On Conditional Propositions and Equivalence in Logical B_3

HASAN KELEŞ[1*], HALİL İBRAHİM ŞAHİN[2]

[1]Department of Mathematics, Karadeniz Technical University, Campus of Kanuni, 61080-Trabzon, TÜRKİYE.

[2]Department of Computer Sciences, Karadeniz Technical University, Campus of Kanuni, 61080-Trabzon, TÜRKİYE.

Abstract: - This article is about logical values. Known logical values are explored in the definition of logic given in B_2. The statement "not both" is analyzed. The logical definition B_3 is given according to the results obtained. The expansion of logical values in B_2 is investigated in logic B_3. New logical values are added to the literature. True corresponds to 1 and false corresponds to 0. If these both cases are not enough. We need to put -1 for ternary logic which is the scope of the paper. Therefore, the existing values of real expressions are obtained. The new definitions, lemmas and theorems covering the three logical values are the core of the paper.

Key-Words: logic, binary logic, ternary logic, uncertainty science, lieology.
AMSMathematics Subject Classification: 03A05, 03B47, 03A05. 68M01.

Received: March 5, 2024. Revised: August 14, 2024. Accepted: December 6, 2024. Published: March 12, 2025.

1 Introduction

Boolean algebra is a subfield of algebra where the value of variables can be true or false. True and false values are usually expressed as 1 and 0 respectively. Unlike basic algebra, where the variable values are numbers and the operations are addition and multiplication, Boolean algebra has the operations "and" denoted by ∧, "or" denoted by ∨, and "not" denoted by ¬. [1]

Boolean algebra is named after George Boole (1815-1864) and is claimed to have been first proposed by Sheffer in 1913 [2, 3]

George Boole is considered the father of Boolean algebra. In 1848, a work called "Mathematical Analysis of Logic" was published. This work opened a new era in mathematics and Boole gained a certain reputation. This pamphlet was also appreciated by de Morgan. This work was to be the harbinger of a work that would appear six years later [3, 4].

Aristoteles, a famous Greek mathematician, created symbolic logic, Aristotelian logic, on a mathematical basis. Later, Boolean improved this idea and therefore it is called Boolean logic and then Boolean algebra, mathematical logic, symbolic logic, after some studies, fuzzy logic etc. were created.

De Morgan (1806-1871) later developed the famous De Morgan theorem on Boolean Algebra. This algebra has been fundamental in the development of digital electronics, and therefore, for all modern programming languages. It is also used in set theory and statistics. Huntington's primary research interest was the foundations of mathematics. He was one of the "American postulate theorists". Hunginton developed postulate for Boolean algebra. His postulates are used for modern algebra circuits [5 - 7].

Huntington established Boolean algebra on an axiomatic basis in 1904. He revisited Boolean Axiomatics in 1933 and proved that Boolean algebra requires a single binary operation (denoted by infix '+' below) that commutes and combines, and a single singular operation, completion, denoted by postfix prime “'”. The only axiom required by Boolean algebra is:

$$(a'+b')'+(a'+b)' = ab+ab'$$

$$= a(b+b') = a.$$

It is known as Huntington's axiom.

Shannon [3] is the first to show that all these axioms and theorems could be used as the basis of electronic circuits as a field of application. In 1937, at the age of 21, he wrote his master's thesis “Symbolic Analysis of Relay and Switching Circuits” at the Massachusetts Institute of Technology, showing that electromagnetic relays could be simplified using Boolean Algebra. In this way, he laid the foundation for the use of electrical switches, which are the building blocks of today's digital computers [2, 3]. Çağman gave the important contribution to logic(fuzzy) with his study on the basic methods used in mathematical proofs in [8].

With the digitization of systems, the rules of Boolean algebra have been used in digital communication systems. Algebraic structures are introduced by Peirce in [1]. The contribution on the basic structures of ternary logic is made by Keleş in [9, 10].

Binary logic has many uses, especially in statistics, economics, communications, electrical-electronic circuits and computer science, such as regression analysis [7].

Boolean algebra is based on 10 fundamental postulates [1,7]. Because of the numbers 0 and 1, each postulate is expressed in pairs. Since the postulates include the characters 0 and 1, their description is usually done with closed and open electrical circuits [6]. Bhandari, Manandhar and Jha gave some constructions on the cartesian product of Fuzzy b-Metric Space [11]. Lampreia, Mestre, Morgado and Navas used fuzzy logic in maintenance system management [12].

