# On a dynamical model of genetic networks

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*Abstract:* We consider the model of a four-dimensional gene regulatory network (GRN in short). This model consists of ordinary differential equations of a special kind, where the nonlinearity is represented by a sigmoidal function and the linear part is present also. The evolution of GRN is described by the solution vector $X(t)$, depending on time. We describe the changes that the system undergoes if the entries of the regulatory matrix are perturbed in some way. The sensitive dependence of solutions on the initial data is revealed by the analysis using the Lyapunov exponents.

*Key-Words:* nonlinear dynamical system, attractor, chaotic regime, Lyapunov exponents, Artificial Neural Networks

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## 1. Introduction

Nonlinear dynamics plays an important role in modern natural science and studies such objects and phenomena as dynamic chaos and various types of self-organization of matter. Problems of nonlinear dynamics have exact analytical solutions only in very rare cases, which is why they have to be studied using a computer experiment. In the study of various natural phenomena, the construction of useful mathematical models with their subsequent study using the exact and graphical methods of modern mathematics is of decisive importance. As mathematical models widely systems of differential equations are used.

The variety of dynamics observed in nonlinear systems can be reduced to simple regimes associated with some repetition for a wide variety of systems by characteristic types of solutions. These characteristic solutions have the important property of invariance. Moreover, many other solutions to the system are attractive to them. Knowledge of such solutions - attractors allows getting an idea of the overall picture of the nonlinear system dynamics [6]. Changing system parameters can significantly change attractor type. In this case, the system has a bifurcation [1]. The bifurcation is a change in the dynamics system, accompanied by the disappearance of some and the appearance of other regimes. Firstly, the stable point goes into the periodic regime (limit cycle) [1], [7], then to the chaotic regime (strange attractor) [2], [8].

Consider the general form of writing the $n$ -dimensional dynamical system, that is expected to model a genetic regulatory network,

$$\begin{cases} x'_1 = \frac{1}{1+e^{-\mu_1(w_{11}x_1+w_{12}x_2+\cdots+w_{1n}x_n-\theta_1)}} - v_1x_1, \\ \dots \\ x'_n = \frac{1}{1+e^{-\mu_n(w_{n1}x_1+w_{n2}x_2+\cdots+w_{nn}x_n-\theta_n)}} - v_nx_n, \end{cases}$$

(1)

where $\mu_i > 0, \theta_i$ and $v_i > 0$ are parameters, and $w_{ij}$ are elements of the $n \times n$ regulatory matrix $W$ [8]. System (1) appears also in the theory of telecommunication networks. [3]

The sigmoidal function $f(z) = \frac{1}{1+e^{-\mu}}$ is used in (1). The sigmoidal function is a mathematical function having a characteristic 'S-shaped' curve or sigmoid curve. They are many: logistic function [1], [2], [7], [8], [11], Gompertz function [9], Hill function, inverse trigonometric functions. A set of coefficients $w_{ij}$ form the so called regulatory matrix

$$W = \begin{pmatrix} w_{11} & \dots & w_{1n} \\ \dots & \dots & \dots \\ w_{n1} & \dots & w_{nn} \end{pmatrix} \quad (2)$$

The set $Q = \left\{x \in R^n : 0 < x_i < \frac{1}{v_i}, i = 1, \dots, n\right\}$ is invariant [5, Definition 2, Section 2.5, Ch.2] with respect to system (1). This follows from the properties of the sigmoidal function (the value range is the interval (0,1)) and can be established by inspection of the vector field, defined by (1).

Systems of the form (1), but with different sigmoidal functions, appear when studying neuronal networks. An example is considered in the next section. This example and comparison with systems from the theory of genetic networks, was one of motivations for this work.

## 2. Materials and methods

Our consideration is numerical and geometrical. All processes take place in a bounded parallelepiped and our main intent is to use the 3D projections of the attractor on different subspaces, to construct the graphs of solutions for understanding and managing the system. Computations, plotting the solution, and the image of the projections of attractors are performed using Wolfram Mathematica. Also, we use Lyapunov exponents (LE) and a high Kaplan–Yorke dimension, guaranteeing chaotic behavior for longtimes. First, we illustrate the usage of Lyapunov exponents considering the modification of the model of Artificial Neuronal Network as given in [17]. The chaotic behavior is confirmed for the specific values of parameters. Next, we consider the four-dimensional genetic model (8) composed of two independent 2D systems. This new 4D system is shown to have an attractor, which is stable under small perturbations. The zero blocks in the regulatory matrix (9) were then filled with non-zero elements and new coupled 4D system was considered with the regulatory matrix (10). Chaos was not confirmed by the analysis using the Lyapunov spectrum. Finally, the system (8) was considered with the regulatory matrix (11), where the entries were randomly selected. Chaotic behavior was observed.

