Bi-Level Inventory Formalization

TODOR STOILOV, KRASIMIRA STOILOVA
Institute of Information and Communication Technologies,
Bulgarian Academy of Sciences,
1113 Sofia Acad. G. Bonchev str. bl.2,
BULGARIA

Abstract: - A bi-level problem is defined for the optimization of inventory management. This formalization allows the simultaneous delivery of different types of goods according to the required volume of consumption for the production of goods. The resources, which are supplied are strictly related to the production volumes, which minimize the inventory costs. The bi-level optimization problem is defined for the case of delivery of food for diet preparations with appropriate content. The costs for delivery are minimized for a given nutrition content. The empirical comparisons and results of the bi-level inventory management give advantages for the derived optimization problem.

Key-Words: - Inventory management, resource allocation, optimization, bi-level formalization, maximization of return, minimization of inventory cost, nutrition content.

1 Introduction

The supply of raw materials and goods is an important phase for production and business management. This production phase is named inventory management and its goal is to minimize the costs that have to be allocated for the supply of resources. The minimization of inventory costs influences strongly the production costs of the goods that the business presents to the market. Following the estimates of [1] for the role of the inventory the optimal decisions for delivery management are:

- The minimum level of spare parts for the production engines;

- No lack of time for raw materials, when the demands for them are changed randomly;

- The losses from storage of excess quantities of raw materials are minimized;

- The production technology is loaded continuously;

- Optimal inventory management decreases the financial expenditure and increases the profit.

For the optimization of inventory management, an appropriate optimization problem must be defined to find the appropriate parameters for the inventory volume. The optimization problem is based on formal relations, which are applied for the definition and solution of optimization problems. The variables and parameters that the inventory problem evaluates are [2]: the optimal volumes of the raw materials, the minimal cost for the delivery,

and the minimal losses for material storage, taking into consideration the physical restriction of the storage capacity for the raw materials, the time for the next needed delivery, according to the demands for production.

The inventory management is tightly connected with the production technology, which provides the volumes for the production of goods. Because the inventory and the production are sequentially related, it is useful to integrate to define a common optimization problem, which integrates simultaneously the requirements for these two important tasks of business management. Having in mind that the market behavior randomly changes the request for the production of goods, such dynamic behavior has to be considered by the inventory management. Such integration of inventory and production is formalized with optimization problems with various peculiarities. In [3] the optimization problem is a linear integer optimization one. The optimization gives the optimal volume of inventory through minimization of the costs. The importance of integrating the inventory and production in a common problem is worked out in [4]. The variables, which are under optimization, are again the volumes of raw materials in the supply. Markov chain formalization for simultaneous optimization of inventory and production is made in [5]. The target of the optimization is to maximize the quality of the production. A special case of production is regarded in [6] by sequential

operations of logistic operations, resource delivery, and production of goods. The goal of this problem is to minimize the total costs of this sequence of operations. The formal definition of the problem is linear integer one. The optimization of the production volumes, according to the demands of the market players is formalized in [7]. The defined problem uses statistical data about the mean values of demands per period and the covariance between the volumes of the production goods and the requested raw materials for that production. The problem is a combinatorial one and it was applied the colony algorithm for its solution. In [8] formal relations between the volumes of inventory and the production intensity are included in a common optimization problem. The problem was applied to different cases for textile, flour, and food production. In [9] the inventory optimization has been extended with considerations about the administrative management of the business entity. Recommendations for the administrative workflows are considered.

Considering this short overview, we can conclude that inventory management is an important part of business management in achieving optimal production policy. For the implementation of optimal inventory solutions, it is needed to derive formal models, which will give quantitative solutions for variables and parameters of the inventory and production tasks.

The goal of this research is to formalize the relationship between inventory and production. This formalization will be used for the definition of a common optimization problem, which can integrate the benefits from the minimization of inventory costs and maximization of the volume of goods. This research applies bi-level optimization, which allows such integration between the requirements of inventory minimization and production maximization. The bi-level problem is giving as optimal variables the volumes of deliveries, based on the production demands. The maximization of the goods leads to profit optimization. The bi-level formalization is based on simultaneous solutions of two subproblems. The upper-level subproblem performs minimization of the inventory costs. As an additional constraint, it is considered for the low-level subproblem the maximization of the production profit. Both subproblems deal with common constraints concerning the volume of inventory and technological requirements for the needed volume of resources for the production of a unit good. The solutions to the defined bi-level problem have been assessed by comparison with a classical inventory problem without considering the

production. The comparisons have been performed for the optimal size of the inventory and the costs for the classical and bi-level problems.