In digital electronics, the three-state buffer is an of the best examples of ternary logic. This logic is used in processor, bus and memory design.

2 Logic B_2

In this section, some definitions, properties and basic rules on simple logic are given.

Boolean Algebra based on, especialy, 10 theorems [1] as follows:

Substitution Rule

$$A+B=B+A, \quad A.B=B.A$$

Merger Rule

$$A+B+C=(A+B)+C=A+(B+C)$$

$$A.B.C=(A.B).C=A.(B.C)$$

Same Force Rule (law of identity)

$$A.A=A, \quad A+A=A$$

and (and) law

$$A.1=A, \ A.0=0$$

or (or) law

$$A+1=1, A+0=A$$

Ineffective Element Rule

$$A.1=A, \ A+0=A$$

DOI: 10.37394/232026.2025.7.2

Complementary Rule

$$A.A'=0,\ A'+A=1$$

Rule of Ingestion

$$A.\left(A+B\right)=A$$

$$A+\left(AB\right)=A$$

Dispersion Rule

$$A\left(B+C\right)=\left(AB\right)+\left(AC\right)$$

$$A+\left(B.C\right)=\left(A+B\right)\left(A+C\right)$$

Double Reversal Rule (Reverse Reversal)

$$\left(A'\right)'=A$$

$$\left[\left(A+B\right)'\right]'=A+B$$

De Morgan Rule

$$\left(A.B\right)'=A'+B'$$
$$\left(A+B\right)'=A'.B'$$

Let D be the set of expressions. There exist at least $X_1, X_2 \in \mathsf{D}$ such that $X = X_1 * X_2$, for some $X \in \mathsf{D}$, where with the operation $*$.

Example. 2.1. Let x^2-1 be the expression**.** This is the product of two independent expressions $x-1, x+1$. This is written as follows.

$$x^2-1=\left(x-1\right)\left(x+1\right), \text{ true.}$$

Algebraic expressions are true or false in the case of equality or inequality. An algebraic expression is true or false in a set other than the given set. For example, the equation $x^2+1=0$ is meaningless in the set of real numbers. But, $i^2+1=0, \left(-i\right)^2+1=0$ are true in the larger set, complex numbers, $x^2+1=0$ is false for all $x^2 \neq -1$.

A proposition is a statement that is either true or false, but not both.

If $-1+1=0$ is true.

If $-1+1=1$ is false.

These statements are propositions. But, $x+1=0$ is not a proposition. Because,

If $x=-1$, then it is true and If $x \neq -1$, then it is wrong.

Logic B_2 is the basic structure formed by the idempotent elements.

In Logic B_2,

$$2^{(\Box,.)}=\left\{x \in \Box \mid x^2=x\right\}=\left\{0,1\right\}.$$

$$P(X)=\begin{cases} 1, if\ X \text{ is } \textit{true} \\ 0, if\ X \text{ is } \textit{false} \end{cases}$$

Definition 2.2. Let D be a set and P be an expression depending on the variable X. If $P(X)$ is a proposition for each $X \in \mathsf{D}$, then P is called a propositional function or a verb with respect to the set D. This set D is the definition set of the proposition P and the set $\left\{0,1\right\}$ is the image set $P(X)$.

That is, an expression $P:\mathsf{D} \to 2^{(\Box,.)}$ is called a proposition in B_2 that satisfies the following two conditions

i. For all $X \in \mathsf{D}$, $P\left(X\right) \in 2^{(\Box,.)}$.

ii. For all $X, Y \in \mathsf{D}$, if $X=Y$, then

$$P\left(X\right)=P\left(Y\right)$$

iii. For all $X \in \mathsf{D}$,
$$P\left(X\right) \neq 0, P\left(X\right) \neq 1.$$

We now give the following.

Lemma 2.3. The equality or inequality of any two mathematical expressions without variables or parameters is a proposition.

Proof.

Two different mathematical expressions expressed in numbers are either true or false. And the comparison of these two statements is not true and false at the same time.

For example, the proposition $(-1)^2 = 1$ is a true proposition and $(-1)^4 = -1$ is false.

Example 2.4. Let D $= \{1\}$ and $P :$ D $\to 2^{(□,.)}$ by $P = \{(1,1),(1,0)\} \subset$ D $\times 2^{(□,.)}$, then this is not a proposition.