## 3. Lyapunov exponents

The Lyapunov exponents are an important tool for the characterization of an attractor of a finite-dimensional nonlinear dynamic system and their excessive sensitivity to initial conditions.[12] The Lyapunov exponent is an approach to detect chaos, and it is ameasure of the speeds at which initially nearby trajectories of the system diverge.[13] Relationships between the Lyapunov exponents and the propertiesand types of attractors:

1. $(LE_1, LE_2, LE_3, LE_4) = (-,-,-,-)$ - stable fixed point;

2. $(LE_1, LE_2, LE_3, LE_4) = (0, -, -, -)$ - periodic solutions (limit cycles);
3. $(LE_1, LE_2, LE_3, LE_4) = (0,0, -, -)$ - quasiperiodic solution (2 torus);
4. $(LE_1, LE_2, LE_3, LE_4) = (+,0, -, -)$ - strange attractor;
5. $(LE_1, LE_2, LE_3, LE_4) = (+, +,0, -)$ - hyperchaotic attractor.

*Properties of Lyapunov exponents:*

1. The number of Lyapunov exponents is equal to the number of phase space dimensions, or the order of the system of differential equations. They are arranged in a descending order.[14]
2. The largest Lyapunov exponent of a stable system does not exceed zero.[13]
3. A chaotic system has at least one positive Lyapunov exponent, and the more positive the largest Lyapunov exponent, the more unpredictable the system is.[13]
4. To have a dissipative dynamical system, the values of all Lyapunov exponents should sum to a negative number.[14]
5. A hyperchaotic system is defined as a chaotic system with at least two positive Lyapunov exponents. Combined with one null exponent and one negative exponent, the minimal dimension for a hyperchaotic system is four.[15]

The formula for the Kaplan–Yorke dimension is

$$D_{KY} = j + \frac{\sum_{i=1}^{j} LE_i}{|LE_j + 1|} \quad (3)$$

## 4. Artificial Neural Networks

In 1943 American neurophysiologist and cybernetician Warren SturgisMcCulloch and American logician Walter Harry Pitts modeled a neuronas a switch that receives input from other neurons and, depending on the total weighted input, is either activated or remains inactive.[16]

**Definition 1.** A dynamical model inspired by the connectivity and behavior of neurons in the brain is an Artificial Neural Network.

One example of which is

$$x'_i = \tanh \sum_{j=1}^{N} a_j x_j - b_i x_i, \quad (4)$$

where $N$ is the number of neurons, each of which represents a dimension of the system.[17] The hyperbolic tangent is a sigmoidal function.

Consider the regulatory matrix

$$W = \begin{pmatrix} 0 & -1 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & -1 \\ 0 & -1 & 1 & 0 \end{pmatrix} \quad (5)$$

and the system

$$\begin{cases} x'_1 = \tanh(x_4 - x_2) - \mathrm{b}x_1, \\ x'_2 = \tanh(x_1 + x_4) - \mathrm{b}x_2, \\ x'_3 = \tanh(x_1 + x_2 - x_4) - \mathrm{b}x_3, \\ x'_4 = \tanh(x_3 - x_2) - \mathrm{b}x_4, \end{cases} \quad (6)$$

at b = 0.03. The initial conditions are

$$x_1(0) = 1.2;\ x_2(0) = 0.4;\ x_3(0) = 1.2;\ x_4(0) = -1. \quad (7)$$

This system is studied numerically (Wolfram Mathematica), providing description of the phase space and images of 2D and 3D projections. The method of Lyapunov (3) exponents was used to analyze the system and to obtain evidences of sensitive dependence of solutions on the initial data.

The attractor as shown in Figure 1and Figure 2 and oscillatory solutions in Figure 3and Figure 4.

