The Pricing Problem of Rainbow Option in Uncertain Financial Market

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Abstract: - In this paper we mainly investigate pricing problems of rainbow option under uncertain financial market. The price of the underlying asset is assumed to obey an uncertain process. Uncertain differential equations are used to build a price model. Furthermore, the differential equations under the uncertain mean-reverting model are solved to deduce the pricing formulas of several rainbow options. Additionally, in order to verify the reasonableness of our pricing formulas, some numerical experiments are designed to show the prices of these options.

Key-Words: - Uncertainty theory; Option pricing; Rainbow option; Uncertain differential equation.

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1 Introduction

An option is a contract in which the rights and obligations of both parties are not equal, and the buyer pays an option fee in exchange for the right to buy or sell an asset at an agreed price on or before the expiration date. Although options have been around since the late 18th century, they are not widely used due to the pricing problem of option fees. Until 1973, the famous Black-Scholes formula on the basis of stochastic differential equations proposed by Black and Scholes [2] made great progress in option pricing theory, and option prices were expressed by various differential equations. After more in-depth research and further expansion of the B-S formula, it has been applied to many other financial derivatives pricing models.

With the development of option pricing theory, options have become the most dynamic derivative financial products, which have been rapidly developed and widely used, and the types of options have also increased very quickly. Among them, options involving two or more risky assets are often referred to as rainbow options. In 2015, the price of two path-dependent derivatives was determined using the disturbance theory in a two-dimensional asset model with random correlation and volatility by Marcos et al. [6]. Wang et al. studied the pricing problem of fragile European options under Markov modulation jump diffusion process in 2017 [21]. In 2020, Edeki et al. use the separated variable transformation method (HSVTM) to give a exact (closed form) solution of the classical Black Scholes option pricing model with time fraction [7]. Later in 2021, Aimi and Guardasoni [1], relying on the collocated Boundary Element method, extended a semi-analytical approach to the barrier option technique to barrier option pricing with earnings dependent on multiple assets. Under the Merton jump-diffusion model, Ghosh and Mishra [9] studied the fast, parallel, and numerically accurate pricing of two-asset American options in 2022.

The traditional option pricing theory have often started from the perspective of probability theory, regarding the underlying asset price as a Wiener process, and construct price models based on this, then conducts further derivation. However, the traditional option pricing theory requires the sample size is large enough to find a distribution function that is close enough to the frequency, but in many cases, for various reasons, we often cannot obtain enough samples or even no sample to find the available probability distributions. In these cases, we need to rely on the expert's belief degree in each uncertain event.

In 2007, for describing the belief degree, Liu [10] proposed an uncertainty theory, which is based on the axiomatic system of regularity, duality, subadditivity and product measure, to solve the mathematical problem that cannot be solved by probability theory due to insufficient sample data or the absence of sample, and further refined the theory in 2010 [12]. At the same year, Chen and Liu [4] studied the existence and uniqueness theorem for uncertain differential equations. In 2013, Liu [14] utilized some paradoxes to prove that the actual price of a stock does not follow any of Ito's stochastic differential equations, which overthrew the view of traditional stochastic financial theory. Therefore, it is reasonable to express asset prices with uncertain differential equations.

Liu [11] first introduced uncertainty theory into the stock model in 2009, he used the geometric Liu process to construct uncertain stock models and derived the pricing formula for European call options. Furthermore, based on uncertain differential equations, Zhu [25] studied the application of uncertain optimal control problem in portfolio selection models. After that, Chen [3] deduced the pricing formula for American options in 2011, Sun and Chen [17] derived the pricing formula for Asian options in 2015. In addition, in 2011, Peng and Yao [16] considered an option pricing model with mean reversion process and derived the pricing formulas for European and American options under the model. Chen et al. [5] proposed an uncertain stock model with periodic dividends. Additionally, Yao [23] proposed an uncertain floating-rate stock model, in which both stock prices and interest rates follow uncertain differential equations. Sun and Su [18] proposed a mean-reverting stock model under floating interest rates. Yang et al. [22] derived a pricing formula for Asian barrier options in 2019. In the same year, Gao et al. [22]deduced a pricing formula of American barrier options. Lu et al. [15] proposed an uncertain stock model based on fractional differential equations, and discussed the pricing of European-style options under this model. Tian et al. [19] determined a barrier option pricing problem under the mean-reverting stock model. In 2021, Wang and Ralescu[20] studied pricing formulas of lookback option for the uncertain Heston volatility model. Gao [8] studied the pricing problem of Asian rainbow options based on uncertain stock models.

In this paper, by leveraging knowledge of uncertainty theory, the price of the underlying asset is treated as an uncertain process, based on uncertain differential equations, while taking into account the mean reversion characteristics of asset prices. After that, the pricing formulas of several types of rainbow options are deduced. This paper is organized as follows: The second part of this paper briefly states some theorems and definitions related to this paper. The third part derives the pricing of put 2 and call 1 type rainbow options. The fourth section studies rainbow call on max option and rainbow call on min option. The fifth part deduces the pricing formula of rainbow put options, including rainbow put on max option and rainbow put on min option. Section sixth of this paper gives some brief conclusions.