Definition 2.5. Let P, Q be any two propositions on an arbitrary set D. The logical operations "." and "+" are defined as follows.

i. $P(X).Q(X) = \begin{cases} 0, \text{ if } P(X) = 0 \text{ or } Q(X) = 0 \\ 1, \text{ if } P(X) = 1,\ Q(X) = 1 \end{cases}$

ii. $P(X) + Q(X) = \begin{cases} 1, \text{ if } P(X) = 1 \text{ or } Q(X) = 1 \\ 0, \text{ if } P(X) = 0,\ Q(X) = 0 \end{cases}$

So we get the following.

For any $X \in$ D				
$P(X)$	$Q(X)$	$Q(X)$ AND $P(X)$	$Q(X)$ OR $P(X)$	$P'(X)$
0	0	0	0	1
0	1	0	1	1
1	0	0	1	0
1	1	1	1	0

Table 1.

If expression X is not true, it is false.

If expression X is not false, it is true.

If an expression is neither false nor true, then the meaning of this expression is absent in Logic B_2.

For all $X, Y \in$ D,

i. If $P(X+Y) = 1$, then $P(X) = 0$ and $P(Y) = 1$ or $P(X) = 1$ and $P(Y) = 0$.

ii. If $P(X+Y) = 0$, then $P(X) = 0$ and $P(Y) = 0$.

Note: Is a lie or imaginary verb true or false? This is quite difficult to distinguish in Logic B_2. Because there are two options there. This expression is neither true nor false.

If there is some $X \in$ D such that $P(X) = 1$ and there exists $Y \in$ D with $X \neq Y$, such that $P(Y) = 0$. This situation is not possible.

Lemma 2.6. Let D a set and P be an expression depending on the variable X. If x_1, x_2 are two different expressions of X in B_2, then $P(x_1) = P(x_2)$.

Corollary 2.7. If $P(x_1) \neq P(x_2)$ such that x_1, x_2 are two different expressions of X in B_2, then X is not a proposition in B_2.

Example 2.8. Let $X : x + 1 = 3$, for $x \in □$, then X is not a proposition. Because,

For all $x \in □$, $P(x) \in 2^{(□,.)}$. But, if $x = 2$, then $P(X) = 1$, if $x \neq 2$, then $P(X) = 0$.

This is explained bellow.

B_2		L or I
1(T)	0(F)	Unknown

Table 2.

Definition 2.9. An expression that is neither true nor false is called a vague (lie, imaginary) statement. A vague expression whose truth or falsity can be calculated by at least one operation called a vague proposition. It is denoted by $P(X)=-1$ in [9].

The expression in Example 1.6 is a vague expression. Because, the truth or falsity of this expression can be calculated as $P(X)=-1$.

3 Logic B_3

This section is started with the values in the uncertainty table. The added value is -1 in ternary logic. This value is corresponded to uncertain, imaginary, lie or false, as in the following table.

B_3		
B_2		
1(T)	0(F)	-1 (L or I)

Table 1.

Definition 3.1. Let $\mathbb{D}$ a set and P be an expression depending on the variable X. If $P(X)$ is a proposition for each $X\in\mathbb{D}$, then P is called a propositional function or a verb with respect to the set $\mathbb{D}$. The set $\mathbb{D}$ is the definition set of the proposition P and the set $P(X)$ is the image set $\{-1,0,1\}$.

The definition of a proposition in Logic B_3 similar to the definition above is given below.

An expression $P:\mathbb{D}\to 3^{(\square,.)}$ is called a proposition in B_3 that satisfies the following conditions

i. For all $X\in\mathbb{D}$, $P(X)\in 3^{(\square,.)}$.

ii. For all $X,Y\in\mathbb{D}$, if $X=Y$, then $P(X)=-1$.

iii. For some $X\in\mathbb{D}$, P(X)=-1

We explain this as follows.

i. For all $X\in\mathbb{D}$, $P(X)\in 3^{(\square,.)}$

ii. $P(X)$ is only one value in $3^{(\square,.)}$ for any $X\in\mathbb{D}$.

iii. If $P=\bigcup_{l=1}^{n}P_l$, $P_i(X)=1$ for $i\neq j$, $P_i\cap P_j=\varnothing$, $i,j\in\{1,2,\ldots,n\}$, $X\in\mathbb{D}$ then $P(X)=P_i(X)=P_j(X)=1$.