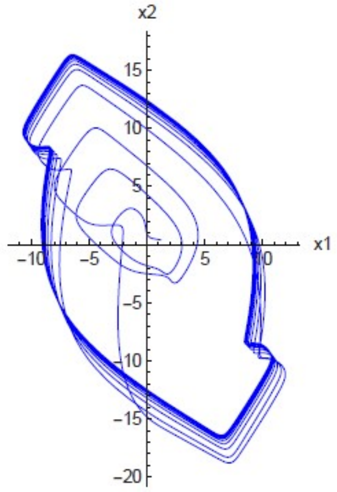


Figure 1.The projection of the attractor on 2D subspace on$(x_1, x_2)$.

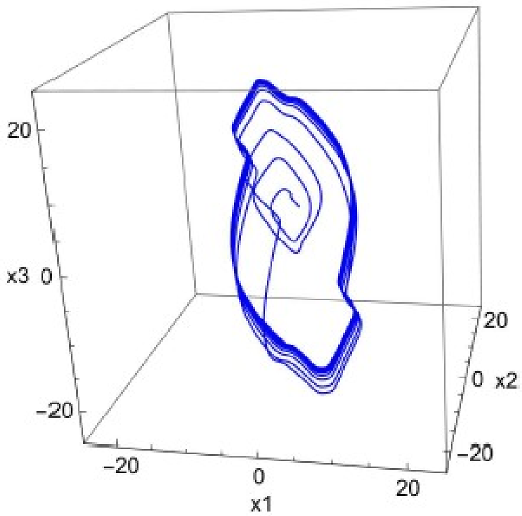


Figure 2.The projection of the attractor on 3D subspace $(x_1, x_2, x_3)$.

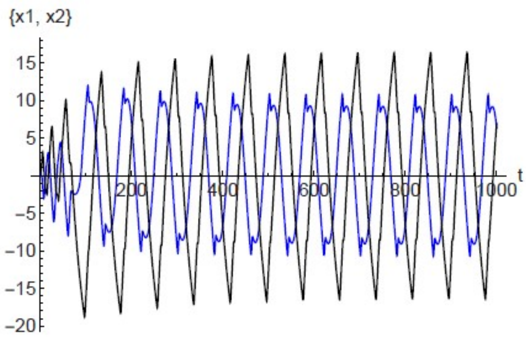


Figure 3. Solutions $(x_1, x_2)$.

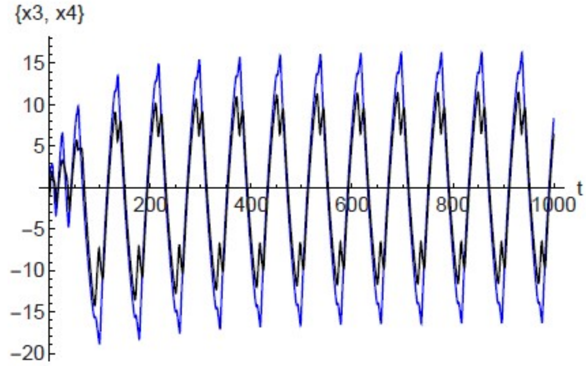


Figure 4. Solutions $(x_3, x_4)$.

It was pointed out in [17] that the minimal dissipative artificial neural network that exhibits chaos has $N = 4$ and is given by (6) at $b = 0.043$ and an attractor as shown in Figure 5 and Figure 6.For specific parameters this system solution has a chaotic trajectory as shown in Figure 7 and Figure 8. [17] Dynamics of Lyapunov exponents are shown in Figure 9.

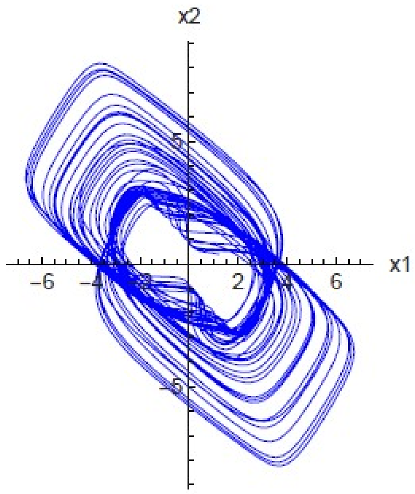


Figure 5.The projection of the attractor on 2D subspace on $(x_1, x_2)$.