2 Preliminaries

Definition 1 *(Liu [11] [13]) Providing that $(\Gamma, \mathcal{L})$ is a measurable space. A set function $\mathcal{M} : \mathcal{L} \to [0, 1]$ is called an uncertain measure if it satisfies the following conditions: (i) (normality axiom) $\mathcal{M}\{\Gamma\} = 1$ for the universal set Γ; (ii) (duality axiom) $\mathcal{M}\{\Lambda\} + \mathcal{M}\{\Lambda^c\} = 1$ for any event Λ; (iii) (subadditivity axiom) $\mathcal{M}\left\{\bigcup_{i=1}^{\infty} \Lambda_i\right\} \leq \sum_{i=1}^{\infty} \mathcal{M}\{\Lambda_i\}$ for every countable sequence of events $\Lambda_1, \Lambda_2, \cdots$.*

Definition 2 *(Liu [11]) The uncertain measure on the product σ-algebra $\mathcal{L}$ is called product uncertain measure defined by the following product axiom: (Product Axiom) Let $(\Gamma_k, \mathcal{L}_k, \mathcal{M}_k)\,(k = 1, 2, \cdots)$ represent uncertainty spaces. The product uncertain measure $\mathcal{M}$ is an uncertain measure satisfying*

$$\mathcal{M}\left\{\prod_{k=1}^{\infty} \Lambda_k\right\} = \bigwedge_{k=1}^{\infty} \mathcal{M}_k\left\{\Lambda_k\right\}.$$

Definition 3 *(Liu [10]) An uncertain variable is a function ξ from an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to the set of real numbers such that $\{\xi \in B\}$ is an event for any Borel set B of real numbers.*

Definition 4 *(Liu [10]) The uncertainty distribution Φ of an uncertain variable ξ is defined by*

$$\Phi(x) = \mathcal{M}\{\xi \leq x\}$$

for any real number x.

Theorem 1 *(Liu [12]) For any events Λ_1 and Λ_2 with $\Lambda_1 \subset \Lambda_2$, we have*

$$\mathcal{M}\left\{\Lambda_1\right\} \leq \mathcal{M}\left\{\Lambda_2\right\}.$$

Theorem 2 *(Liu [14]) A function $\Phi^{-1} : (0, 1) \to \Re$ is the inverse uncertainty distribution of an uncertain variable ξ if and only if it is continuous and*

$$\mathcal{M}\left\{\xi \leq \Phi^{-1}(\alpha)\right\} = \alpha$$

for all $\alpha \in (0, 1)$.

Definition 5 *(Liu [10]) Let ξ be an uncertain variable. Then the expected value of ξ is defined by*

$$E[\xi] = \int_0^{+\infty} \mathcal{M}\{\xi \geq x\}dx - \int_{-\infty}^{0} \mathcal{M}\{\xi \leq x\}dx$$

Theorem 3 *(Liu [10]) Assume that there is uncertainty distribution ϕ for an uncertain variable ξ. If $E[\xi]$ exists,then*

$$E[\xi] = \int_{-\infty}^{+\infty} x d\phi(x).$$

Furthermore if ϕ is regular, then we also have

$$E[\xi] = \int_0^1 \phi^{-1}(\alpha)d\alpha.$$

Theorem 4 *(Yao and Chen [24]) Suppose that there are the solution X_t and α-path X_t^α for an uncertain differential equation*

$$dX_t = f(t, X_t)dt + g(t, X_t)dC_t. \tag{1}$$

Then, the time integral $\int_0^s Y(X_t)dt$ *possesses an inverse uncertainty distribution*

$$\psi_s^{-1}(\alpha) = \int_0^s Y(X_t^\alpha)dt, s > 0$$

where $Y(x)$ *is a function owning strictly increasing feature.*

Theorem 5 *(Liu [12]) Let* $\xi_1, \xi_2, \cdots, \xi_n$ *be independent uncertain variables with regular uncertainty distributions* $\phi_1, \phi_2, \cdots, \phi_n$*, respectively. If* $f(\xi_1, \xi_2, \cdots, \xi_n)$ *is strictly increasing with respect to* $\xi_1, \xi_2, \cdots, \xi_m$ *and strictly decreasing with respect to* $\xi_{m+1}, \xi_{m+2}, \cdots, \xi_n$*, then* $f(\xi_1, \xi_2, \cdots, \xi_n)$ *has an inverse uncertainty distribution*

$$\psi_s^{-1}(\alpha) = f(\phi_1^{-1}(\alpha), \cdots, \phi_m^{-1}(\alpha), \phi_{m+1}^{-1}(1-\alpha), \cdots, \phi_n^{-1}(1-\alpha)).$$

