On the New Exponentiated-Sujatha Distribution and its Applications

BADMUS NOFIU IDOWU, AROMOLARAN ADEYEMI DAVIDSON
Department of Statistics
University of Lagos and Yaba College of Technology
Akoka, Yaba, Lagos
NIGERIA

Abstract: - In this article, we present a new distribution called the new Exponentiated-Sujatha distribution. This is an extension from the one-parameter Sujatha distribution by [1] is developed using the Exponentiation method. Some of its statistical properties like reliability function, hazard rate function, reversed hazard rate function, cumulative hazard rate function, moments, moment generating function, and order statistics of the proposed distribution were established. In the same vein, the estimation of the parameters of the distribution is found based on the maximum likelihood technique. Then, the applicability and tractability of the distribution are tested on two numerical illustrations. Therefore, the new distribution performance supersedes and has better fits than other extant distributions considered.

Key-Words: - Cumulative Hazard, Exponentiation, moments, Order Statistics, Reserved Hazard Rate, Sujatha distribution

Received: June 15, 2023. Revised: February 21, 2024. Accepted: March 19, 2024. Published: May 31, 2024.

1 Introduction

Excessive skewness and kurtosis are common in the data from applied sciences including engineering, medicine, insurance, and finance, amongst others, due to their nature. The Sujatha distribution (SD) was introduced by [1] with one parameter. Meanwhile, the new Exponentiated-Sujatha (NE-S) distribution is a two-parameter and developed to control the level of skewness and kurtosis that may be displayed by any data set from any one of the above-listed fields/disciplines. We discussed some of its statistical properties including reliability, hazard rate, reversed hazard, cumulative hazard rate moments, and order statistics. Also, we present the estimation of parameters using the method and maximum likelihood estimation.

However, researchers have worked on the convolution of Sujatha and Akash with other distributions including [2] came up with discrete Poisson Sujatha (DPS) distribution, [3] discussed a three-parameter Sujatha distribution, [4] dealt with generalized inverse power Sujatha distribution, [5] introduced discrete Poisson-Akash (DPA) distribution, [6] investigated exponentiated Akash (EA) distribution, [7] examined a generalized Poisson-Akash (GPA) distribution, [8] written the exponentiated power Akash (EPA) distribution. [9], [10] and [11] proposed a two-parameter Sujatha (2-PS), a new two-parameter Sujatha (N2-PS), and another two-parameter Sujatha (A2-PS) distribution. Then, [12] and [13] discussed the zero-truncated Poisson Sujatha (ZTPS) and size-biased Poisson Sujatha S-BPS) distribution.

Also, [14] derived a quasi-Sujatha QS) distribution, [15] worked on a generalization of Sujatha (GS) distribution and its applications [16] presented exponentiated Anadhana distribution (EA), [17] laid their hands on a new exponentiated distribution with applications to engineering science (ED), [18] did justice to the new generalization of Pranav distribution (PD), amongst others. In a nutshell, the new exponentiated Sujatha distribution can handle and fit most of the non-normal data sets. This is an advantage it has over other distributions considered in the study. The distribution function and density function of Sujatha distribution as established by [1] are given in (1) below:

$$F(x;\,\theta) = 1 - \left[1 + \frac{\theta x(\theta x+\theta+2)}{\theta^2+\theta+2}\right] e^{-\theta x}\,,\; x > 0, \theta > 0 \qquad (1)$$

and

$$f(x;\, \theta) = \frac{\theta^3}{\theta^2+\theta+2}(1+x+x^2)e^{-\theta x}, \;\; x > 0, \; \theta > 0 \qquad (2)$$

where θ is the scale parameter and can be found in [1].

A continuous random variable X having exponential distribution with density function and distribution is given as follows:

$$G(x;\, \lambda) = P(X \le x) = [F(x;\, \theta)]^{\lambda}; \;\; U\epsilon\, Z' \; \lambda > 0 \qquad (3)$$

$$g(x;\, \lambda) = \lambda [F(x;\, \theta)]^{\lambda-1} f(x;\, \lambda), \qquad \lambda > 0 \qquad (4)$$

Hence, we obtain the distribution function and density distribution of the new exponentiated Sujatha distribution by substituting (1) and (2) into (3) and (4) respectively:

$$G(x;\, \theta, \lambda) = \left[1 - \left[1 + \frac{\theta x(\theta x+\theta+2)}{\theta^2+\theta+2}\right]e^{-\theta x}\right]^{\lambda}, \;\; x, \theta, \lambda > 0 \qquad (5)$$

and

$$g(x;\, \theta, \lambda) = \frac{\lambda\theta^3(1+x+x^2)}{\theta^2+\theta+2}\left[1 - \left[1 + \frac{\theta x(\theta x+\theta+2)}{\theta^2+\theta+2}\right]e^{-\theta x}\right]^{\lambda-1} e^{-\theta x}, x, \theta, \lambda > 0 \qquad (6)$$

where θ and λ are scale and shape parameters. These can be found in [1], [6], and [19].

Here, are the pdf and cdf plots of the NE-S distribution at various values in Fig.1 below:

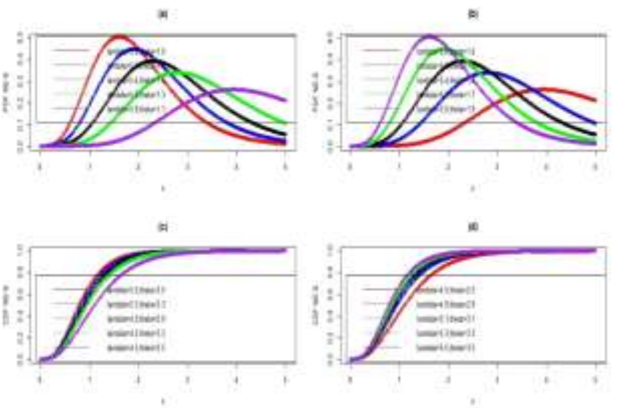

Fig. 1: The pdf plot (a and b), and CDF plot (c and d) of the NE-S distribution

2 Some Statistical Properties

2.1 Reliability Function (RF)

In reliability analysis, the reliability function is one of the important functions that does not fail earlier in time (t) which is the reliability function. This is given by

$$RF(x;\, \theta, \lambda) = 1 - P(X \le x) = 1 - G(x;\, \theta, \lambda) \;\; (7)$$

