On the stability of the mixing flow dynamical system perturbed with a logistic term

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Abstract: - This paper continues some recent work in a powerful mathematical domain, with applications in connected scientific fields, namely the stability of dynamical systems. In fluid mechanics, stabilizing a dynamical system is a challenging task and it can be done by various ways.

Stabilizing a dynamical system could be often easier if we approach *controllable systems*, because in this form, there can be imposed some bounds on its behavior, by studying the improvement of the operators that describe the system.

In this paper, the mixing flows dynamical systems are taken into account, more exactly the kinematics of mixing flows. The stability analysis of the mixing flow is taken into account, in the case of perturbation with a logistic term. The results can be extended to some other versions of the model.

Key-Words: - Kinematics of mixing; Dynamical systems in control; Lyapunov stability; Lyapunov function; Computational methods; Algebraic methods

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1 Introduction

1.1. General lines

In fluid dynamics, the turbulence is a widespread phenomena; also, it is a basic feature for most systems with few or infinity freedom degrees. It can be defined as chaotic behavior of the systems with few freedom degrees, which are far from the thermodynamic equilibrium. A turbulent flow can generally exhibit all of the following features: [1]

1. random behavior;
2. sensitivity to initial conditions;
3. extremely large range of length and time scales;
4. enhanced diffusion and dissipation;

In this area two important zones are distinguished: [1]

- The theory of transition from laminar smooth motions to irregular motions;
- Characteristic studies of turbulent systems.

Osborne Reynolds' experiments, briefly described in [1] and Reynolds' seminal paper [2] of 1894 are among the most important results produced on the subject of turbulence.

The Reynolds' number was identified as the only physical parameter involved in transition to turbulence in a simple incompressible flow over a smooth surface. Moreover, since turbulence is too complicated to allow a detailed understanding, Reynolds introduced the decomposition of flow variables into mean and fluctuating parts. After that, a lot of studies were produced to obtain some predictable techniques based on his viewpoint.

In hydrodynamics, the transition problem starts at the end of last century, with the pioneering works of Reynolds and Lord Rayleigh. The method of considering the linear stability of basic laminar flow until infinitesimal turbulences was highlighted as a good investigation. Nonlinearity can act in the sense of stabilizing the flow and therefore the primary state is replaced with another stable motion which is considered as secondary flow. This one can be further replaced with a tertiary stable flow, and the process goes on. It is thus obtained a *bifurcations sequence*, and Couette-Taylor flow is a widespread example in this sense [3].

1.2.The kinematics of mixing framework

A flow has the general mathematical formula:

$$x = \Phi_t(X), X = \Phi_{t=0}(X) \quad (1)$$

In the continuum mechanics the relation (1) is named *flow*, and it is a diffeomorphism of class C^k. It must satisfy the following equation [4]:

$$J = \det(D(\Phi_t(X))) = \det\left(\frac{\partial x_i}{\partial X_j}\right) \quad (2)$$

Here D denotes the derivation with respect to the reference configuration, in this case **X**. The equation (2) implies two particles, X_1 and X_2, which occupy the same position **x** at a moment. In the kinematics of mixing, it is taken into account a basic flow (which can be water) containing a biologic material mixed in it. Therefore the basic measure is the *deformation gradient* **F**, with the relation:

$$\mathbf{F} = (\nabla_X \Phi_t(\mathbf{X}))^T,\ F_{ij} = \left(\frac{\partial x_i}{\partial X_j}\right) \quad (3)$$

where ∇_X denotes differentiation with respect to **X**. According to equation (2), **F** is non- singular.
After defining the basic deformation of a material filament and the corresponding relation for the area of an infinitesimal material surface, the deformation measures are defined: the *length deformation* λ and *surface deformation* η, with the following relations [4]:

$$\lambda = (\mathbf{C} : \mathbf{MM})^{1/2},\ \eta = (\det F) \cdot (\mathbf{C}^{-1} : \mathbf{NN})^{1/2} \quad (4)$$