2 Inventory Management without Considering the Production Phase

In general, inventory management uses parameters for its optimization, which concern the costs of holding the resources in a warehouse, [10]. The optimization of the inventory addresses the minimization of the costs. The components, which are taken in the total inventory cost, are [11]:
- The costs of the raw materials;
- The transportation costs from the supplier to the inventory warehouse;
- The initial cost K [BGN/per request] for the activation of the delivery from a supplier;
- The cost h for holding the unit inventory quantity in a warehouse, [BGN/per unit resource].
- The costs, which take place from losses of materials, during their stay in a warehouse (during their storage).

The inventory management contains in a common considers the initial costs K and the holding ones h. For simplicity of the inventory problem, the costs from losses are included in the category h of the holding costs. Thereafter the total inventory costs are formalized by the addition.

$$\text{Total costs} = K + hy,$$

where y is the volume of raw material in the inventory.

The solution of the inventory optimization is the volume y for the delivery. The evaluation of y is performed under the constraint that it satisfies the requested demand for goods D. The sequence of the sequential implementation of deliveries is graphically illustrated in Figure 1.
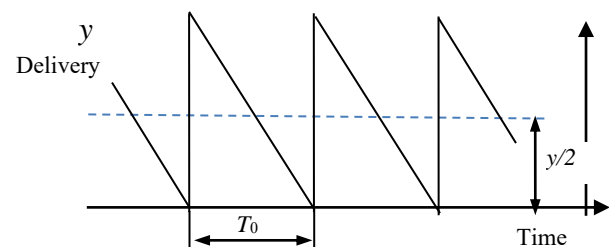

Fig. 1: Dynamical changes of the inventory level y

The inventory volume y decreases during the production of goods. The volume of the demand D.

defines the slope of decrease. The higher the demand, the slope of the graphic of y is sharper.

The inventory must be periodically implemented because the warehouse has restricted capacity and cannot accommodate large volumes of raw materials. If the delivery lacks delay, a new inventory must be performed, when its current value is zero, $y=0$. In Figure 1 this graphically is illustrated by a sequential vertical increase of the inventory y and its steepest decrease because of its consumption by the demand for consumption.

The sequence of inventory is performed with period T_0, which depends on the demand D and the available maximal value of inventory y or

$$T_0 = D/y . \tag{1}$$

This relation is applied by the formal inventory modeling EOQ (Economic-Order-Quantity) model [11]. In this case, the demand D is assumed to be known and static in value. The mean holding costs for one inventory period is the multiplication $h \times (0+y)/2$, considering the full warehouse with y inventory and 0 levels at the end of the inventory cycle, Figure 1.

Thereafter the total inventory (TIC) for one T_0 is

$$TIC(x) = \frac{K + h\left(\frac{y}{2}\right)T_0}{T_0} \tag{2}$$

After the substitution of (1) in (2) the costs for one inventory cycle are expressed with the main parameters of the inventory, K, h, D, and the argument y

$$TIC(y) = \frac{KD}{y} + h\frac{y}{2} . \tag{3}$$

From (3) the optimal volume of the inventory y^{opt} can be evaluated by minimization of (3) towards y. Because this minimization is without constraints, the solution is found from the first derivative of $TIC(y)$ towards y or

$$y^{opt} = arg\left\{\frac{dTIC(y)}{dy} = -\frac{KD}{y^2} + \frac{h}{2}\right\} \tag{4}$$

or

$$y^{opt} = \sqrt{\frac{2KD}{h}} . \tag{5}$$

The solution (5) is given as a result of the EOQ modelling of the inventory process. The volume of inventory y^{opt} is delivered on each inventory cycle with period T_0.

In the general case of inventory for a set of resources y_j, $j=1,..,m$, problem (4) cannot be used directly. A decomposition to a set of optimization problems is performed and additional constraints appear considering the restricted capacity d of the warehouse to accommodate this set of resources. The inventory problems become a nonlinear form:

$$\min_{y} \left\{RC(y) = \sum_{j=1}^{m}\left(\frac{K_j D_j}{y_j} + h_j \frac{x_y}{2}\right)\right\} \tag{6}$$
$$\sum_{j=1}^{m} a_j y_j \leq d ,$$

where a_j is the relative space, which the resource y_i needs to be accommodated in the warehouse, d is the warehouse capacity.

The solution of (6) has to be performed with algorithms for nonlinear constraint optimization.