Theorem 1: A random variable X has a reliability function if and only if its statistical expression is given by (7)

Proof.

$$RF(x;\, \theta, \lambda) = 1 - G(x;\, \theta, \lambda)$$

$$RF = 1 - \left[1 - \left[1 + \frac{\theta x(\theta x+\theta+2)}{\theta^2+\theta+2}\right]e^{-\theta x}\right]^{\lambda} \qquad (8)$$

2.2 Hazard Rate Function (HRF)

This is another important function in reliability analysis, which is defined as a conditional density in which the event of interest has not occurred before time t. The density function of the distribution at and (6) divided by the RF in (8)

Theorem 2: A random variable X is said to have a hazard rate function if expression (6) is divided by (8), that is $g(x;\, \theta, \lambda)/[1 - G(x;\, \theta, \lambda)]$.

Proof.

$$HRF(x;\, \theta, \lambda) = \frac{g(x;\, \theta, \lambda)}{RF(x;\, \theta, \lambda)} \;;\;\; x, \theta, \lambda > 0 \qquad (9)$$

$$HRF(x;\, \theta, \lambda) = \frac{\frac{\lambda\theta^3(1+x+x^2)}{\theta^2+\theta+2}\left[1 - \left[1 + \frac{\theta x(\theta x+\theta+2)}{\theta^2+\theta+2}\right]e^{-\theta x}\right]^{\lambda-1} e^{-\theta x}}{1 - \left[1 - \left[1 + \frac{\theta x(\theta x+\theta+2)}{\theta^2+\theta+2}\right]e^{-\theta x}\right]^{\lambda}} \qquad (10)$$

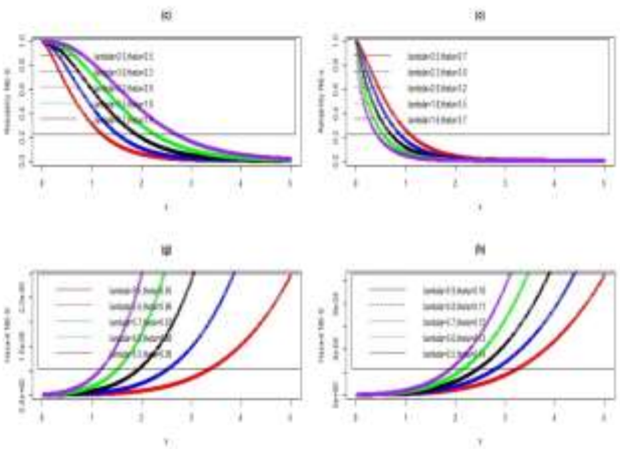

Fig. 2: The reliability plot (e and f), and hazard rate plot (g and h) of the NE-S distribution

2.3 Reversed Hazard Rate Function (RHRF)

The RHRF is defined as the ratio between the life density to its distribution function. Hence, it plays a vital role in analysing survival data which applies to any part of physical sciences. [20] and [21]

Theorem 3: Let $x \geq 0$ be a random variable represented by equation (6) divided by (5) as:

$g(x;\ \theta,\lambda)/G(x;\ \theta,\lambda)$

Proof:

$$RHRF(x;\ \theta,\lambda) = \frac{g(x;\theta,\lambda)}{G(x;\theta,\lambda)}\ ;\ \ x,\theta,\lambda > 0 \qquad (11)$$

$$RHRF(x;\ \theta,\lambda) = \frac{\frac{\lambda\theta^3(1+x+x^2)}{\theta^2+\theta+2}\left[1-\left[1+\frac{\theta x(\theta x+\theta+2)}{\theta^2+\theta+2}\right]e^{-\theta x}\right]^{\lambda-1}e^{-\theta x}}{\left[1-\left[1+\frac{\theta x(\theta x+\theta+2)}{\theta^2+\theta+2}\right]e^{-\theta x}\right]^{\lambda}} \qquad (12)$$

2.4 Cumulative Hazard Rate Function (CHRF)

This function supplying the total accumulated hazard of loss of experiencing the event of interest has been gained by advancing to time t.

Theorem 4: Consider the (6) in which $1 - G(x;\ \theta,\lambda)$ is the reliability function of the NE-S distribution. Assume that RF is known and $G(x;\ \theta,\lambda)$.

Proof:

$$CHRF(x;\ \theta,\lambda) = -\ln[1 - G(x;\ \theta,\lambda)];\ x,\theta,\lambda > 0$$

$$= -\ln \mathrm{RF}(x;\ \theta,\lambda);\ \ x,\theta,\lambda > 0 \qquad (13)$$

$$= 1 - \left[1 - \left[1 + \frac{\theta x(\theta x+\theta+2)}{\theta^2+\theta+2}\right]e^{-\theta x}\right]^{\lambda} \qquad (14)$$

where, $CHRF = -\ln \mathrm{RF}(x;\ \theta,\lambda)$ is in (8), [22].

3 Moments

There are moments about the mean and origin, here we will focus on moments about the origin, $E(X^S)$. A continuous random variable X, the S^{th} moments about the origin $E(X^S)$ of the NE-S distribution is obtained as follows:

$$E(X^S) = \int_0^\infty x^S g(x;\ \theta,\lambda)dx \qquad (15)$$

$$= \int_0^\infty x^S \frac{\lambda\theta^3(1+x+x^2)}{\theta^2+\theta+2}.$$

$$\left[1 - \left[1 + \frac{\theta x(\theta x+\theta+2)}{\theta^2+\theta+2}\right]e^{-\theta x}\right]^{\lambda-1} e^{-\theta x}dx$$

$$= \int_0^\infty \frac{\lambda\theta^3 x^S e^{-\theta x}}{\theta^2+\theta+2}.$$

$$\left[1 - \left[1 + \frac{\theta x(\theta x+\theta+2)}{\theta^2+\theta+2}\right]e^{-\theta x}\right]^{\lambda-1} dx$$

$$+ \int_0^\infty \frac{\lambda\theta^3 x^{S+2}}{\theta^2+\theta+2}e^{-\theta x}.$$

$$\left[1 - \left[1 + \frac{\theta x(\theta x+\theta+2)}{\theta^2+\theta+2}\right]e^{-\theta x}\right]^{\lambda-1} dx$$