where **C** (=$\mathbf{F}^T \cdot \mathbf{F}$) is the *Cauchy-Green deformation tensor,* and the vectors **M**, **N** are the orientation versors in length and surface respectively.
Very often, in practice is used the scalar form of (4), details can be found in [4].
A central study point in the kinematics of mixing is the deformation efficiency, which can be naturally quantified. In this context, the basic qualitative quantities in the kinematic of mixing are defined; the first one is the *deformation efficiency in length* [4]:

$$e_\lambda = \frac{D(\ln \lambda)/Dt}{(\mathbf{D}:\mathbf{D})^{1/2}} \le 1 \quad (5)$$

Similarly, there is defined the *deformation efficiency in surface*, e_η *:* in the case of an isochoric flow (the jacobian equal 1), the following equation holds:

$$e_\lambda = \frac{D(\ln \lambda)/Dt}{(\mathbf{D}:\mathbf{D})^{1/2}} \le 1 \quad (6)$$

where **D** is the deformation tensor, obtained by decomposing the velocity gradient in its symmetric and non-symmetric part.

The flows with a special form of the deformation gradient **F** have a great interest, because in this class there are contained the *Constant Stretch History Motion* –CSHM flows. [4,5]

2 Computational stability analysis. Recent results

2.1. Computational Lyapunov analysis

When considering a differential equation modelling the evolution of a phenomenon, we must always take into account the fact that the reproduction of the initial conditions is never entirely identical. Therefore is very important to study how *small variations in the initial conditions* will introduce small variations in the phenomenon evolution.

Stability is a well-known property of the solutions of differential equations in R^n of the form $\dot{\boldsymbol{x}} = \boldsymbol{f}(t, \boldsymbol{x})$ by which, given a “reference" solution $x^*(t, t_0^*,\ x_0^*)$, any other solution $x(t, t_0, x_0)$ starting close to $x^*(t, t_0^*,\ x_0^*)$ remains close to $x^*(t, t_0^*,\ x_0^*)$ for long times.

Although the converse theorems provided a great help for this problems, starting with 1950’s, they are not very constructive in practice, since they use the solution trajectory of the system to construct the Lyapunov function, but the solution trajectories are generally not known [6].

The Lyapunov theorem is of great importance in system theory, giving the possibility of establishing stability or asymptotic stability of equilibrium points without explicitly computing trajectories. [6,7].

Theorem 1 (Lyapunov). Let $x_e = 0$ be an equilibrium point for the system (1). Let $V : R^n \to R$ be a positive definite continuously differentiable function.

1. If $\dot{V}: R^n \to R$ is negative semi-definite, then x_e is stable;
2. If $\dot{V}$ is negative definite, then x_e is asymptotically stable.

The theorem gives the existence of a Lyapunov function but does not provide a method to compute one. If for linear systems, this issue arises naturally, in general computing a Lyapunov function is an open problem giving rise to different methods to construct it.

The two basic Lyapunov criteria are very useful for finding a Lyapunov and control Lyapunov function. The first one is based on eigenvalues analysis, and the second one on the monotonicity of the function V [7].

The first Lyapunov criterion is based on the eigenvalues analysis.
Let us consider the following continuous-time nonlinear system:

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}(t), u(t)) \quad (7)$$

In the vicinity of the equilibrium point (x_0, u_0), let us consider the corresponding linearized system:

$$\dot{\tilde{x}}(t) = A\tilde{x}(t) + B\tilde{u}(t) \quad (8)$$

This criterion has three distinct cases for the eigenvalues λ_i of the matrix A [7]:

(i) If $Re\,\lambda_i < 0$ for all i, then $(x_0,\ u_0)$ is asymptotically stable;
(ii) If there exits at least one i such as $Re\,\lambda_i > 0$ then $(x_0,\ u_0)$ is unstable;
(iii) If there exits at least one i such as $Re\,\lambda_i = 0$ and for all other λ_j, $j \neq i$, $Re\,\lambda_j < 0$, then we cannot conclude anything about the stability of $(x_0,\ u_0)$. In this case we say that the criterion is not effective

We concern if Lyapunov functions always exist. How could we find such a function? For the first question the answer is generally positive but, finding a Lyapunov function is not immediate, since the converse theorems assume the knowledge of the solutions of the system (7) [6,7]. Therefore, refining the definition of Lyapunov function and establishing a more specific context for it is very necessary. Between the specific directions of Lyapunov function research, two are very wide used in applications: the control Lyapunov functions and sum of squares (SOS) Lyapunov functions [7,8,9]. We recall briefly in what follows the definition of control Lyapunov functions.

Similar with the system (7), we can define a control system as follows:

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u}) \quad (9)$$

where $\boldsymbol{u} \in U \subset R^m$ is the control. The control is an *open-loop control* if u is function of time, ***u** = **u**(t)* and *closed-loop* if **u = k(x).** The closed-loop control is in fact the *feedback control*. If the feedback has been fixed, **u**=k(**x**) and the equilibrium in the origin has a desired stability property, then we have *a feedback stabilized system* [7,10].
The system (9) is called *locally, asymptotically null-controllable*, [8] if for every ρ in a neighborhood of the origin there is an open-loop control u such that the solution of the system with initial value ρ tends asymptotically towards the origin. In this context, Sontag [10] introduced the *control Lyapunov function* (CLF) as follows:

$$inf_{u \in U} \nabla V(\boldsymbol{x}) \cdot \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u}) \le -\gamma(\|\boldsymbol{x}\|) \quad (10)$$

Where V is a positive definite function and γ is a comparison function [8]. Asymptotic null-controllability cannot be characterized by smooth control Lyapunov functions. Therefore, more general definitions of differentiability like the Dini- or the proximal sub-differential must be taken into account. Details about the refinements of the relation (10) can be found in [10].

2.2 Recent results

Lyapunov functions are not always “energy-like” functions, but they have some similarities with energy-like functions in certain contexts. In the stability theory, Lyapunov functions are used to analyse the stability of an equilibrium point of a dynamical system. Energy functions often have a similar role in physical systems; still Lyapunov functions can be more general.

For a linear system case $\dot{x} = Ax$, finding a Lyapunov function implies finding a matrix P such that $A^T P + PA$ is negative definite [11]. Then the associated Lyapunov function is given by $V(x) = x^T P x$. But although this seems not very complicated, it was seen only recently that, in this context, both $V(x)$, $\dot{V}(x)$ are *sum of squares functions*!

In the case of non-linear systems, there are involved supplementary requirements for the quantities implied in calculus. Therefore related mathematical methods provided recently a useful help. Between them, the convex analysis through the semidefinite-programming provided very useful algorithms.

The sum of squares (SOS) technique has significant impact not only in optimization, but also in convex analysis and especially in control theory. The SOS technique generalizes a computational appliance in control theory, “Linear Matrix Inequalities” – LMI. Parrilo and Ahmadi had important contributions in this field. [9]. With this technique, the models can be easier handled and most solutions can be found numerically.

In recent papers, [12,13,14] it was taken into account the stability of the basic form for the mixing flow dynamical system. When studying the mixing flow phenomena, one starts from the widespread kinematic 2d model [4]:

$$\begin{cases} \dot{x_1} = Gx_2 \\ \dot{x_2} = KGx_1, \end{cases} \quad -1 < K < 1, \;\; G \in R \tag{11}$$

Although this is a linear model, when associating the corresponding initial condition

$$x_1(0)=x_1(t=0)=X_1; x_2(0)=x_2(t=0)=X_2 \tag{12}$$

it is obtained a complex solution for the Cauchy problem (11)-(12) [3,4]. The geometric standpoint is very interesting, the streamlines of the above model satisfy the relation $x_2^2 - K \cdot x_1^2 = const$ and this is corresponding to some ellipses with the axes rate if $\left(\frac{1}{|K|}\right)^{1/2}$ K is negative, and to some hyperbolas with the angle $\beta = arctan\left(\frac{1}{|K|}\right)^{1/2}$ between the extension axes and x_2, if K is positive [4].