Definition 6 *(Yao and Chen [24]) Let* α *be a number between* 0 *and* 1*. An uncertain differential equation*

$$dX_t = f(t, X_t)\, dt + g(t, X_t)\, dC_t$$

is said to have an α*-path* X_t^α *if it solves the corresponding ordinary differential equation*

$$dX_t^\alpha = f(t, X_t^\alpha)\, dt + |g(t, X_t^\alpha)|\, \Phi^{-1}(\alpha)dt$$

where $\Phi^{-1}(\alpha)$ *is the inverse standard normal uncertainty distribution, i.e.,*

$$\Phi^{-1}(\alpha) = \frac{\sqrt{3}}{\pi} \ln \frac{\alpha}{1-\alpha}.$$

Theorem 6 *(Yao and Chen [24]) Let* X_t *and* X_t^α *be the solution and* α*-path of the uncertain differential equation*

$$dX_t = f(t, X_t)\, dt + g(t, X_t)\, dC_t,$$

respectively. Then

$$\mathcal{M}\{X_t \le X_t^\alpha, \forall t\} = \alpha,$$
$$\mathcal{M}\{X_t > X_t^\alpha, \forall t\} = 1 - \alpha.$$

where X_t *possesses an inverse uncertainty distribution*

$$\Psi_t^{-1}(\alpha) = X_t^\alpha$$

3 Put 2 and Call 1 option

A put 2 and call 1 option(PCO) include two assets, among which the expected price of asset 1 rises and the expected price of asset 2 falls, the owner of the P-CO have rights to replace the bearish Asset 2 with the bullish Asset 1 on the expiration date. It means that, upon expiration, if the two assets fall and rise as expected, and their difference is greater than the option fee, the greater the difference. The higher the return of the holder. The yield of the option depends on the the difference between the two assets at maturity. We will get the pricing formula of PCO through rigorous derivation in this section. In addition, we will obtain the option price through a numerical algorithm.

We assume that the underlying asset prices of option with a maturity time T obey different uncertain differential equations. At the same time, considering that theoretically the stock price cannot always rise or fall, its mean-reversion characteristic is inevitable. We propose the following model:

$$\begin{cases} dZ_t = rZ_t dt \\ dS_t = u_1(m_1 - a_1 S_t)\, dt + \sigma_1 S_t dC_{1t} \\ dV_t = u_2(m_2 - a_2 V_t)\, dt + \sigma_2 V_t dC_{2t} \end{cases} \tag{2}$$

where Z_t on behalf of the bond price, S_t and V_t represent, respectively, the price of Asset 1 and Asset 2, C_{1t} and C_{2t} are independent Liu processes, σ_1 and u_1 are respectively, the log-diffusion and log-drift of S_t, σ_2 and u_2 are respectively, the log-diffusion and log-drift of V_t. Moreover, r is the riskless interest rate and m_i/a_i represents the mean reversion speed.

Because C_{1t} and C_{2t} are independent Liu process, so S_t and V_t are independent of each other. From Definition 6 we know that the α-path S_t^α and V_t^α of S_t and V_t, respectively, satisfy

$$dS_t^\alpha = u_1(m_1 - a_1 S_t^\alpha)\, dt + |\sigma_1 S_t^\alpha| \Phi^{-1}(\alpha)dt, \tag{3}$$

and

$$dV_t^\alpha = u_2(m_2 - a_2 V_t^\alpha)\, dt + |\sigma_2 V_t^\alpha| \Phi^{-1}(\alpha)dt. \tag{4}$$

The solutions of (3) and (4), respectively, are

$$S_t^\alpha = S_0 \frac{u_1 m_1 - \exp\left(-u_1 a_1 + \Phi^{-1}(\alpha)\sigma_1\right) t}{u_1 a_1 - \Phi^{-1}(\alpha)\sigma_1}, \tag{5}$$

and

$$V_t^\alpha = V_0 \frac{u_2 m_2 - \exp\left(-u_2 a_2 + \Phi^{-1}(\alpha)\sigma_2\right) t}{u_2 a_2 - \Phi^{-1}(\alpha)\sigma_2}. \tag{6}$$

From Theorem 6, we know S_t^α and V_t^α are also inverse uncertain distributions of S_t and V_t, respectively.

Firstly, from the holder's point of view, the payoff at expiration date T is

$$(S_T - V_T)^+.$$

Suppose the option fee the holder paid at time 0 is f_{pc}. Then the net return of the holder at the initial moment is

$$-f_{pc} + \exp(-rT)\left(S_T - V_T\right)^+.$$

Secondly, assuming bank is the seller of the option, then the payoff of bank at time T is

$$-(S_T - V_T)^+.$$

Thus at the initial moment, the bank owns net return

$$f_{pc} - \exp(-rT)\,(S_T - V_T)^+ .$$

For fairness, the expected returns of buyers and sellers should be equal, so we have

$$\begin{aligned} & -f_{pc} + \exp(-rT)E\left[(S_T - V_T)^+\right] \\ = \ & f_{pc} - \exp(-rT)E\left[(S_T - V_T)^+\right]. \end{aligned} \tag{7}$$

Obviously, the PCO price is

$$f_{pc} = \exp(-rT)E\left[(S_T - V_T)^+\right].$$

Theorem 7 *Suppose a PCO option for Model (2) has a maturity time T. Then, the price of option is*

$$\begin{aligned} f_{pc} = \exp(-rT)\int_0^1 & (S_0 h_1(\alpha) \\ & -V_0 h_2(1-\alpha))\,d\alpha, \end{aligned} \tag{8}$$

where for $i = 1, 2$,

$$h_i(\alpha) = \frac{u_i m_i - \exp\left(-u_i a_i + \Phi^{-1}(\alpha)\sigma_i\right)T}{u_i a_i - \Phi^{-1}(\alpha)\sigma_i}. \tag{9}$$