By using binomial expansion,[1] and [6], we have

$$\left[1 - \left[1 + \frac{\theta x(\theta x+\theta+2)}{\theta^2+\theta+2}\right]e^{-\theta x}\right]^{\lambda-1}$$

$$= \sum_{i=1}^{\infty}\binom{\lambda-1}{i}(-1)^i\left[1 + \frac{\theta x(\theta x+\theta+2)}{\theta^2+\theta+2}\right]^i e^{-i\theta x}$$

$$= \sum_{i=0}^{\infty}\binom{\lambda-1}{i}(-1)^i\sum_{j=0}^{i}\binom{i}{j}\sum_{k=0}^{j}\binom{j}{k}\frac{2^k(\theta x)^{2j-k}e^{-i\theta x}}{(\theta^2+\theta+2)^j}$$

$$E(X^S) = P_{i,j,k}\frac{\lambda.2^k\theta^{2j-k+3}}{(\theta^2+\theta+2)^{j+1}}\int_0^\infty x^{S+2j-k}.e^{\theta x(i+1)}dx$$

$$+ P_{i,j,k}\frac{\lambda.2^k\theta^{2j-k+3}}{(\theta^2+\theta+2)^{j+1}}\int_0^\infty x^{S+2j-k+2}.e^{\theta x(i+1)}dx$$

where, $P_{i,j,k} = \sum_{i=0}^{\infty} x^n e^{-\lambda x}dx = \frac{\Gamma(n+1)}{\lambda^{n+1}}$ and $\Gamma(\lambda) = (\lambda-1)!$

Hence,

$$E(X^S) = P_{i,j,k}\left[\frac{\lambda.2^k\theta^{2j-k+3}}{(\theta^2+\theta+2)^{j+1}}\cdot\frac{(S+2j-k)!}{(i+1)^{S+2j-k+1}}\right]$$

$$+P_{i,j,k}\left[.\ \frac{(S+2j-k+2)!}{(i+1)^{S+2j-k+3}}\right] \qquad (16)$$

Equation (16) becomes the moment of the NE-S distribution.

3.1 Moment Generating Function (MGF)

The MGF of the NE-S distribution in (6) can be obtained as

$$M_x(t) = E(e^{tx}) = \int_0^\infty e^{(tx)} g(x;\ \theta,\lambda)\,dx \qquad (17)$$

$$= \int_0^\infty e^{(tx)}\frac{\lambda\theta^3(1+x+x^2)}{\theta^2+\theta+2}$$

$$\left[1-\left[1+\frac{\theta x(\theta x+\theta+2)}{\theta^2+\theta+2}\right]e^{-\theta x}\right]^{\lambda-1}$$

$$= \int_0^\infty \frac{\lambda\theta^3}{\theta^2+\theta+2}.e^{-x(\theta-t)}$$

$$\left[1-\left[1+\frac{\theta x(\theta x+\theta+2)}{\theta^2+\theta+2}\right]e^{-\theta x}\right]^{\lambda-1}dx$$

$$+\int_0^\infty \frac{\lambda\theta^3x^2}{\theta^2+\theta+2}.e^{-x(\theta-t)}$$

$$\left[1-\left[1+\frac{\theta x(\theta x+\theta+2)}{\theta^2+\theta+2}\right]e^{-\theta x}\right]^{\lambda-1}dx$$

Again, we employ binomial expression as follows:

$$\left[1-\left[1+\frac{\theta x(\theta x+\theta+2)}{\theta^2+\theta+2}\right]e^{-\theta x}\right]^{\lambda-1}$$

$$=\sum_{i=0}^\infty\binom{\lambda-1}{i}(-1)^i\left[1+\frac{\theta x(\theta x+\theta+2)}{\theta^2+\theta+2}\right]^i e^{-i\theta x}$$

$$=\sum_{i=0}^\infty\binom{\lambda-1}{i}(-1)^i\sum_{j=0}^{i}\binom{i}{j}\sum_{k=0}^{j}\binom{j}{k}\frac{2^k(\theta x)^{2j-k}e^{-i\theta x}}{(\theta^2+\theta+2)^j}$$

Meanwhile, setting $e^{(tx)} = \sum_{i=0}^\infty\frac{(tx)!}{h!}$ and substituting it, we get

$$M_x(t) = P_{i,j,k}\sum_{h=0}^\infty\frac{(t)^h}{h!}.\frac{\lambda.2^k\theta^{2j-k+3}}{(\theta^2+\theta+2)^{j+1}}\int_0^\infty x^{2j-k+1}e^{-\theta x(i+j)}\,dx$$

$$+P_{i,j,k}\sum_{h=0}^\infty\frac{(t)^h}{h!}.\frac{\lambda.2^k\theta^{2j-k+3}}{(\theta^2+\theta+2)^{j+1}}\int_0^\infty x^{h+2j-k+1}e^{-\theta x(i+j)}\,dx$$

where, $P_{i,j,k}\sum_{h=0}^\infty\binom{\lambda-1}{i}(-1)^i\sum_{j=0}^{i}\binom{i}{j}\sum_{k=0}^{j}\binom{j}{k}$

Also, recall that $\int_{i=0}^\infty x^n e^{-\lambda x}dx = \frac{\Gamma(n+1)}{\lambda^{n+1}}$ and $\Gamma(\lambda) = (\lambda-1)!$

Consequently,

$$M_x(t) = \sum_{h=0}^\infty\left(\frac{t}{\theta}\right)^h\left[P_{i,j,k}\frac{\lambda.2^k\theta^{-2}}{(\theta^2+\theta+2)^{j+1}}.\frac{(2j-k+h)!}{h!\,(i+j)^{2j-k+h+3}}\right]$$

$$+\sum_{h=0}^\infty\left(\frac{t}{\theta}\right)^h\left[P_{i,j,k}\frac{\lambda.2^k}{(\theta^2+\theta+2)^{j+1}}.\frac{(2j-k+h+2)!}{h!\,(i+j)^{2j-k+h+3}}\right]$$

Thus,

$$M_x(t) = P_{i,j,k}\sum_{h=0}^\infty\left(\frac{t}{\theta}\right)^h\frac{\lambda.2^k}{(\theta^2+\theta+2)^{j+1}}\left[\frac{\theta^{-2}(2j-k+h)!}{h!\,(i+j)^{s+2j-k+1}}\right.$$

$$\left.+\frac{(2j-k+h+2)!}{h!(i+j)^{s+2j-k+1}}\right] \qquad (18)$$

Therefore, (18) becomes the MGF of the NE-S distribution.