For all $X,Y\in\{-1,0,1\}$,

i. If $P(X+Y)=1$, then $P(X)=0$ and $P(Y)=1$ or $P(X)=1$ and $P(Y)=0$.

ii. If $P(X+Y)=0$, then $P(X)=1$ and $P(Y)=-1$ or $P(X)=-1$ and $P(Y)=1$.

iii. If $P(X+Y)=-1$, then $P(X)=0$ and $P(Y)=-1$ or $P(X)=-1$ and $P(Y)=0$.

Let us give the following theorem without proof.

Theorem 3.2([9]). For any $k\in\square^{+}\setminus\{1\}$ the following statements hold.

i. $(2k+1)^{(\square,.)}=3^{(\square,.)}$.

ii. $(2k)^{(\square,.)}\subset 3^{(\square,.)}$.

Definition 3.3. Let P,Q be any two propositions on the set $\mathbb{D}$. The logical operations "." and "+" are defined as follows.

i. $$P(X).Q(X)=\begin{cases}0, if\ P(X)=0 \text{ or } Q(X)=0\\ Q(X),\ if\ P(X)=1\\ -1,\ if\ P(X)=-1, Q(X)=-1\end{cases}$$

ii. $$P(X)+Q(X)=\begin{cases}P(X), if\ P(X)=Q(X)\\ Q(X),\ if\ P(X)=0\\ 1,\ if\ P(X)=1 \text{ or } Q(X)=1\end{cases}$$

Addition and multiplication operations in B_3 are given in the tables below.

.	−1	0	1
−1	−1	0	−1
0	0	0	0
1	−1	0	1

Table 2.

+	−1	0	1
−1	−1	−1	1
0	−1	0	1
1	1	1	1

Table 3.

Property 3.4. Let P,Q be any two propositions for any $X \in \mathsf{D}$. The following hold.

i. $P(X)Q(X)=Q(X)P(X)$.

ii. If $Q(X)=1$, then $P(X)Q(X)=P(X)$.

iii. If $Q(X)=0$, then $P(X)Q(X)=Q(X)$.

iv. $P(X)+Q(X)=Q(X)+P(X)$.

Proof. From the tables the proof are immediate.

Lemma 3.5. For all $X,Y,Z \in \mathsf{D}$,

i. if $X=Y=Z$, then $P(X)=P(Y)=P(Z)$.

ii. If $P(X)=-1$, then there are some X,Y such that $P(X)=1$, $P(Y)=0$ for $X \neq Y$.

Proof. From the tables the proof is clear.

Definition 3.6. Let P be any propositions and any $X \in \mathsf{D}$,

i. if $P(X)=1$, then $P(X)$ is called a tautology.

ii. if $P(X)=0$, then $P(X)$ is called a contradiction.

iii. if $P(X)=-1$, then $P(X)$ is called a liarology or vaguosity (uncertainty science).

Lemma 3.7. Let $P(X),Q(X)$ be any two propositions. Then the following hold.

i. If $P(X)=Q(X)$, then the truth value of $P(X)\Rightarrow Q(X)$ is 1.

ii. If $P(X)\neq Q(X)$, then the truth value of $P(X)\Rightarrow Q(X)$ is equal to truth value of $Q(X)$.

Proof. Let $P(X)$, $Q(X)$ be any two propositions.

i. If $P(X)=Q(X)$, then the result is unchanged as the truth value of $P(X)$ is the same as the truth value of $Q(X)$. In this case, the truth value of $P(X)\Rightarrow Q(X)$ is preserved. That is 1.

iii. If $P(X)\neq Q(X)$, then the truth value of $P(X)\Rightarrow Q(X)$ is determined by $Q(X)$. It is equal to truth value of $Q(X)$.

$P(X)$	$Q(X)$	$P(X) => Q(X)$
-1	-1	1
-1	0	0
-1	1	1
0	-1	-1
0	0	1
0	1	1
1	-1	-1
1	0	0
1	1	1

Table 4.

The following abbreviations for the proposition $P(X)$ and $P'(X)$ used in the table are given below.