To this broad isochoric flow, one can easily associate the corresponding 3d dynamical system [3]:

$$\begin{cases} \dot{x}_1 = G \cdot x_2 \\ \dot{x}_2 = K \cdot G \cdot x_1 \quad ,-1 \langle K \langle 1, c = const. \\ \dot{x}_3 = c \end{cases} \tag{13}$$

with the third component standing for the movement velocity of the system.

In the 3d case the nonlinear system has a complicate behaviour, the influence of the parameters leading to a far from equilibrium model. The perturbed model was taken into account and it was found again a strong sensitivity with respect to parameters [3].

In 2d case, some perturbations of the mixing flow dynamical system were taken into account. In a first stage, the feedback linearization [7] of the model in a slightly perturbed form was performed, namely for the system:

$$\begin{cases} \dot{x_1} = Gx_2 + x_1 \\ \dot{x_2} = KGx_1 - x_2 \end{cases} \quad -1 < K < 1, \;\; G \in R \tag{14}$$

and an interesting conclusion was found, namely it was found a different parameters' distribution for the feedback linearized form of the model. Also, a SOS Lyapunov function was found *both* for the initial and for the feedback linearized model [12].

In a next stage, a control Lyapunov function was found for the model (14) in a controlled form [14]. It is important to notice that this can be realized in feasible conditions for the parameters.

Further, in [13] it was realized a good comparison analysis concerning the existence of a Lyapunov function, for the initial form versus the feedback linearized form, for the mixing flow model perturbed with a logistic type term.

3 Results for the mixing flow dynamical system perturbed with a logistic type term

The mixing flow dynamical system in 3d case is a nonlinear model. If we start to perturb it, then we get a strongly nonlinear model. In [3] it was analysed the solution of the model in the non-perturbed case and with a perturbation with a logistic type term. So the following model was taken into account:

$$\begin{cases} \dot{x}_1 = G \cdot x_2 \\ \dot{x}_2 = K \cdot G \cdot x_1 + G \cdot (x_2 - x_1) \\ \dot{x}_3 = c(const) \end{cases} \tag{15}$$

For this model a complex solution was found, with the expression:

$$\begin{cases} x_1 = \left(\frac{X_1}{2} + \frac{2 \cdot X_2 - X_1}{2 \cdot \sqrt{4K-3}}\right) \cdot \exp(G \cdot K_1 \cdot t) + \\ \left(\frac{X_1}{2} - \frac{2 \cdot X_2 - X_1}{2 \cdot \sqrt{4K-3}}\right) \cdot \exp(G \cdot K_2 \cdot t) \\ x_2 = K_1 \cdot \left(\frac{X_1}{2} + \frac{2 \cdot X_2 - X_1}{2 \cdot \sqrt{4K-3}}\right) \cdot \exp((G \cdot K_1 \cdot t)) + \\ K_2 \cdot \left(\frac{X_1}{2} - \frac{2 \cdot X_2 - X_1}{2 \cdot \sqrt{4K-3}}\right) \cdot \exp(G \cdot K_2 \cdot t) \\ x_3 = c \cdot t + X_3 \end{cases} \tag{16}$$

In (16) the following notations were made in order to simplify the expressions:

$$K_1 = \frac{1+\sqrt{4K-3}}{2}, K_2 = \frac{1-\sqrt{4K-3}}{2}$$

Taking into account the stability analysis for a model like (15) is easy to observe that this is a challenging task. Therefore, starting with the 2d case would bring useful information.