4 Order Statistics (OS)

OS is a useful concept in statistical theory and modelling. Its applicability covers many areas like estimating parameter distributions, insurance policies, modelling, etc. However, the density function of r^{th} order statistics is given by

$$g_X(x) = \frac{n!}{(r-1)!\,(n-r)!} G(x;\ \theta,\lambda)^{r-1}$$

$$[1 - g(x;\ \theta,\lambda)]^{n-r}$$

$$g_X(x) = \frac{n!}{(r-1)!\,(n-r)!} \sum_{i=0}^{n-r} \binom{n-r}{i} (-1)^i$$

$$G(x;\ \theta,\lambda)^{r-1+i} g(x;\ \theta,\lambda) \qquad (19)$$

By substituting $G(x;\ \theta,\lambda)^{r-1}$ and $g(x;\ \theta,\lambda)$ in (19), we get

$$g_X(x) = \frac{\lambda\theta^3 n!\,(1+x+x^2)}{(\theta^2+\theta+2)(r-1)!\,(n-r)!} \sum_{i=0}^{n-r} \binom{n-r}{i} (-1)^i$$

$$e^{-\theta x}.\left[1 - \left[1 + \frac{\theta x(\theta x+\theta+2)}{\theta^2+\theta+2}\right] e^{-\theta x}\right]^{\lambda(r-1+i)}$$

$$\left[1 - \left[1 + \frac{\theta x(\theta x+\theta+2)}{\theta^2+\theta+2}\right] e^{-\theta x}\right]^{\lambda-1}$$

$$g_X(x) = \frac{\lambda\theta^3 n!\,(1+x+x^2)e^{-\theta x}}{(\theta^2+\theta+2)(r-1)!\,(n-r)!} \sum_{i=0}^{n-r} \binom{n-r}{i} (-1)^i$$

$$\left[1 - \left[1 + \frac{\theta x(\theta x+\theta+2)}{\theta^2+\theta+2}\right] e^{-\theta x}\right]^{\lambda(r+i)-1}$$

Also, using binomial expansion

$$\left[1 - \left[1 + \frac{\theta x(\theta x+\theta+2)}{\theta^2+\theta+2}\right] e^{-\theta x}\right]^{\lambda(r-1+i)}$$

$$= \sum_{j=0}^{\infty} \binom{\lambda(n+i)-1}{j} (-1)^i$$

$$\left[1 + \frac{\theta x(\theta x+\theta+2)}{\theta^2+\theta+2}\right]^j e^{-j\theta x}$$

$$= \sum_{j=0}^{\infty} \binom{\lambda(n+i)-1}{j} (-1)^i \sum_{k=0}^{j} \binom{j}{k} \sum_{l=0}^{k} \binom{k}{l}.$$

$$\frac{2^l(\theta x)^{2k-1}}{(\theta^2+\theta+2)^k} e^{-j\theta x}$$

[19] and [21]. Hence, the density function of r^{th} OS is obtained as

$$g_X(x) = \left(A\,.\sum_{k=0}^{j} \binom{j}{k} \sum_{l=0}^{k} \binom{k}{l}. \frac{\lambda n!\,\theta^{2k-l+2}(1+x+x^2)x^{2k-l}e^{-\theta x(j+1)}}{(\theta^2+\theta+2)(r-1)!\,(n-r)!} \right)$$

where,

$$A = \sum_{i=0}^{n-r} \binom{\lambda(n-p)}{i} (-1)^i \sum_{j=0}^{\infty} \binom{\lambda(n+i)-1}{j} (-1)^i$$

Furthermore, the distribution function of ther^{th} OS is given by

$$G_X(x) = \sum_{i=r}^{n} \binom{n}{i} G(x;\ \theta,\lambda)^i [1 - G(x;\ \theta,\lambda)]^{n-i}$$

$$G_X(x) = \sum_{i=r}^{n} \sum_{j=0}^{n-i} \binom{n}{i} \binom{n-i}{i} (-1)^i G(x;\ \theta,\lambda)^{i+j} \qquad (20)$$

By substituting, we have

$$G_X(x) = \sum_{i=r}^{n} \sum_{j=0}^{n-i} \binom{n}{i} \binom{n-i}{i} (-1)^i$$

$$\left[1 - \left[1 + \frac{\theta x(\theta x+\theta+2)}{\theta^2+\theta+2}\right] e^{-\theta x}\right]^{\lambda(i+j)}$$

Here, we employ binomial expression to obtain

$$\left[1 - \left[1 + \frac{\theta x(\theta x+\theta+2)}{\theta^2+\theta+2}\right] e^{-\theta x}\right]^{\lambda(i+j)}$$

$$= \sum_{k=0}^{\infty} \binom{\lambda(i+j)}{k} (-1)^k \left[1 + \frac{\theta x(\theta x+\theta+2)}{\theta^2+\theta+2}\right]^k e^{-k\theta x}$$

$$= \sum_{k=0}^{\infty} \binom{\lambda(i+j)}{k} (-1)^k \sum_{l=0}^{k} \binom{k}{l} \sum_{m=0}^{l} \binom{l}{m}.$$

$$\frac{2^n(\theta x)^{2l-m}}{(\theta^2+\theta+2)^l} e^{-k\theta x}$$

Therefore,

$$G_X(x) = \left(B \cdot \sum_{l=0}^{k} (-1)^k \binom{k}{l} \sum_{m=0}^{l} \binom{l}{m} \cdot \frac{2^n (\theta x)^{2l-m} e^{-k\theta x}}{(\theta^2+\theta+2)^l} \right)$$

where,

$$B = \sum_{i=r}^{n} \sum_{j=0}^{n-i} \sum_{k=0}^{\infty} \binom{n}{i} \binom{n-i}{i} (-1)^i \binom{\lambda(i+j)}{k}$$

