] Lions, J.-L.; Magenes, E., Non-Homogeneous Boundary Problems and Applications, («Problèmes aux Limites Non Homogènes et Applications»), Vol. 1 et 2, Dunod, Paris (1968).

[15] Lions, J.-L. et J. Peetre., On a class of interpolation spaces, («Sur une classe d'espaces d'interpolation»). *Inst. Hautes Etudes C N° 19*, Paris (1964), 5-68.

[16] H. Brézis, *Functional analysis, theory and application, («Analyse fonctionnelle, théorie et application»)*, Paris, Masson, (1998), Sciences Sup., ISBN: 978-2100493364.

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