Proof. First of all, we can get S_T^α and $-V_T^{1-\alpha}$, which are, respectively, inverse uncertain distribution of S_T and $-V_T$. Because S_T and $-V_T$ are independent of each other, so the uncertain variable

$$(S_T - V_T)^+$$

has an inverse uncertain distribution

$$\left(S_T^\alpha - V_T^{1-\alpha}\right)^+ \tag{10}$$

by Theorem 5, where

$$S_T^\alpha = S_0 \frac{u_1 m_1 - \exp\left(-u_1 a_1 + \Phi^{-1}(\alpha)\sigma_1\right)T}{u_1 a_1 - \Phi^{-1}(\alpha)\sigma_1},$$

$$V_T^{1-\alpha} = V_0 \frac{u_2 m_2 - \exp\left(-u_2 a_2 + \Phi^{-1}(1-\alpha)\sigma_2\right)T}{u_2 a_2 - \Phi^{-1}(1-\alpha)\sigma_2}.$$

Finally, we can obtain the pricing formula of PCO

$$\begin{aligned} f_{pc} &= \exp(-rT)E\left[(S_T - V_T)^+\right] \\ &= \exp(-rT)\int_0^1 \left(S_T^\alpha - V_T^{1-\alpha}\right)^+ d\alpha \end{aligned} \tag{11}$$

by Theorem 3. Instituting the expressions of S_T^α and $V_T^{1-\alpha}$ into (11) gets the conclusion (8). The proof is completed.

To calculate the option price, according to the definition of integral, we divide the integrating interval into 100 sub intervals to calculate the integral sum for approximating the integral. On this basis, we design the following algorithm:

Algorithm 1 (Option price for model (2))

Step 1. Input the values of parameters: S_0, V_0, m_1, m_2, a_1, a_2, σ_1, σ_2, u_1, u_2 and T.
Step 2. Let α start at $\alpha_0 = 0.01$ and grow to 0.99 at a step of 0.01 to obtain

$$\alpha_j = \alpha_{j-1} + 0.01,\ j = 1, 2, \cdots, 99.$$

Step 3. Calculate $h_i(\alpha_j)$ by (9) for $i = 1, 2$, $j = 1, 2, \cdots, 99$.
Step 4. Calculate

$$\begin{aligned} f_{pc} = \exp(-rT) \times 0.01 \times \sum_{j=1}^{99} & (S_0 h_1(\alpha_j) \\ & -V_0 h_2(1-\alpha_j)). \end{aligned}$$

Example 1 Set $S_0 = 5$, $V_0 = 4$, $m_1 = m_2 = 4$, $a_1 = a_2 = 1$, $\sigma_1 = 0.01$, $\sigma_2 = 0.01$, $u_1 = 0.06$, $u_2 = -0.04$, and $T = 1$. Then the Put 2 and Call 1 option price for model (2) is obtained to be $f_{pc} = 9.6367$ by Algorithm 1.

4 Rainbow call on option

In this section we will study rainbow call on options (RCO), which include rainbow call on max options (RCMAO) and rainbow call on min options (RCMIO). By solving the asset-price model and further derivation, we obtain the pricing formulas of RCO. Moreover, some numerical experiments are designed to calculate the option prices.

We assume that the underlying asset prices of option with a maturity time T obey different uncertain differential equations. At the same time, considering that theoretically the stock price cannot always rise or fall, the following model is proposed:

$$\begin{cases} dZ_t = rZ_t dt \\ dS_{1t} = u_1\left(m_1 - a_1 S_{1t}\right)dt + \sigma_1 S_{1t} dC_{1t} \\ dS_{2t} = u_2\left(m_2 - a_2 S_{2t}\right)dt + \sigma_2 S_{2t} dC_{2t} \\ \cdots \\ dS_{nt} = u_n\left(m_n - a_n S_{nt}\right)dt + \sigma_n S_{nt} dC_{nt} \end{cases} \tag{12}$$

where Z_t is a bond price, r is a riskless interest rate, S_{it} represents the price of Asset i, C_{it} represents independent Liu processes, u_i and σ_i are, respectively, the log-drift and log-diffusion of S_{it} $(i = 1, 2, \ldots, n)$, m_i/a_i represents the mean reversion speed.

The changes of S_{it} are independent of each other because C_{it} $(i = 1, 2, \ldots, n)$ are independent Liu process. The α-pathes of systems in Model (12) are

$$\begin{cases} Z_t^\alpha = Z_0 \exp(rt) \\ S_{it}^\alpha = S_{i0} h_i(\alpha) \end{cases} \tag{13}$$

where

$$h_i(\alpha) = \frac{u_i m_i - \exp\left(-u_i a_i + \Phi^{-1}(\alpha)\sigma_i\right) t}{\mu_i a_i - \Phi^{-1}(\alpha)\sigma_i} \tag{14}$$

for $i = 1, 2, \cdots, n$.