5 Estimation of Model Parameter

Suppose $X_1, X_2, \dots, X_n$ is a random sample of size n from NE-S distribution, then the log-likelihood function of the parameters is written as

$$LoL(x;\, \theta, \lambda) = \prod_{i=1}^{n} ln[g(x;\, \theta, \lambda)]$$

$$= \prod_{i=1}^{n} \left\{ \frac{\lambda\theta^3}{\theta^2+\theta+2} \left(1 + x_i + x_i^2\right) \cdot \left[1 - \left[1 + \frac{\theta x_i(\theta x_i + \theta + 2)}{\theta^2+\theta+2}\right] e^{-\theta x_i}\right]^{\lambda-1} e^{-\theta x_i} \right\}$$

$$= ln\left\{ \left(\frac{\lambda\theta^3}{\theta^2+\theta+2}\right)^n \prod_{i=1}^{n} \left(1 + x_i + x_i^2\right) e^{-\theta \sum x_i} \cdot \prod_{i=1}^{n} \left[1 - \left[1 + \frac{\theta x_i(\theta x_i + \theta + 2)}{\theta^2+\theta+2}\right] e^{-\theta \sum x_i}\right]^{\lambda-1} \right\}$$

$$LoL = n[\ln(\lambda) + 3\ln(\theta) - \ln(\theta^2+\theta+2)] + \sum_{i=1}^{n} ln\left(1 + x_i + x_i^2\right) - \theta \sum_{i=1}^{n} x_i + (\lambda - 1) \cdot \sum_{i=1}^{n} ln\left[1 - \left[1 + \frac{\theta x_i(\theta x_i + \theta + 2)}{\theta^2+\theta+2}\right] e^{-\theta \sum x_i}\right]$$

$$-\theta \sum_{i=1}^{n} x_i + (\lambda - 1) \sum_{i=1}^{n} ln\left[1 - \left[1 + \frac{\theta x_i(\theta x_i+\theta+2)}{\theta^2+\theta+2}\right] e^{-\theta \sum x_i}\right] \quad (21)$$

Then, we solve the nonlinear likelihood equations simultaneously by taking the partial differentiation of (21) concerning θ and λ as we have them below

$$\frac{\partial LoL}{\partial(\theta,\ \lambda)} = n[\ln(\lambda) + 3\ln(\theta) - \ln(\theta^2+\theta+2)] + \sum_{i=1}^{n} ln\left(1 + x_i + x_i^2\right) - \theta \sum_{i=1}^{n} x_i + (\lambda - 1) \sum_{i=1}^{n} ln \left[1 - \left[1 + \frac{\theta x_i(\theta x_i + \theta + 2)}{\theta^2+\theta+2}\right] e^{-\theta \sum x_i}\right]$$

$$\frac{\partial LoL}{\partial(\theta)} = \frac{n[3(\theta^2+\theta+2) - 2\theta^2 + 1]}{\theta(\theta^2+\theta+2)+1} - \sum_{i=1}^{n} x_i + \frac{(\lambda-1)\left[\frac{2\theta^2 x_i(2x_i{}^2 - 1 - \theta x_i - 2) + 4\theta x_i{}^2}{(\theta^2+\theta+2)}\right]}{1 - \left[1 + \frac{\theta x_i(\theta x_i+\theta+2)}{\theta^2+\theta+2}\right] e^{-\theta x_i}} \quad (22)$$

$$\frac{\partial LoL}{\partial(\lambda)} = \frac{n}{\lambda} + \sum_{i=1}^{n} ln\left[1 - \left[1 + \frac{\theta x_i(\theta x_i+\theta+2)}{\theta^2+\theta+2}\right] e^{-\theta x_i}\right] \quad (23)$$

The equations (22 and 23) above may not be solved directly, due to the inability to be expressed in closed forms. Hence, the MLE's $(\hat{\theta},\ \hat{\lambda})$ of $(\theta,\ \lambda)$ were solved using Newton-Raphson iteration (algorithm) available in R-software. The results are shown in Tables 5 and 7 below.

5.1 Monte Carlo Simulation Study

Here, in this section, we present the maximum likelihood estimates of the two parameters of the NE-S distribution through a Monte Carlo simulation study. In the simulation study, we generate 1000 data from NE-S distribution using some initial values for each parameter as: $\theta = 0.10, \lambda = 0.15, \theta = 0.10, \lambda = 0.25, \theta = 0.20, \lambda = 0.25$ and $\theta = 0.20, \lambda = 0.50$, for different sample size of $K = 25, 50, 75$ and 100. Also, for each estimate $\hat{\varphi} = (\hat{\theta},\ \hat{\lambda})$, we compute and obtain the values of the following:

(i) Average Bias (AB) $(\hat{\varphi}) = \frac{1}{N} \sum_{i=1}^{k} (\hat{\varphi}_1 - \varphi_0)$ (24)

(ii) Mean Square Error (MSE) =

$$\frac{1}{N}\sum_{i=1}^{k}(\hat{\varphi}_1 - \varphi_0)^2 \quad (25)$$

(iii) Root Mean Square Error (RMSE) =

$$\sqrt{\frac{1}{N}\sum_{i=1}^{k}(\hat{\varphi}_1 - \varphi_0)} \quad (26)$$

See [23], [24] and [25].

Table 1. Monte Carlo Simulation Study for AB, MSE, and RMSE of the MLE when parameter $\theta = 0.10, \lambda = 0.15$, and $n = 25, 50, 75$, and 100.