For the production case, one unit of good needs corresponding volumes of different resources. Because the goal of the research is the integration of the inventory and production, additional constraints have to be added to the inventory problem, which formalizes the relations between the needed set of resources for the production of one unit of good. Thereafter the goal of such an integrated optimization problem is to find the optimal volume of inventory resources, which maximizes the production. The approach of this research is for this integral optimization problem to be defined in a bi-level form. The upper level of the problem will minimize the inventory costs and evaluate the optimal volumes of the inventory. The low-level problem will maximize the production profit using the evaluated inventory volumes from the upper-level problem.

3 Definition of the Bi-level Optimization Problem

The low-level problem is the production one, which performs maximization of a goal function $F(x)$ for production profit, where x is a vector with n components for the number of products $x = (x_1, \dots, x_n)$. To decrease the computational workload for the bi-level problem, the function $F()$ is used in linear form $F = c^T x$, where the coefficients $c = (c_1, , c_n)$ concern the profit given by good x_i. The inventory resources, used for the production of one unit of good of type x_i are formalized by the set of inequalities:

$$\sum_{j=1}^{m} b_{j,i} y_j \geq x_i \quad j = 1, \dots, m; i = 1, \dots, n \tag{7}$$

where $b_{j,i}$ is a part of resource type j for the production of a good type i.

Additional constraints are added to the production problem, due to the limited capacity f of the warehouse to store the production x

$$\sum_{i=1}^{n} g_i x_i \leq f \tag{8}$$

where g_i is the space for a unit good of type i to be stored in the warehouse.

The formal description of the production problem takes an analytical form

$$\max_{x}\{F(x(y))\} \quad (9)$$

and the constraints (7) and (8).

The bi-level problem is defined by the integration of the inventory problem (6) and the production one (9). The upper and lower problems in the bi-level formulation have a common set of inequalities (7), which give the relation between the arguments x and y.

Therefore, the bi-level optimization problem is analytically written in the form

$$\min_{y}\left\{RC(y(x)) = \sum_{j=1}^{m}\left(\frac{K_j D_j}{y_j} + h_j \frac{y_j}{2}\right)\right\} \quad (10)$$

$$\sum_{j=1}^{m} a_j y \leq d$$

$$\sum_{j=1}^{m} b_{j,i} x_j \geq y_i \quad j = 1, \dots, m; i = 1, \dots, n$$

subject to
$$\max_{x}\{F(x(y))\}$$

$$\sum_{i=1}^{n} g_i y_i \leq f$$

$$\sum_{j=1}^{m} b_{j,i} x \geq y_i \quad j = 1, \dots, m; i = 1, \dots, n.$$

The Bi-level solutions of (10) are both the production volume of goods x and the inventory volumes y. They are evaluated by minimization of the total inventory costs $TIC(y)$ and maximization of the production profit $F(x(y))$.

4 Solution and Comparison of the Bi-level Results

The bi-level optimization problem is defined for the case of delivery of foods having appropriate nutrition content. The costs for delivery are minimized for a given nutrition content. The bi-level problem finds the optimal content of fat and carbohydrates from foods of milk and beans. These relations are needed for making appropriate diets for human beings and animals. The constraints of the bi-level problem are defined according to the data from [12], [13] and summarized in Table 1.

Table 1. Content of the foods

Foods	Fat, x_1	Carbohydrate, x_2
Milk, y_1	4.59	4.20
Beans, y_2	1.6	40

In the preparation of a diet, a combination of foods must be mixed with a content of fat and carbohydrates $x=(x_1, x_2)$. This content has to be obtained from the foods of milk and beans $y=(y_1, y_2)$. The bi-level problem must evaluate the minimal inventory cost for delivery of foods that will satisfy the required content of the diet with fat and carbohydrates.

For the inventory part of the bi-level problem, it is needed to estimate the initial values of the delivery K, holding cost per unit food h, and the demand D. The linear set of relations (7) between the two sets of arguments x and y are numerically defined with the data from Table 1 in the form

$$A\,y \geq x \quad (11)$$

where A=[4.59 4.20; 1.6 40] from Table 1.

The constraints of type (8), concerning the storage holding of the food due to warehouse limitation, are analytically defined in linear form as

$$A1\,y \leq b(1),$$

where the components of (8) are chosen numerically to $A1$=[0.4; 0.6], $b(1)$=1.

The parameters of the inventory part of the bi-level problem are estimated to K=[40; 5], h=[25; 10]. They correspond to the delivery and holding of the food. The value of the demand D results from the relations between foods and their content given in Table 1. The demand for milk is evaluated according to the coefficients for Fat and Carbohydrate as a sum per column:

$D(1)= 4.59 + 1.6 = 6.19$, $D(2)= 4.20 + 40 = 44.20$

where A=[4.59 4.20; 1.6 40] from Table 1.