We assume $\Psi_{1t}^{-1}(\alpha)$, $\Psi_{2t}^{-1}(\alpha)$, $\cdots$, $\Psi_{nt}^{-1}(\alpha)$ are, respectively, inverse uncertain distributions of S_{1t}, S_{2t}, $\cdots$, S_{nt}. Then we have

$$\Psi_{it}^{-1}(\alpha) = S_{it}^\alpha = S_{i0} h_i(\alpha) \tag{15}$$

by Theorem 6 for $i = 1, 2, \cdots, n$.

4.1 Rainbow call on max option

The holder of the RCMAO can buy the highest priced asset contained in the option at the strike price K at time T. It means at time T, when the price of the highest priced asset is greater than K, the higher price of the highest priced asset in the rainbow option, the higher the yield for the option holder. The payoff of the option depends on the price of the highest priced asset in the rainbow option on the expiration date.

Firstly, assuming we are the holder of the option, then our payoff at expiration date is

$$\left(\max_{1\le i\le n} S_{iT} - K\right)^+.$$

Suppose the option fee we paid at time 0 is f_{1c}. The net return we own at time 0 is

$$-f_{1c} + \exp(-rT)\left(\max_{1\le i\le n} S_{iT} - K\right)^+.$$

Secondly, assuming bank is the seller of the option, then the payoff of bank at time T is

$$-\left(\max_{1\le i\le n} S_{iT} - K\right)^+.$$

At time 0, the bank charged an option fee, hence the bank's net return is

$$f_{1c} - \exp(-rT)\left(\max_{1\le i\le n} S_{iT} - K\right)^+.$$

From the consideration of fairness, the expected returns of buyers and sellers should be equal, it means

$$\begin{aligned} &-f_{1c} + \exp(-rT)\left(\max_{1\le i\le n} S_{iT} - K\right)^+ \\ &= f_{1c} - \exp(-rT)\left(\max_{1\le i\le n} S_{iT} - K\right)^+. \end{aligned}$$

Thus we can conclude that the price of RCMAO is

$$f_{1c} = \exp(-rT) E\left[\left(\max_{1\le i\le n} S_{iT} - K\right)^+\right]$$

Theorem 8 *Suppose that a RCMAO for Model (12) has a maturity time T and an exercise price K. Then, its option price is*

$$f_{1c} = \exp(-rT)\int_0^1 \left(\max_{1\le i\le n} S_{i0} h_i(\alpha) - K\right)^+ d\alpha$$

where $h_i(\alpha)$ is shown as (14) for $i = 1, 2, \cdots, n$.

Proof. By Theorems 5 and 6, the uncertain variable

$$\left(\max_{1\le i\le n} S_{iT} - K\right)^+$$

has an inverse uncertainty distribution

$$\left(\max_{1\le i\le n} S_{iT}^\alpha - K\right)^+.$$

Finally, we can obtain the pricing formula of RCMAO is

$$\begin{aligned} f_{1c} &= \exp(-rT) E\left[\left(\max_{1\le i\le n} S_{iT} - K\right)^+\right] \\ &= \exp(-rT)\int_0^1 \left(\max_{1\le i\le n} S_{iT}^\alpha - K\right)^+ d\alpha \\ &= \exp(-rT)\int_0^1 \left(\max_{1\le i\le n} S_{i0} h_i(\alpha) - K\right)^+ d\alpha. \end{aligned}$$

The proof is completed.

4.2 Rainbow call on min option

The buyer of RCMIO pays an option premium in exchange for the right to buy the lowest priced asset in the rainbow option at the strike price K on the expiry date T. Therefore the buyer's benefit depends on the lowest price of each asset in the rainbow option on the expiration date.

Firstly, assuming we are the holder of the option, then our payoff at expiration date T is

$$\left(\min_{1\le i\le n} S_{iT} - K\right)^+.$$

Suppose the option fee we paid at time 0 is f_{2c}. Then our net return at time 0 is

$$-f_{2c} + \exp(-rT)\left(\min_{1\le i\le n} S_{iT} - K\right)^+.$$

Secondly, assuming bank is the seller of the option, then the bank's payoff at time T is

$$-\left(\min_{1\le i\le n} S_{iT} - K\right)^+.$$

At time 0, the bank charged an option fee, hence the bank's net return is

$$f_{2c} - \exp(-rT)\left(\min_{1\le i\le n} S_{iT} - K\right)^+.$$

From the consideration of fairness, the expected returns of buyers and sellers should be equal, it means

$$\begin{aligned}&-f_{2c} + \exp(-rT)\left(\min_{1\le i\le n} S_{iT} - K\right)^+\\ &= f_{1c} - \exp(-rT)\left(\min_{1\le i\le n} S_{iT} - K\right)^+.\end{aligned}$$

Thus we can conclude that the price of RCMIO is

$$f_{2c} = \exp(-rT)E\left[\left(\min_{1\le i\le n} S_{iT} - K\right)^+\right].$$

Theorem 9 *Suppose that a RCMIO for Model (12) has a maturity time T and an exercise price K. Then, its price is*

$$f_{1c} = \exp(-rT)\int_0^1 \left(\min_{1\le i\le n} S_{i0}h_i(\alpha) - K\right)^+ d\alpha$$

where $h_i(\alpha)$ is shown as (14) for $i = 1, 2, \cdots, n$.