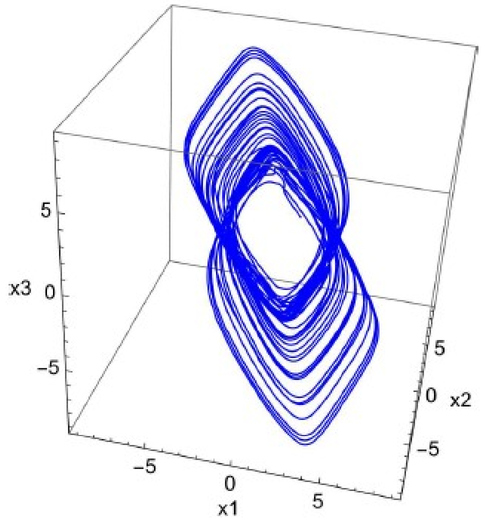


Figure 6.The projection of the attractor on 3D subspace $(x_1, x_2, x_3)$.

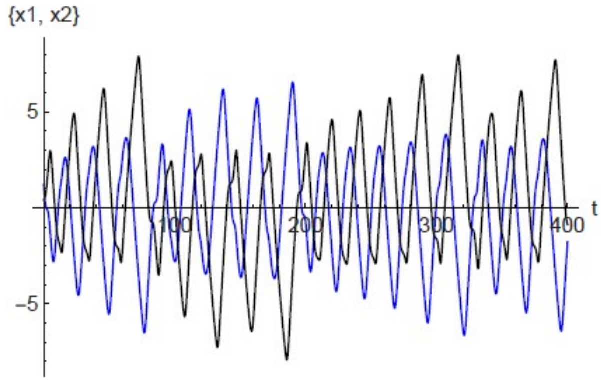


Figure 7.Solutions $(x_1, x_2)$.

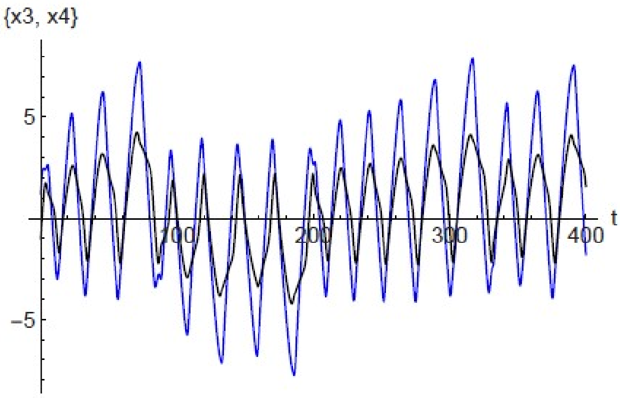


Figure 8. Solutions $(x_3, x_4)$.

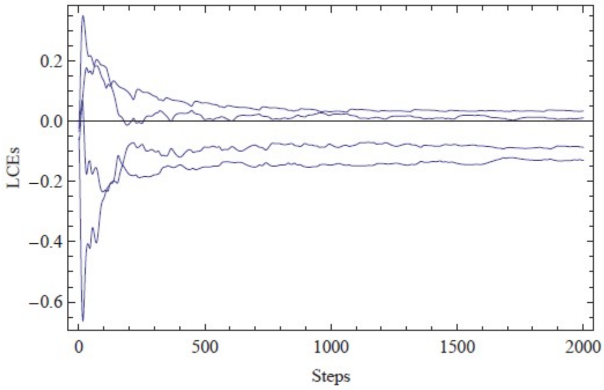


Figure 9. $(LE_1, LE_2, LE_3, LE_4) = (0.03, 0.01, -0.09, -0.13)$- strange (chaotic) attractor, Kaplan–Yorke dimension is $D_{KY} = 2.62$.

In the next section, following this example, we investigate our problem.

## 5. Results

Consider the four-dimensional system

$$\begin{cases} x'_1 = \dfrac{1}{1 + e^{-\mu_1(w_{11}x_1 + w_{12}x_2 + w_{13}x_3 + w_{14}x_4 - \theta_1)}} - v_1 x_1, \\ x'_2 = \dfrac{1}{1 + e^{-\mu_2(w_{21}x_1 + w_{22}x_2 + w_{23}x_3 + w_{24}x_4 - \theta_2)}} - v_2 x_2, \\ x'_3 = \dfrac{1}{1 + e^{-\mu_3(w_{31}x_1 + w_{32}x_2 + w_{33}x_3 + w_{34}x_4 - \theta_3)}} - v_3 x_3, \\ x'_4 = \dfrac{1}{1 + e^{-\mu_4(w_{41}x_1 + w_{42}x_2 + w_{43}x_3 + w_{44}x_4 - \theta_4)}} - v_4 x_4. \end{cases}$$