For the aim of the present paper, the 2d case is taken into account for the mixing flow perturbed with a logistic type term:

$$\begin{cases} \dot{x_1} = Gx_2 \\ \dot{x_2} = KGx_1 + G(x_2 - x_1), \end{cases} \quad -1 < K < 1, \; G \in R \tag{17}$$

As it is about a nonlinear model, finding a Lyapunov function (in the form CLF or SOS) is quite difficult, since it is difficult to fulfill the criteria. Therefore, the stability analysis is approached according to the eigenvalues criterion.
In order to have the system in the form (7), we consider it formally in a "controlled" form, adding a control on the first component:

$$\begin{cases} \dot{x_1} = Gx_2 + px_2 \\ \dot{x_2} = KGx_1 + G(x_2 - x_1), \end{cases} \quad -1 < K < 1, \; G \in R, p \in R \tag{18}$$

It is obvious that the origin is solution for (18).
The matrix associated to linearized system around the origin is:

$$A = \begin{pmatrix} 0 & G + p \\ KG - G & G \end{pmatrix} \tag{19}$$

Consequently, the characteristic equation is

$$\lambda^2 - G\lambda - G(K-1)(G+p) = 0 \tag{20}$$

According to the eigenvalues criterion for stability, we impose for the discriminant of (20) the condition $\Delta_p < 0$ which implies, after the calculus, the condition

$$\Delta_p = (4K - 3 - 4p)G^2 + 4KpG < 0 \tag{21}$$

This is equivalent to

$$G[(4K - 3 - 4p)G + 4Kp] < 0 \tag{22}$$

From the above inequality, taking into account that $G \in R, -1 < K < 1,$ we have two possibilities:

$$i)\; G < 0, \quad F(G, K, p) > 0 \tag{23}$$

$$ii)\; G > 0, \quad F(G, K, p) < 0$$

$$F(G, K, p) = (4K - 3 - 4p)G + 4Kp$$

It is easy to evaluate each of the situations in (23), starting from each hypothesis for G and then evaluate G itself from the second inequality.
After all calculus, taking into account the basic conditions for G and K, we obtain the situations:

$$i)\; 0 < K < 1, \quad p < K - \frac{3}{4} \tag{24}$$

$$ii)\; 0 < K < 1, \quad p < \frac{3}{4} - K$$

So from the inequalities (24), we deduce that the control p depends on the parameter K.
The above relations are feasible from the parameters standpoint, therefore we can assess that in the form (18) of the mixing flow model, it is feasible to apply the eigenvalues criterion.
Going further and calculating the eigenvalues, we find:

$$\lambda_1 = \frac{G}{2} - i\frac{\sqrt{-\Delta_p}}{2} \tag{25}$$

$$\lambda_2 = \frac{G}{2} + i\frac{\sqrt{-\Delta_p}}{2}$$

Thus, the eigenvalues criterion is realized, namely in the conditions (24) for the parameters, we have the situations:

a) If G<0 then the origin is asymptotically stable;
b) If G>0 then the origin is asymptotically unstable

Conclusions

The mixing flow dynamical system perturbed with a logistic type term provides a strongly nonlinear model. In [13] it was realized a Lienard type construction of the model and based on it, a Lyapunov function was found both for the initial and the feedback linearized model.

In the present paper the stability analysis is approached for the mixing flow model perturbed with a logistic term. Namely, it is found that in a controlled form and some feasible conditions for the parameters, the first stability criterion is feasible and some simple feasible conditions for the stability of the origin are found.

Thus, although a nonlinear model, the dynamical system modeling the kinematics of mixing can reach the stability. This enables us to take into account the construction of a Lyapunov function, in a suitable form for the controlled model. Also, further versions of the model and some 3d versions of vortex models will be taken into account in order to complete the panel of events for the mixing flow theory.

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