N	Parameter	AB	MSE	$RMSE$
25	$\theta = 0.10$	0.5105	0.3080	0.5550
	$\lambda = 0.15$	-0.1501	0.0225	0.1502
50	$\theta = 0.10$	0.5143	0.2893	0.5379
	$\lambda = 0.15$	-0.1501	0.0225	0.1502
75	$\theta = 0.10$	0.5136	0.2800	0.5292
	$\lambda = 0.15$	-0.1501	0.0225	0.1502
100	$\theta = 0.10$	0.5088	0.2719	0.5215
	$\lambda = 0.15$	-0.1501	0.0225	0.1502

Table 2. Monte Carlo Simulation Study for AB, MSE, and RMSE of the MLE when parameter $\theta = 0.10, \lambda = 0.25$, and $n = 25, 50, 75$, and 100.

N	Parameter	AB	MSE	$RMSE$
25	$\theta = 0.10$	0.7728	0.6466	0.8041
	$\lambda = 0.15$	-0.2502	0.0625	0.2503
50	$\theta = 0.10$	0.7853	0.6432	0.8020
	$\lambda = 0.15$	-0.2502	0.0625	0.2503
75	$\theta = 0.10$	0.8080	0.6728	0.8203
	$\lambda = 0.15$	-0.2502	0.0625	0.2503
100	$\theta = 0.10$	0.8392	0.7251	0.8515
	$\lambda = 0.15$	-0.1502	0.0625	0.1503

Table 3. Monte Carlo Simulation Study for AB, MSE, and RMSE of the MLE when parameter $\theta = 0.20, \lambda = 0.25$, and $n = 25, 50, 75$ and 100.

N	Parameter	AB	MSE	$RMSE$
25	$\theta = 0.20$	0.7940	0.6889	0.8300
	$\lambda = 0.25$	-0.2502	0.0625	0.2503
50	$\theta = 0.20$	0.7628	0.6171	0.7855
	$\lambda = 0.25$	-0.2502	0.0625	0.2503
75	$\theta = 0.20$	0.7557	0.6006	0.7750
	$\lambda = 0.25$	-0.2502	0.0625	0.2503
100	$\theta = 0.20$	0.7358	0.5712	0.7561
	$\lambda = 0.25$	-0.1502	0.0625	0.2503

Table 4. Monte Carlo Simulation Study for AB, MSE, and RMSE of the MLE when parameter $\theta = 0.20, \lambda = 0.50$, and $n = 25, 50, 75$ and 100.

N	Parameter	AB	MSE	$RMSE$
25	$\theta = 0.20$	1.4835	2.2767	1.5089
	$\lambda = 0.50$	-0.5000	0.2500	0.5000
50	$\theta = 0.20$	1.4821	2.2369	1.4956
	$\lambda = 0.50$	-0.5000	0.2500	0.5000
75	$\theta = 0.20$	1.5070	2.3123	1.5206
	$\lambda = 0.50$	-0.5000	0.2500	0.5000
100	$\theta = 0.20$	1.4504	2.1732	1.4742
	$\lambda = 0.50$	-0.5000	0.2500	0.5000

Tables 1, 2, 3, and 4 reveal the Monte Carlo simulation results for the AB, MSE, and RMSE of the NE-S distribution for different parameter values. We notice from the results in Tables 1 to 4 that parameter θ is positively biased and λ is negatively biased. But both parameters are positive under MSE and RMSE in Tables 1 to 3, and as the sample size n increases the values of the parameter estimates decrease (tend $\rightarrow 0$). Except in Table 4 where the values are more than 1 as the value of λ increased.

5.2 Applications and Goodness of Fit

The goodness of fit of the new exponentiated-Sujatha distribution (NE-S) using MLEs has been illustrated with two real lifetime data sets and the fit is examined with exponentiated Akash, two-parameter exponential Akash, Akash, and Sujatha distributions. The two real lifetime data sets have been considered for both the goodness of fit and model selection criterion of considered distributions.

First Data set: The data set represents the strength of 1.5cm glass fibers measured at the National Physical Laboratory, England. It was given in [26], but taken from [2] research paper.

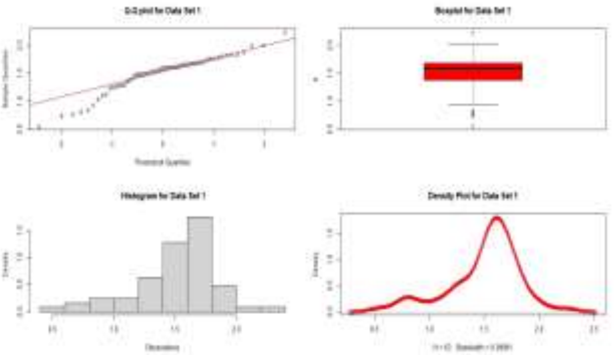

Fig. 3. Fitted QQ-Normal, Boxplot, Histogram, and Density Plots for Data Set 1.

Second Data set: This data set is the strength data of the glass of the aircraft window reported by [27]. While it has been used by [17] and [21] respectively in their papers.

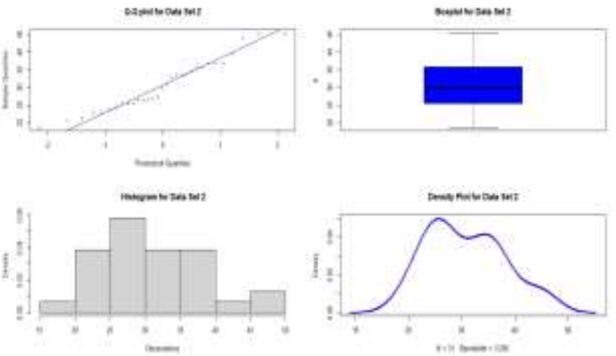

Fig. 4. Fitted QQ-Normal, Boxplot, Histogram, and Density Plots for Data Set 2.

Table 5. MLEs, (* denote Standard Errors) and Statistics: (W*), (AD*), (K-S), and (P denotes P-value) from Data Set 1.