(8)

Periodic solutions can exist in systems of the form (8). Consider the system (3) with the matrix

$$W = \begin{pmatrix} 0.5 & 2 & 0 & 0 \\ -2 & 0.5 & 0 & 0 \\ 0 & 0 & 0.5 & 2 \\ 0 & 0 & -2 & 0.5 \end{pmatrix} \quad (9)$$

and $\mu_1 = \mu_2 = \mu_3 = \mu_4 = 10$ ; $v_1 = v_2 = v_3 = v_4 = 1$ ; $\theta_1 = 1.25, \theta_2 = -0.75, \theta_3 = 1.25, \theta_4 = -0.75$ .The initial values$x_1(0) =$

$0.42, x_2(0) = 0.39, x_3(0) = 0.4, x_4(0) = 0.395.$

This system consists of two identical but independent two-dimensional systems. The 4D system has, by the appropriate choice of $\theta$-s, the critical point of the type an unstable 4D focus at the center of a unit cube. Both two 2D systems have a stable limit cycle. The $(x_1, x_2)$ and $(x_3, x_4)$ projections coincide with the 2D limit cycles. The $(x_2, x_4)$projectionis depicted in Figure 10. Any of these two two-dimensional systems has a single critical point of the type unstable focus. This is the result of Andronov-Hopf bifurcation in 2D systems. The unique critical point of the type stable focus loses its stability, and instead, an attractor in the form of a limit cycle emerges. The limit cycles (Figure 10) with the same periods exist. There are single critical points inside both limit cycles, and both critical points are of the type unstable focus.

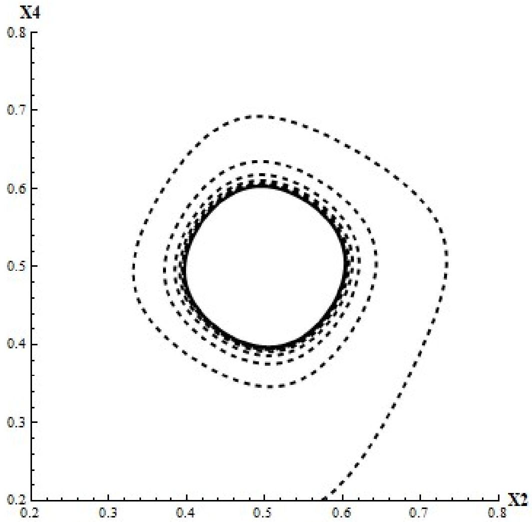


Figure 10. The attractor of (8), projection on $(x_2, x_4)$.

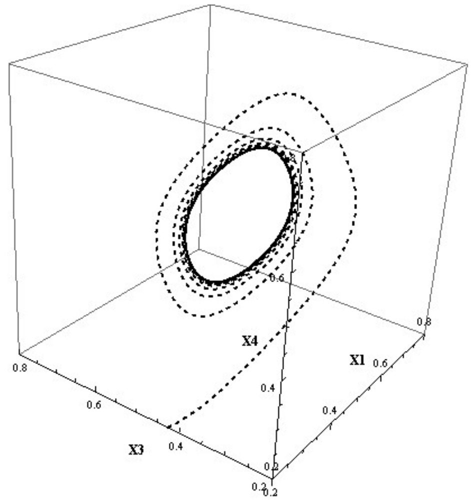


Figure 11. The attractor of (8), projection on $(x_1, x_3, x_4)$.

Now fill in all zero elements of the regulatory matrix (9) with values *0.1*, so the regulatory matrix becomes

$$W = \begin{pmatrix} 0.5 & 2 & 0.1 & 0.1 \\ -2 & 0.5 & 0.1 & 0.1 \\ 0.1 & 0.1 & 0.5 & 2 \\ 0.1 & 0.1 & -2 & 0.5 \end{pmatrix} \quad (10)$$

There is exactly one critical point at (0.5472, 0.448, 0.5472, 0.448). The standard linearization analysis provides the characteristic numbers $\lambda_{1,2} = -0.0099 \pm 4.944i$; $\lambda_{3,4} = 0.4852 \pm 4.944i$.Consider the system (8), which is no longer uncoupled, with the same parameters and initial conditions for a solution. The dynamics of Lyapunov exponents are shown in Figure 14.