$P(X) P'(X)$			$P(X) \vee P'(X)$		$P(X) \wedge P'(X)$	
	$P_1'(X)$	$P_2'(X)$	$P(X) \vee P_1'(X)$	$P(X) \vee P_2'(X)$	$P(X) \wedge P_1'(X)$	$P(X) \wedge P_2'(X)$
-1	0	1	0	1	-1	-1
-1	0	1	0	1	-1	-1
-1	0	1	0	1	-1	-1
0	-1	1	0	1	-1	0
0	-1	1	0	1	-1	0
0	-1	1	0	1	-1	0
1	-1	0	1	1	-1	0
1	-1	0	1	1	-1	0
1	-1	0	1	1	-1	0

Table 5.

Theorem 3.8. Let $P(X)$ be any an proposition and let P_1 and P_2 be defined as above. Then the following hold.

i. $P(X) \wedge P_1'(X)$ is the liarology.

ii. If $P(X) \in 2^{(\square ,\cdot)}$, then $P(X) \wedge P_2'(X)$ is the contradiction.

iii. $P(X) \vee P_2'(X)$ is the tautology.

Proof. The proof of this theorem is clear by Definition 3.6.

4 Discussions

The accuracy values of binary logical values and ternary logical values are compared. Some results in binary logic are realized in ternary logic. The following additional additions are obtained for any $X \in D$,

i. If $P(X) \neq -1, P(X) \neq 0$, then $P(X) = 1$.

ii. If $P(X) \neq -1, P(X) \neq 1$, then $P(X) = 0$.

iii. If $P(X) \neq 0, P(X) \neq 1$, then $P(X) = -1$.

Binary logic is embedded in triple logic. The main conclusion of the study is necessitated the development in binary logic.

5 Results

The values of triple logic are up for debate. A new logical value is presented other than the existing logic values in the binary matrix. This value is -1. This new logical structure is introduced with references. This structure gives rise to many calculations and operations. These are open problems.

Acknowledgement:
It is an optional section where the authors may write a short text on what should be acknowledged regarding their manuscript.

References:

[1] M. Morris Mano, *Digital Design*, Pearson, 2018.

[2] Siobhan Roberts, The Forgotten Father of the Information Age. New York. 2023.

[3] James Ioan , "Claude Elwood Shannon 30 April 1916 – 24 February 2001". *Biographical Memoirs of Fellows of the Royal Society*. 2009, 55: 257–265. doi:10.1098/rsbm.2009.0015.

[4] Horgan John , *Claude Shannon: Tinkerer, Prankster, and Father of Information Theory*, IEEE. Retrieved September 28, 2023.

[5] Harry L. Lewis, An Investigation of the Laws of Thought on Which Are Founded the Mathematical Theories of Logic and Probabilities (1854), *In Ideas that Created the Future: Classic Papers of Computer Science*, MIT Press, 2021, pp.27-44.

[6] Nahin Paul J, *The Logician and the Engineer: How George Boole and Claude Shannon Created the Information Age,* Princeton University Press. 2012, ISBN 978-0691176000. JSTOR j.cttq957s.

[7] Richard Johnsonbaugh, *Discrete Mathematics*, Pearson Publishing, Chicago, 2018.

[8] Naim Çağman, (2024). Fuzzy soft set theory and its applications, *Iranian Journal of Fuzzy Systems,* 8 (3), 20111, 37-147.

[9] Hasan Keleş, *On The Basic Structure of Logic B_3*, Bidge Publications, 4-20, Ankara, 2024.

[10] Hasan Keleş, On Generalized Lattice B2, *Journal of Applied and Pure Mathematics*, vol.5, no.1, 2023, pp.1-8.

[11] Thaneshwar Bhandari, K. B. Manandhar And Kanhaiya Jha, Some Topological Properties on The Cartesian Product of Fuzzy b-Metric Space, *International Journal of Computational and Applied Mathematics & Computer Science*, vol.5, no.1, 2023, pp.1-8. DOI: 10.37394/232028.2024.4.10

[12] Suzana Lampreia, Inês Mestre, Teresa Morgado, Helena Navas, *Dynamic Maintenance based on Fuzzy Logic,* Volume 21, 2024, pp. 2578-2590. DOI: 10.37394/23207.2024.21.211

Contribution of Individual Authors to the Creation of a Scientific Article (Ghostwriting Policy)

The authors equally contributed in the present research, at all stages from the formulation of the problem to the final findings and solution.

Sources of Funding for Research Presented in a Scientific Article or Scientific Article Itself

No funding was received for conducting this study.

Conflict of Interest

The authors have no conflicts of interest to declare that are relevant to the content of this article.