Proof. The proof is similar to that of Theorem 8.

To calculate the option price, we design the following algorithm according to the definition of definite integral:

Algorithm 2 (Option price for model (12))

Step 1. Input the values of parameters: S_{10}, S_{20}, …, S_{n0}, m_1, m_2, …, m_n, a_1, a_2, …, a_n, T, σ_1, σ_2, …, σ_n, u_1, u_2, …, u_n and K.

Step 2. Let α start at $\alpha_0 = 0.01$ and grow to 0.99 at a step of 0.01 to obtain

$$\alpha_j = \alpha_{j-1} + 0.01,\ j = 1, 2, \cdots, 99.$$

Step 3. Calculate $h_i(\alpha_j)$ by (14) for $i = 1, 2, \cdots, n$, $j = 1, 2, \cdots, 99$.

Step 4. Calculate

$$\begin{aligned}f_{1c} &= \exp(-rT)\times 0.01\\ &\quad\times\sum_{j=1}^{99}\left(\max_{1\le i\le n} S_{i0}h_i(\alpha_j) - K\right)^+\end{aligned}$$

and

$$\begin{aligned}f_{2c} &= \exp(-rT)\times 0.01\\ &\quad\times\sum_{j=1}^{99}\left(\min_{1\le i\le n} S_{i0}h_i(\alpha_j) - K\right)^+\end{aligned}$$

Example 2 Set $n = 5$, $S_{10} = 5$, $S_{20} = 4$, $S_{30} = 3$, $S_{40} = 2$, $S_{50} = 1$, $m_1 = m_2 = m_3 = 1$, $m_4 = m_5 = 2$, $a_1 = a_2 = a_3 = 0.1$, $a_4 = a_5 = 0.5$, $\sigma_1 = \sigma_2 = ... = \sigma_5 = 0.5$, $u_1 = 0.05$, $u_2 = 0.04$, $u_3 = 0.03$, $u_4 = 0.02$, $u_5 = 0.01$, and $T = 1$, $K = 10$. Then the RCMAO price is obtained to be $f_{1c} = 32.2840$ and the RCMIO price is $f_{2c} = 2.5957$ by Algorithm 2.

5 Rainbow put on option

In this section, we will reveal the pricing formula of rainbow put on option (RPC), which include rainbow put on max option (RPMAO) and rainbow put on min option (RPMIO). After further derivation, the pricing formula of options is obtained. As in the previous section, we will also use some numerical experiments to verify the rationality of the formula.

5.1 Rainbow put on max option

The holder of RPMAO has the right to sell the highest-priced asset contained in the option at the strike price K on time T. It means on the expiration date, when the strike price K is higher than the price of the highest priced asset in RPMAO, the lower price of the highest priced asset in the option, the higher the yield for the option holder. So the return of the option depends on the highest price among all asset in the RPMAO on the expiration date.

Assume that RPMAO possesses a maturity time T and an exercise price K for Model (12).

Firstly, assuming we are the holder of the option, then our payoff at expiration date T is

$$\left(K - \max_{1\le i\le n} S_{iT}\right)^+.$$

Suppose the option fee we paid at time 0 is f_{3c}, then at time 0, the net return we have is

$$-f_{3c} + \exp(-rT)\left(K - \max_{1\le i\le n} S_{iT}\right)^+.$$

Secondly, form the perspective of seller, the payoff of the option seller on expiration date T is

$$-\left(K - \max_{1\le i\le n} S_{iT}\right)^+.$$

At the initial moment, the seller of option charged an option fee, hence the seller's net return is

$$f_{3c} - \exp(-rT)\left(K - \max_{1\le i\le n} S_{iT}\right)^+.$$

From the consideration of fairness, the expected returns of buyers and sellers should be equal. It means

$$\begin{aligned} &-f_{3c} + \exp(-rT)\left(K - \max_{1\le i< n} S_{iT}\right)^+ \\ &= f_{3c} - \exp(-rT)\left(K - \max_{1\le i\le n} S_{iT}\right)^+. \end{aligned}$$

Thus we can conclude that the price of RPMAO is

$$f_{3c} = \exp(-rT)E\left[\left(K - \max_{1\le i\le n} S_{iT}\right)^+\right].$$

Theorem 10 *Suppose that a RPMAO option for Model (12) has a maturity time T and an exercise price K. Then, its price is*

$$\begin{aligned} f_{3c} = \ & \exp(-rT)\int_0^1 (K \\ & - \max_{1\le i\le n} S_{i0}h_i(1-\alpha)\Big)^+ d\alpha, \end{aligned}$$

where $h_i(\alpha)$ is shown as (14) for $i = 1, 2, \cdots, n$.

Proof. By Theorems 5 and 6, the uncertain variable

$$\left(K - \max_{1\le i\le n} S_{iT}\right)^+$$

has an inverse uncertainty distribution

$$\left(K - \max_{1\le i\le n} S_{iT}^{1-\alpha}\right)^+.$$

Finally, we can obtain the pricing formula of RPMAO is

$$\begin{aligned} f_{3c} = \ & \exp(-rT)E\left[\left(K - \max_{1\le i< n} S_{iT}\right)^+\right] \\ = \ & \exp(-rT)\int_0^1 \left(K - \max_{1\le i\le n} S_{iT}^{1-\alpha}\right)^+ d\alpha \\ = \ & \exp(-rT)\int_0^1 (K \\ & - \max_{1\le i\le n} S_{i0}h_i(1-\alpha)\Big)^+ d\alpha. \end{aligned}$$

The proof is completed.