Distr.	Estimate	W*	AD*	K-S
NE-S Proposed	$\hat{\lambda} = \mathbf{0.4694}$ $(\mathbf{0.0749}*)$ $\hat{\theta} = \mathbf{0.8790}$ $(\mathbf{0.1028}*)$	**0.5814**	**3.1875**	**0.4205** $\mathbf{4.227e^{-10P}}$
EA (2018)	$\hat{\alpha} = 0.6303$ $(0.1010*)$ $\hat{\theta} = 1.2236$ $(0.1163*)$	0.5667	3.1080	0.4729 $1.164e^{-12P}$
TPEA (2017)	$\hat{\alpha} = 0.9859$ $(0.0431*)$ $\hat{\theta} = 0.3456$ $(0.0441*)$	0.5858	3.2117	0.4885 $1.176e^{-13P}$
AD (2015)	$\hat{\theta} = 1.5093$ $(0.1063*)$	18.6311	122.0569	0.9660 $2.2e^{-16P}$
SD (2015)	$\hat{\theta} = 1.4829$ $(0.1081*)$	0.4181	2.2949	0.7549 $2.2e^{-16P}$

Table 6. Model selection criteria for the fitted models with Data Set 1.

Distr	2LoL	AIC	CAIC	BIC	HQIC
NE-S	**81.8760**	**167.7513**	**167.9530**	**172.0370**	**169.4371**
EA	120.1927	244.3861	244.5863	248.6732	246.0722
TPEA	112.4924	228.9842	229.1836	233.2696	230.6695
AD	107.1628	216.3266	216.3917	218.4699	217.1698
SD	97.1920	196.3837	196.4577	198.5269	197.2267

Table 7. MLEs, (* denote Standard Errors) and Statistics: (W*), (AD*), (K-S), and (P denotes P-value) from Data Set 2.

Distr.	Estimate	W*	AD*	K-S
NE-S Proposed	$\hat{\lambda} = \mathbf{4.2420}$ $(\mathbf{2.9117}*)$ $\hat{\theta} = \mathbf{0.1771}$ $(\mathbf{0.0350}*)$	$\mathbf{0.0796}$	$\mathbf{0.4431}$	$\mathbf{1.0000}$ $\mathbf{3.331e^{-16P}}$
EA (2018)	$\hat{\alpha} = 5.3971$ $(3.4214*)$ $\hat{\theta} = 0.1908$ $(0.0324*)$	0.2220	1.6020	1.0000 $3.331e^{-16P}$
TPEA (2017)	$\hat{\alpha} = 0.1908$ $(0.3242*)$ $\hat{\theta} = 5.4032$ $(3.4254*)$	0.2201	1.5914	1.0000 $3.331e^{-16P}$

AD (2015)	$\hat{\theta} = 0.0972$ $(0.0101 *)$	10.8994	62.2919	0.9879 $3.331e^{-13P}$
SD (2015)	$\hat{\theta} = 0.0964$ $(0.0100 *)$	2.4721	11.7847	1.0000 $3.331e^{-13P}$

Table 8. Model selection criteria for the fitted models with Data Set 2

Dist	2LoL	AIC	CAIC	BIC	HQIC
NE-S	**121.6406**	**247.2823**	**247.7096**	**250.1506**	**248.2171**
EA	224.8546	453.7087	454.1383	456.5768	454.6442
TPEA	224.8546	453.7087	454.1383	456.5768	454.6442
AD	141.7561	285.5122	285.6486	286.9465	285.6501
SD	141.4546	284.9085	285.0475	286,3433	285.3769

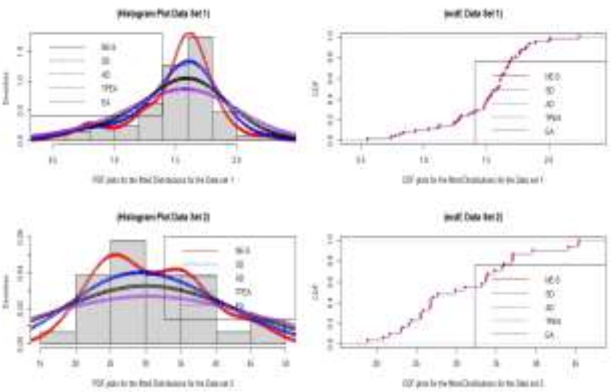

Fig. 5. The PDF and CDF Plots of the fitted distributions for Data Sets 1 & 2.

Discussion and Results

In the discussion and results, we made the comparison by considering the following criteria include, goodness of fit statistics like W*, AD*, and K-S (Cramer von Mises, Anderson Darling and Kolmogorov Smirnov) and model selection such as log-likelihood (-2LoL), AIC, BIC, CAIC and HQIC (Akaike Information Criterion, Consistent Akaike Information Criterion, Bayesian Information Criterion and Hannan Quantile Information criterion) using two different data sets for illustrations. According to the results in Tables 5, 6, 7, and 8 above, values of the NE-S distribution from W*, AD*, K-S, AIC, CAIC, BIC, and HQIC, respectively, smaller than the extant distributions considered such as EA, TPEA, AD, and SD distributions.

The state of being of high rank of the proposed distribution depends on its performance over others (its smallest values) in the illustrations. Also, the claim can be confirmed through the PDF and CDF plots of the fitted distributions displayed in Figure 5. Fig.1 & Fig.2, showcase the PDF, CDF, reliability, and hazard rate function of the NE-S distribution. While, Fig.3 & Fig.4 show the characteristics of data sets by plotting the QQ-Normal, boxplot, histogram, and their density plots.

Conclusion

In this study, we derived a new distribution namely the new exponentiated-Sujatha (NE-S) distribution. Also, we were able to discuss some of its properties like reliability, hazard rate, reversed hazard function, cumulative hazard rate, moments and moment generating function, and order statistics. Then, it is revealed in the results in Tables 5, 6, 7, and 8, and Fig.5 that the proposed distribution performed better in the two illustrations used than other distributions considered in the study.

Acknowledgement:
The author acknowledged the two senior colleagues and the three reviewers who helped proofread and did justice to the manuscript to the final stage.

References:

[1] Shanker, R. (2016). Sujatha Distribution and Its Applications. *Statist. Transition New Ser.*, Vol. 17(3): 1 – 20.