The bi-level problem targets minimization of the costs for the delivery of milk and beans y and maximization of the content of fat and carbohydrate x. Following these assumptions, the goal function of the production problem is chosen in quadratic form:

$$\min_{x}\{-x^T Q2\,x + y^T Q1\,y).$$

Both matrices $Q1$ and $Q2$ consider the relative weight of the different components x and y for the production and delivery processes. Their numerical values are chosen to

$Q1$= [20 0;0 1]; $Q2$ = [50 0;0 5].

The constraint (8) for the storage of the products x is defined as linear inequalities:
$$A2\,x \le b(2)$$

where it is estimated to $A2$=[0.7; 0.1], $b(2)$ = 16.

The bi-level arguments x and y are subject to lower bounds, which preserve the solutions to give zero values for the inventory. Therefore the bi-level problem has constraints of the form $x \ge LBx$, $y \ge LBy$. The corresponding values The values of LBx , LBy have been estimated by assessing the sensitivity of the solutions from the amount of fat and carbohydrate content, generating different sets of possible. The analytical bi-level problem has the form becomes:

$$\min_{y}\ \left\{ \frac{K(1)D(1)}{y(1)} + h(1)\,\frac{y(1)}{2} + \left(\frac{K(2)D(1)}{y(2)}\right. \right.$$
$$\left. \left. + h(2)\,\frac{y(2)}{2} \right\} \right.$$
$$Ay \ge x \qquad\qquad (12)$$

$$A1\,y \le b(1)$$

where

$$\min_{x}\ \{ -x^T\,Q2\,x +\, y^T\,Q1\,y)$$
$$Ay \ge x$$
$$A2\,x \le b(2)$$
$$x \ge LBx , \quad y \le UBy.$$

The solution of (12) is evaluated with UBy=[10; 10], LBx=[0; 0] for the low values for fat and carbohydrates, and the limitations and limitations of the warehouse capacities as b=[1; 16] . The bi-level solutions are:

- y^{opt}=[0.13; 2.31] , x^{opt}=[10.28; 88.01],
- the costs for the inventory of the foods for milk and beans, evaluated with the goal function of (10) is $F(y^{opt}(x^{opt}))$ =558.33,
- the production profit from the production problem evaluated with the goal function (9) is
- $f(x^{opt}(y^{opt}))=x^{Topt}\,Q2\,x^{opt}$= 44018,
- the total benefit of the bi-level problem with the costs $P\,(y^{opt}(x^{opt}))$ from the integration of the delivery and production is

$$P\,(y^{opt}(x^{opt}))= f(x^{opt}(y^{opt})) - F(y^{opt}(x^{opt}))=43460$$

The bi-level solutions are assessed by comparing the inventory problem (6), which does not consider the production requirements. The solution of the inventory problem (6) gives

yn^{opt}= [0.34; 1.98] .

These volumes of foods are used to evaluate the content of fat and carbohydrates. The calculations are performed according to relations (12) With this food resource for milk and beans the corresponding amounts of fat and carbohydrate xin according to the relations between x and y, defined by constraints (11) gives:

$$xin^{opt} = A\,yn^{opt} = [9.91;\ 79.94].$$

This production is less from the bi-level one x^{opt}=[10.28; 88.01],

By the same approach the costs for the inventory is calculated by the goal function of (6) $Fn(yn^{opt}(xin^{opt}))$ =452,58,

The total profit with the solutions yn^{opt} and the calculated xin^{opt} is found as:
$$fn(xin^{opt}(yn^{opt}))=\ xin^{Topt}\,Q2\,xin^{opt}=36869.$$

The total benefit $Pn\,(yn^{opt}(xin^{opt}))$ from of the inventory problem (6) is:
$$Pn\,(yin^{opt}(xin^{opt}))=$$
$$fn(xin^{opt}(yin^{opt})) - Fn(yin^{opt}(xin^{opt}))=36416.$$

The comparison between the total benefits for the bi-level problem (12) and the inventory-only problem (6) gives superior results for the bi-level optimization,
$$Benefit= P\,(y^{opt}(x^{opt}))- Pn\,(yin^{opt}(xin^{opt}))=\textbf{7044}$$

This comparison gives advantages for the bi-level formalization (12) in comparison with the simple inventory (6). The benefits come from the tight connections between the amount of production and the amount of needed resources. The resource is delivered without excess, which makes an additional profit for the production.