5.2 Rainbow put on min option

The holder of RPMIO has the right to sell the lowest-priced asset contained in the option at the strike price K on the expiration date T, it means on the expiration date, when the strike price K is higher than the price of the lowest priced asset contain in RPMIO, the higher price of the lowest priced asset in the option, the lower the yield for the option holder. So the return of the option depends on the lowest price among all asset in the RPMIO on the expiration date.

Similarly, assume that RPMIO price obey Model (12).

Firstly, the payoff of the option holder on expiration date T is

$$\left(K - \min_{1\le i\le n} S_{iT}\right)^+.$$

Suppose the option fees we paid at time 0 is f_{4c}. Then at time 0 we have the net return

$$-f_{4c} + \exp(-rT)\left(K - \min_{1\le i\le n} S_{iT}\right)^+.$$

Secondly, from the perspective of seller, the payoff of the option seller on expiration date T is

$$-\left(K - \min_{1\le i\le n} S_{iT}\right)^+.$$

At the initial moment, the seller of option charged an option fee, hence the seller's net return is

$$f_{4c} - \exp(-rT)\left(K - \min_{1\le i\le n} S_{iT}\right)^+.$$

From the consideration of fairness, the expected returns of buyers and sellers should be equal, it means

$$\begin{aligned} &-f_{4c} + \exp(-rT)\left(K - \min_{1\le i\le n} S_{iT}\right)^+ \\ &= f_{4c} - \exp(-rT)\left(K - \min_{1\le i\le n} S_{iT}\right)^+. \end{aligned} \tag{16}$$

In summary, we can conclude that the price of RPMIO is

$$f_{4c} = \exp(-rT)E\left[\left(K - \min_{1\le i\le n} S_{iT}\right)^+\right].$$

Theorem 11 *Suppose that a RPMIO for Model (12) has a maturity time T and an exercise price K. Then, its price is*

$$\begin{aligned} f_{4c} = {} & \exp(-rT)\int_0^1 (K \\ & - \min_{1\le i\le n} S_{i0}h_i(1-\alpha)\Big)^+ d\alpha, \end{aligned}$$

where $h_i(\alpha)$ is shown as (14) for $i = 1, 2, \cdots, n$.

Proof. The proof is similar to that of Theorem 10.

To calculate the option price, we design an algorithm as follows:

Algorithm 3 (Option price for model (12))

Step 1. Input the values of parameters: S_{10}, S_{20}, $\ldots$, S_{n0}, m_1, m_2, $\ldots$, m_n, a_1, a_2, $\ldots$, a_n, T, σ_1, σ_2, $\ldots$, σ_n, u_1, u_2, $\ldots$, u_n and K.
Step 2. Let α start at $\alpha_0 = 0.01$ and grow to 0.99 at a step of 0.01 to obtain

$$\alpha_j = \alpha_{j-1} + 0.01,\ j = 1, 2, \cdots, 99.$$

Step 3. Calculate $h_i(\alpha_j)$ by (14) for $i = 1, 2, \cdots, n$, $j = 1, 2, \cdots, 99$.
Step 4. Calculate

$$\begin{aligned} f_{3c} = {} & \exp(-rT) \times 0.01 \\ & \times \sum_{j=1}^{99}\left(K - \max_{1\le i\le n} S_{i0}h_i(1-\alpha_j)\right)^+ \end{aligned}$$

and

$$\begin{aligned} f_{4c} = {} & \exp(-rT) \times 0.01 \\ & \times \sum_{j=1}^{99}\left(K - \min_{1\le i\le n} S_{i0}h_i(1-\alpha_j)\right)^+. \end{aligned}$$

Example 3 Set $n = 5$, $S_{10} = 5$, $S_{20} = 4$, $S_{30} = 3$, $S_{40} = 2$, $S_{50} = 1$, $m_1 = m_2 = m_3 = 1$, $m_4 = m_5 = 2$, $a_1 = a_2 = a_3 = 0.1$, $a_4 = a_5 = 0.5$, $\sigma_1 = \sigma_2 = ... = \sigma_5 = 0.5$, $u_1 = 0.05$, $u_2 = 0.04$, $u_3 = 0.03$, $u_4 = 0.02$, $u_5 = 0.01$, and $T = 1$, $K = 10$. Then the RPMAO price is $f_{3c} = 9.5073$ and the RPMIO price is $f_{4c} = 29.6979$ by Algorithm 3.

6 Conclusion

In this paper, uncertain differential equations are used to describe the price of underlying assets. From the perspective of uncertainty theory, combined with the stock price mean regression model, five types of rainbow options are studied. After that, through strict derivation, we obtain the pricing formulas of five types of rainbow options. Finally, we verified the rationality of the option pricing formula through some numerical experiments.