[2] Shanker, R. (2016). The discrete Poisson-Sujatha distribution. *International Journal Probability and Statistics*. 5(1): 1 - 9.

[3] Nwikpe, B. J and Iwok, I. A. (2020). A Three Parameter-Sujatha distribution with Application. *European Journal Statistics and Probability*. Vol. 15(3): 11 – 25.

[4] Okoli, O. M., Osuji, G and Onyekwere, C. K. (2021). Generalized Inverse Power Sujatha distribution with Applications. *Asian Journal Probability and Statistics and Probability*. Vol. 8(3): 22 - 34.

[5] Shanker, R. (2017). The discrete Poisson-Akash distribution. *International Journal Probability and Statistics.* 6(1): 1 - 10.

[6] Okereke, E. W and Uwaeme, O. R. (2018). Exponentiated Akash Distribution and Its Applications. *Journal of the Nigerian Statistical Association.* 30: 1 – 13.

[7] Shanker, R., Shukla, K. K and Leonida, T. A. (2018). A Generalized of Poisson-Akash Distribution Properties and Applications. International *Journal of Statistics and Applications.* 8(5): 249 – 258.

[8] Nwankwo, M. P., Bright, C. N and Okechukwu, J. O. (2023). The Exponentiated Power Akash Distribution: Properties, Regression, and Applications to Infant Mortality Rate and COVID-19 Patients' Life Cycle. *Annals of Biometrics & Biometric Applications.* 5(4): 1 – 13.

[9] Mussie, T and Shanker, R. (2018a). A two-parameter Sujatha distribution. 7(3) (2018), 188–197. *Biometrics Biostat. Internat. J.* 7(3): 188 – 197.

[10] Mussie, T and Shanker, R. (2018b) A New two-Parameter Sujatha distribution with properties and applications. *Turkiye Klinikleri J. Biostat.* 10(2): 96 – 113.

[11] Mussie, T and Shanker, R. (2019). Another Two-Parameter Sujatha Distribution with Properties and Applications. *Journal of Mathematical Sciences and Modelling.* 2(1): 1 – 13

[12] Shanker, R. and Hagos, F. (2015). Zero-Truncated Poisson-Sujatha distribution with Applications. *Journal Ethiopian Statistics. Association.* 24, 55 – 63.

[13] Shanker, R. and Hagos, F. (2016). Size-biased Poisson-Sujatha distribution with Applications. *American Journal Mathematics Statistics.* 6(4): 145-154.

[14] Shanker, R. (2016). A quasi Sujatha distribution. International Journal of Probability Statistics. 5(4): 89 – 100.

[15] Shanker, R., Shukla, K. K and Hagos F. (2017). A Generalization of Sujatha Distribution and Its Applications with Real Lifetime Data. *Journal of Institute of Science and Technology.* 22(1): 66 – 83.

[16] Gannie, R. A and Rajagopalan, V. (2022). Exponentiated Anadhana Distribution with Properties and Applications in Engineering Science. *Journal of Scientific Research.* 66(1): 316 – 326.

[17] Rather, A. A and Subramanian, C. (2020). A New Exponentiated Distribution with Engineering Science Applications. *Journal of Statistics, Applications and Probability.* 9(1): 127 – 137.

[18] Mohiuddin, M., Kannan, R., Alrashed, A. A and Fasil, M. (2022). A New Generalization of Pranav Distribution with Application Model Real Life Data. *Journal of Statistics Applications & Probability.* 11(3): 991 – 1011.

[19] Uwaeme, O. P., Akpan, N. P and Orumie, U. C. (2018). An Extended Pranav Distribution. *Asian Journal of Probability and Statistics.* 2(4): 1 - 15. Retrieved from Article no. AJPAS 46556.

[20] Kayid, M., Alnalawati, H and Almad, I. A. (2011). Testing behavior of the Reversed Hazard Rate. *Applied Mathematical Modeling.* 35(5), 2508 - 2515.

[21] Gannie, R. A., Rajagopalan, V and Abdulalmi, S. (2021). The Weighted Power Shanker Distribution with Characteristics and Applications of Real-life Tree Data. *Journal of Statistics & Probability.* 10(1): 245 - 265.

[22] Helal, T. S., Elsehetry, A. M and Elshaarawy, R. S. (2022). Statistical Properties of Weighted Shanker Distribution. *Journal of Business and Environmental Sciences.* 1(1): 141 - 153.

[23] Omoruyi, F. A., Omeje, I. L., Anabike, I. C and Obulezi, O. J. (2023). A New Variant of Rama Distribution with Simulation Study and Application to Blood Cancer Data. European *Journal of Theoretical and Applied Sciences.* Vol. 1(4). 389 – 409.

[24] Tuoyo, D. O., Opene, F. C and Ekhosuehi, N. (2021). The Topp-Leone Weibull Distribution: Its Properties and Application. Earthline *Journal of Mathematical Sciences.* Vol. 7(2), 381 – 401.

[25] Ray, M and Shanker, R. (2023). A Compound of Exponential and Shanker Distribution with an Application. *Journal of Scientific Research of the Banaras Hindu University.* Vol. 67(4), 39 – 46.

[26] Smith, R. L. and Naylor, J. C. (1987). A Comparison of Maximum Likelihood and Bayesian Estimators for the Three-Parameter Weibull Distribution. *Applied Statistics.* 36: 358 - 369.

[27] Fuller, E. J., Frieman, S., Quinn, J., Quinn, G. and Carter, W. (1994). Fracture mechanics approach to the design of glass aircraft windows. *A case study, SPIE Proc* 2286. 419 - 430.

Contribution of Individual Authors to the Creation of a Scientific Article (Ghostwriting Policy)

Nofiu Idowu Badmus initialized the research ideas, wrote: the introduction, materials and methods, statistical properties, and carried out the analysis of data with two illustrations using R-software., Adeyemi Davidson Aromolaran did the Monte Carlo simulation study, discussed the results, gave concluding remark and arranged the references accordingly.

Sources of Funding for Research Presented in a Scientific Article or Scientific Article Itself
No funding was received for conducting this study.

Conflict of Interest
The authors have no conflicts of interest to declare that are relevant to the content of this article.