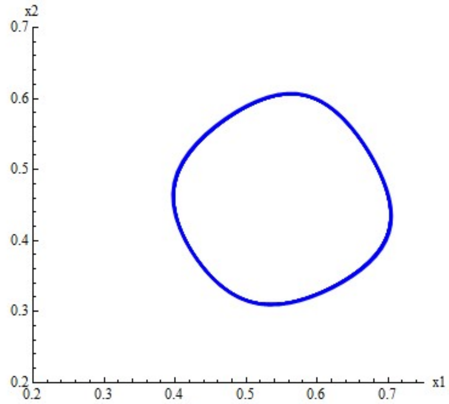


Figure 12. The projection of the attractor (10) on 2D subspace$(x_1, x_2)$.

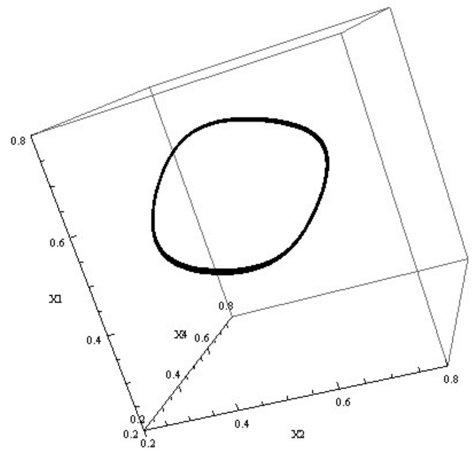


Figure 13. The projection of the attractor (10) on 3D subspace $(x_1, x_2, x_4)$.

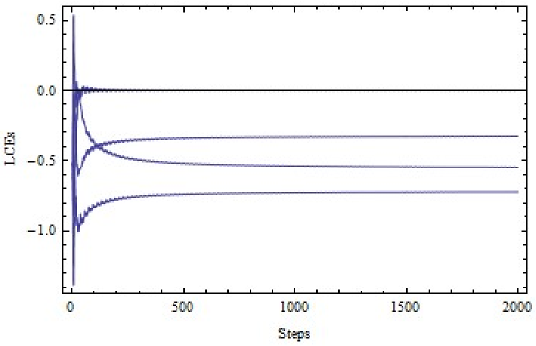


Figure14. $(LE_1, LE_2, LE_3, LE_4) = (0, -0.33, -0.55, -0.72)$ - periodic solutions.

Consider the system (8) with the regulatory matrix

$$W = \begin{pmatrix} 0.8 & 2 & -0.8 & 0.5 \\ -2 & 0.3 & 0.4 & -0.7 \\ -0.5 & 0.2 & 1.8 & 2 \\ 0.8 & -0.7 & -2 & 1.8 \end{pmatrix} \quad (11)$$

andthe parameters $\mu_1 = \mu_2 = \mu_3 = \mu_4 = 10$ ; $v_1 = v_2 = v_3 = v_4 = 1$ and $\theta_i$ , where $i = 1,2,3,4$ are calculated as

$$\begin{cases} \theta_1 = \frac{w_{11}+w_{12}+w_{13}+w_{14}}{2}, \\ \theta_2 = \frac{w_{21}+w_{22}+w_{23}+w_{24}}{2}, \\ \theta_3 = \frac{w_{31}+w_{32}+w_{33}+w_{34}}{2}, \\ \theta_4 = \frac{w_{41}+w_{42}+w_{43}+w_{44}}{2}. \end{cases} \quad (12)$$

$\theta_1 = 1.25, \theta_2 = -1,\ \theta_3 = 1.75,\ \theta_4 = -0.05$.The initial values$x_1(0) = 0.4, x_2(0) = 0.6,\ x_3(0) = 0.39, x_4(0) = 0.38$.

The critical point is at (0.5,0.5,0.5,0.5). The standard linearization analysis provides the characteristic numbers $\lambda_{1,2} = -0.44 \pm 4.603i$; $\lambda_{3,4} = 4.33 \pm 5.135\, i$. The graphs of chaotic solutions are shown in Figure 15. The projection on three-dimensional subspaces is depicted in Figure 16.The dynamics of Lyapunov exponents are shown in Figure 17.