The research results of this paper are to build the asset price model through uncertain differential equations. Considering the fact that fractional order differential equations have achieved many successful practices in economics and other fields in recent years, we will further consider introducing uncertain fractional order differential equations to build the asset price model, and study the pricing of options on this basis.

References:

[1] A. Aimi and C. Guardasoni, Multi-asset barrier options pricing by collocation BEM (with Matlab(R) Code), *Axioms*, Vol.10, No.4, 2021, Article 301.

[2] F. Black and M. Scholes, The pricing of options and corporate liabilities, *The Journal of Political Economy*, Vol.81, No.1, 1973, pp. 637-659.

[3] X. Chen, American option pricing formula for uncertain financial market, *International Journal of Operations Research*, Vol.8, No.2, 2011, pp. 32-37.

[4] X. Chen and B. Liu, Existence and uniqueness theorem for uncertain differential equations, *Fuzzy Optimization and Decision Making*, Vol.9, No.1, 2010, pp. 69-81.

[5] X. Chen, Y. Liu and D. A. Ralescu, Uncertain stock model with periodic dividends, *Fuzzy Optimization and Decision Making*, Vol.12, No.1, 2013, pp. 111-123.

[6] M. Escobar, B. Götz, D. Neykova and R. Zagst, Pricing two-asset barrier options under stochastic correlation via perturbation, *International Journal of Theoretical and Applied Finance*, Vol.18, No.3, 2015, pp. 1550018_1-44.

[7] S. O. Edeki, G. O. Akinlabi, F. O. Egara, A. C. Nzeadibe, Closed-form solutions of the time-fractional standard Black-Scholes model for option pricing using He-separation of variable approach, *WSEAS Transactions on Environment and Development*, Vol.16, 2020, pp. 172-179.

[8] R. Gao, W. Wu and J. Liu, Asian rainbow option pricing formulas of uncertain stock model, *Soft Computing*, Vol.25, No.1, 2021, pp. 8849-8873.

[9] A. Ghosh and C. Mishra, High-performance computation of pricing two-asset American options under the Merton jump-diffusion model on a GPU, *Computers and Mathematics with Applications*, Vol.105, No.1, 2022, pp. 29-40.

[10] B. Liu, *Uncertainty Theory*, Springer-Verlag, 2007.

[11] B. Liu, Some research problems in uncertainty theory, *Journal of Uncertain Systems*, Vol.3, No.1, 2009, pp. 3-10.

[12] B. Liu, *Uncertainty Theory: A Branch of Mathematics for Modeling Human Uncertainty*, Springer-Verlag, 2010.

[13] B. Liu, Why is there a need for uncertainty theory, *Journal of Uncertain Systems*, Vol.6, No.1, 2012, pp. 3-10.

[14] B. Liu, Toward uncertain finance theory, *Journal of Uncertainty Analysis and Applications*, Vol.1, No.1, 2013, pp. 1-15.

[15] Z. Lu, H. Yan and Y. Zhu, European option pricing model based on uncertain fractional differential equation, *Fuzzy Optimization and Decision Making*, Vol.18, No.2, 2019, pp. 199-217.

[16] J. Peng and K. Yao, A new option pricing model for stocks in uncertainty markets, *International Journal of Operations Research*, Vol.8, No.2, 2011, pp. 18-26.

[17] J. Sun and X. Chen, Asian option pricing formula for uncertain financial market, *Journal of Uncertainty Analysis and Applications*, Vol.3, No.1, 2015, Article 11.

[18] Y. Sun and T. Su, Mean-reverting stock model with floating interest rate in uncertain environment, *Fuzzy Optimization and Decision Making*, Vol.16, No.2, 2017, pp. 235-255.

[19] M. Tian, X. Yang and Y. Zhang, Barrier option pricing of mean-reverting stock model in uncertain environment, *Mathematics and Computers in Simulation*, Vol.166, 2019, pp. 126-143.

[20] W. Wang and D. A. Ralescu, Valuation of lookback option under uncertain volatility model, *Chaos, Solitons and Fractals*, Vol.153, No.1, 2021, Article 111566.

[21] W. Wang, X. Su, S. Gan and L. Qian, Pricing vulnerable European options under a Markov-modulated jump diffusion process, *WSEAS Transactions on Mathematics*, Vol.16, 2017, pp. 123-132.

[22] X. Yang, Z. Zhang and X. Gao, Asian-barrier option pricing formulas of uncertain financial market, *Chaos, Solitons and Fractals*, Vol.123, No.1, 2019, pp. 79-86.

[23] K. Yao, Uncertain contour process and its application in stock model with floating interest rate, *Fuzzy Optimization and Decision Making*, Vol.14, No.4, 2015, pp. 399-424.

[24] K. Yao and X. Chen, A numerical method for solving uncertain differential equations, *Journal of Intelligent and Fuzzy Systems*, Vol.25, No.3, 2013, pp. 825-832.

[25] Y. Zhu, Uncertain optimal control with application to a portfolio selection model, *Cybernetics and Systems: An International Journal*, Vol.41, No.7, 2010, pp. 535-547.

Contribution of individual authors to the creation of a scientific article (ghostwriting policy)

Mingchong Liao completed the draft writing.
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