5 Conclusion

This research derives a bi-level optimization problem, which takes additional requirements towards inventory management. These additional requirements concern the integration between the inventory and the production relations. The bi-level optimization has been performed for the case of diet definitions. The be-level formalization has advantages because it applies simultaneous optimization with two goal functions in hierarchical order. The second goal function enters appropriately into the constraints of the first goal function. Thus

the evaluated optimal solution respects the minimum of inventory costs and maximum of the production of the bi-level problem. The bi-level problem demonstrates superiority from an economic point of view. There are no excess resources for production, which decreases the costs for the inventory of resources. Numerically this has been quantified by comparisons of the costs for the bi-level problem and the inventory one. The potential future development of this bi-level formalization can be the additions to the requirements of the inventory and production processes. Formally, this will give additional constraints for the bi-level problem. Thus, the optimization solutions can satisfy more complicated and/or specific requirements, which are close to the physical and material environments.

References:
[1] J. A. Buzacott and R.Q. Zhang, Inventory Management with Asset-Based Financing, *Management Science*, 50(9),2004, pp. 1274-1290. https://doi.org/10.1287/mnsc.1040.0278.

[2] N. Vandeput, *Inventory Optimization: Models and Simulations*, Berlin, Boston: De Gruyter, 2020. https://doi.org/10.1515/9783110673944.

[3] M.A. Rodriguez, G. Corsano, A. Vecchietti, J.M. Montagna, Simultaneous optimization of production planning and inventory management of polyurethane foam plant, *Optimization and Engineering,* vol. 19, 2018, pp. 97–123. https://doi.org/10.1007/s11081-017-9364-3.

[4] M. A. Rodríguez, A. Vecchietti, Integrating Purchase Contract Decisions with Inventory Management Optimization in the Supply Chain, Editor(s): R. M. B. Alves, C. A. O. Nascimento, E. C. Biscaia, *Computer Aided Chemical Engineering*, Elsevier, vol. 27, 2009, pp. 315-320, https://doi.org/10.1016/S1570-7946(09)70273-1p.

[5] K. A. Tasias, Simultaneous optimization of inventory, maintenance, and quality for production systems subject to multiple mean and variance shifts, *Communications in Statistics - Theory and Methods*, 2022, DOI: 10.1080/03610926.2022.2150522

[6] G.M. Kopanos, L. Puigjaner, Simultaneous Optimization of Production and Logistics Operations in Semicontinuous Process Industries. In: *Solving Large-Scale Production Scheduling and Planning in the Process Industries*. Springer, Cham., 2019 https://doi.org/10.1007/978-3-030-01183-3_7.

[7] H. AL-Khazraji, C. Cole, W. Guo, Optimization and Simulation of Dynamic Performance of Pro-duction-Inventory Systems with Multivariable Controls. *Mathematics*, vol. 9(5), 568, 2021. https://doi.org/10.3390/math9050568

[8] R. M. Munyao, V. O. Omulo, M. W. Mwithiga, B. Chepkulei, Role of inventory management practices on performance of production department. A case of manufacturing firms. *International Journal of Economics, Commerce and Management*, United Kingdom vol. III(5), May 2015, pp. 1625-1656, [Online]. http://ijecm.co.uk/wp-content/uploads/2015/05/35107.pdf (Accessed Date: October 30, 2024).

[9] A. O. Agu. Obi-Anike, H. Ozioma, E. C. Nnate, Effect of inventory management on the organizational performance of the selected manufacturing firms, *Singaporean Journal of BuSineSS economicS, and management StudieS (SJBem)*, vol. 5, No. 4, 2016, [Online]. https://singaporeanjbem.com/pdfs/SG_VOL_5_(4)/6.pdf (Accessed Date: October 30, 2024).

[10] S. Padmanava, *Introduction to inventory management*. 2015, DOI: 10.13140/RG.2.2.14914.99522.

[11] H. A. Taha, *Operations Research. An Introduction*. Tenth Edition, Pearson, 2017. ISBN: 9780134444017, p.844.

[12] Belcho Hristov, Calories and Calorimetric Table, Food 24/07/2012 / Wellness and nutrition, [Online]. https://www.belchohristov.com/kalorimetrichna-tablica-na-hranite/ (Accessed Date: October 30, 2024).

[13] Lekbook. "Table of proteins, fats, carbohydrates and calories in some foods", [Online]. https://lekbook.com/dieti/kalorii (Accessed Date: October 30, 2024).

Contribution of Individual Authors to the Creation of a Scientific Article (Ghostwriting Policy)

- Todor Stoilov compiled the research methodology, formal analysis, collected the data included in the simulations, and writing the original draft.
- Krasimira Stoilova carried out the optimization and the simulation and editing the manuscript.

Conflict of Interest
The authors have no conflicts of interest to declare.