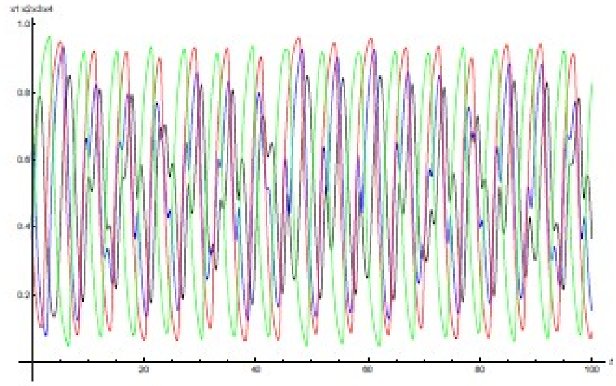

Figure 15.The graphs of $x_i(t), i = 1,2,3,4$.

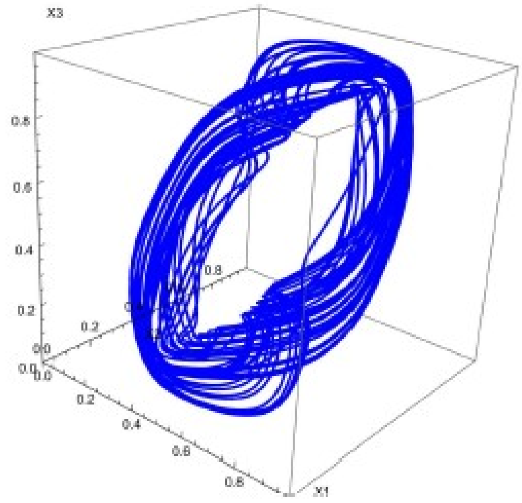


Figure 16.The projection on 3D subspace$(x_1, x_2, x_3)$.

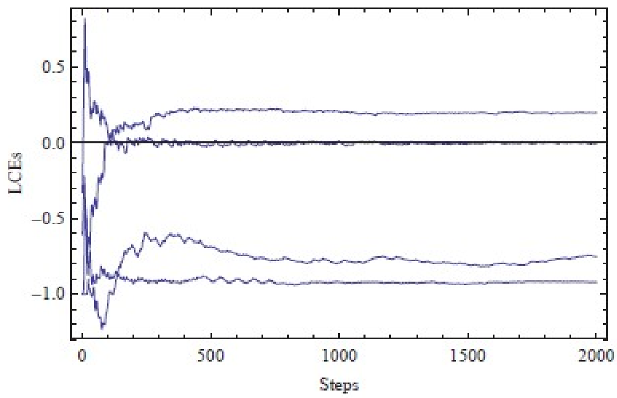


Figure 17.$(LE_1, LE_2, LE_3, LE_4) = (0.2, 0, -0.75, -0.92)$ – a chaotic solution.

## 6. Discussion

We have constructed the attractor for a 4D dynamical system, arising in the theory of genetic networks. This attractor is generated by two limit cycles in the different 2D systems of the same GRN kind (both limit cycles are identical and have equal periods). A single critical point is not attractive (the complex characteristic values have positive real parts). The 4D regulatory matrix at the beginning has a block form and the 4D system is therefore uncoupled. By filling the zero spaces in the regulatory matrix with non-zero elements, the system was made coupled. The attractor still exists for sufficiently small perturbations. This confirms the structural stability of a 4D system. The GRN system with the regulatory matrix (6) is shown to exhibit chaotic behavior. This is confirmed by the analysis using the Lyapunov spectrum. By using a similar procedure, regulatory matrices of any dimension can be constructed from lesser blocks. Combinations of any attractors of lower dimensions are possible. Filling zero spaces in the regulatory matrix of block form often produces attractors of a new structure.

## 7. Conclusions

In this note, we considered the creation of a new 4D attractor from the two 2D periodic attractors. Even for the 2D case, there are several types of attractors in GRN systems, namely, a single stable critical point, several stable critical points, a limit cycle. All of them in various combinations can be used to construct attractors in GRN systems of higher dimensions. They, in turn, can be used to construct more complicated ones in higher dimensions. Also, attractors generated by 3D systems can be used to produce higher dimensional ones. After mechanically combining several low-dimensional matrices into a single block matrix, this matrix can be perturbed by filling zero zones with non-zero entries. The resulting systems can reveal different behaviors, including the generation of a unique attractor. The chaotic behavior of solutions seems to be generic in higher dimensional GRN type systems. Their investigation is possible by a combination of qualitative and numerical methods. A similarity of systems of differential equations arising in models of GRN and neuronal networks, is acknowledged.

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All authors have contributed equally to creation of this article.

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## Conflicts of Interest

The authors declare no conflict of interest.