Optimal Exchange Rates, Savings, Reserves, and Loans: A Malliavin Calculus Approach

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Abstract: We present novel formulae for the determination of the optimal values of the exchange rate, reserves, savings, and external (foreign currency) debt. Based on current account formulae, a stochastic differential equation is developed that relates the net worth of an economy to its exchange rate, savings, reserves, and the level of external debt. Using the martingale optimality principle, the optimal values for the exchange rate, interest rate, reserves, and the external debt are derived. Applied to Egypt, we find that its actual nominal exchange rate has stayed above its optimal value since the early 1990s, except for two years, and that the actual external debt-to-net worth ratio is higher than the optimal one for most of the time.

Key-Words: - Exchange Rate, Savings, Reserves, Foreign Loans, Interest Rate, Martingales, Martingale Optimality Principle, Optimal Stochastic Control, Optimal Debt.

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1 Introduction

A number of domestic structural factors determine a country's current account balance, deficit or surplus. If a deficit reflects higher investment than savings, this can be beneficial for the respective economy's long-run growth. On the other hand, a deficit resulting from higher imports than exports may indicate competitiveness shortcomings. For current account surplus countries, while a surplus helps strengthen the exchange rate and reduce dependency on external finance, if demographic factors and/or falling levels of investment underpin this surplus, these features can be negative for their long-run economic growth, [1].

The relation between current account deficits and financial crises has extensively been analyzed by applied-economic researchers, as it constitutes a major challenge for policymakers in view of the enormous costs associated with financial crises. There are several types of financial crises, including currency crises [2], [3], [4], capital market access sudden stops [6], [7], banking crises [8], and balance-of-payments/foreign xchange/sovereign debt crises [9], [10], [11]. Sometimes crises are correlated with each other and emerge as a twin crisis, e.g., banking and balance-of-payments crises, [6] or triple crisis, e.g., currency-banking-balance of payments crises, [12], which usually becomes difficult to disentangle. Several theoretical models suggest three generations of financial crisis models for the fundamental explanation of crises, [13], [14], [15]. Other theoretical models are based on stochastic control, [16], [17].

The large costs of financial crises arise from the vast disruptions that they generate, including economic slowdown, output losses, unemployment, financial instability, political instability, and social problems, [8], 18], [19], [20], [21], [22], [23]. These economic consequences lead to loss of confidence among investors, often causing decreased investment levels and capital outflows. Further, such consequences can become even more pronounced with the simultaneous or sequential occurrence of other crises. The latest example of a cascading crisis is the sovereign debt crisis in 2010–2019 that was succeeded by the COVID-19-induced supply-lines disruption crisis in 2020-2023 and was compounded by the Russia-Ukraine war-induced global inflation crisis in 2022-2024. Due to these multidimensional consequences, developing a mathematical model that encompasses the leading factors of a multifaceted financial crisis is a major challenge for modelers and policymakers, with the difficulties being amplified by the dynamic nature of the factors affecting a financial crisis. As consequences are intertwined and increase in frequency and intensity, they can multiply each other's impact in a chaotic manner.

The construction of a mathematical model of a financial crisis typically involves four steps: (1) the primary step in formulating such a model is to define a financial crisis; (2) the explanatory variables need then to be selected, i.e., those variables that are likely to lead to a financial crisis if they cross a threshold or interact adversely; (3) a theoretical model should subsequently be developed that relates the different variables. The setup could be statistically-based, econometrically-based, or economics-based.; and (4) various econometric/statistical methodologies are considered to find the desired estimates. In our case, we start with a model based on conventional economic fundamentals [16] and then expand it by incorporating sovereign asset and liability management principles and balance-of-payments/foreign exchange/sovereign debt crisis considerations, [24]. The starting point is the current-account-deficit accounting equations. Stochastic calculus is used to formulate a relation between foreign debt and other economic variables such as exchange rate, reserves, savings, current account, investments, and debt/loans. We subsequently use the martingale optimality principle to find the optimal level of debt, reserves, savings, exchange rate, and domestic interest rate that are consistent with non-crisis outcomes.

In particular, we expand the model in [16] and study the interaction between the current account, real exchange rate, foreign reserves, domestic savings, domestic interest rate, inflation, and external debt in an environment where the return on capital, the domestic interest rate, the foreign interest rate, and the return on reserves are stochastic variables. Our model of such dynamic interaction reveals that an "overvalued" exchange rate, implying that the cost of an identical basket of goods is more expensive domestically than abroad at the prevailing nominal exchange rate, leads to a steady rise in the external debt (Statically, however, an overvalued exchange rate leads to a lower level of external debt in domestic currency terms). In turn, the accumulation of debt due to ensuing trade account deficits and the interest rate payments on the debt exert downward pressure on the exchange rate, which may lead to a currency (balance of payments) crisis. Specifically, a significant depreciation of the currency increases the debt burden in terms of local currency and increases the probability of a debt crisis, [16].

In this study, we examine the determination of the optimal real exchange rate that stems from the maximization of a utility function as opposed to the equilibrium real exchange rate. We find that the optimal real exchange rate is proportional to the return on domestic investments and inversely related to the U.S. interest rate. This result was obtained using the Martingale Optimality Principle, [25], [26], [27], [28], [29], [30], [31].

The novel analysis of this paper is a major contribution in the relevant literature. Although the link between the real (and nominal) exchange rate, productivity, and external debt is crucial for the economic and financial stability of a country, we are not aware of any studies that have analyzed these interlinks, ([10] estimate optimal levels of foreign debt for Egypt during 1985-2008, without taking explicitly into account the dynamic interactions between foreign debt and the exchange rate).. Since both exchange rates and external debt are dynamic concepts, representation of their interlinks warrants a suitable dynamic approach. Typically, the analysis of the dynamic interaction between variables is carried out in optimal value terms. This paper develops such a framework of dynamic behavior. Such framework, appropriately calibrated for specific country conditions, can serve as a policy tool for informing government authorities in their assessments of the state of their external debt and drawing conclusions on followed exchange rate policies.

The paper is organized as follows: Section 2 presents the problem formulation of the dynamic interaction between the exchange rate, reserves, savings, and external (foreign currency) debt. Section 3 highlights the optimization problem and gives the optimal values of the exchange rate, reserves, savings, and external (foreign currency) debt. Section 4 discusses the results and provides some concluding remarks on the applicability and shortcomings of the proposed framework, as well as it points out some areas for future research. An Appendix presents all necessary derivations.

2 Problem Formulation

Our prototype model is a simplification of a complex economy that focuses on shocks (economic and financial disturbances) that have led to crises. The model is analytically tractable, with all derived equations having an economic interpretation. However, introduction of more realistic assumptions would generate a less transparent solution and economic interpretation. The prototype model is proposed as a "benchmark" model. We assume four sources of uncertainty: (1) the value of GDP and the return on capital, (2) the interest rate on foreign loans and bonds, (3) the interest rate paid for domestic savings, and (4) the interest paid for the foreign reserves. It is important to recognize that there might be a correlation between some or all of sources of uncertainty.

Adopting a stochastic calculus formulation [16], the net worth or wealth, X(t), in nominal terms is defined as:

$$X(t) = K(t) + R(t) - L(t) \tag{1}$$

and $dX(t) = dK(t) + dR(t) - dL(t)$

where
K(t) is the capital owned by the residents of the country
L(t) is the country's external debt, denominated in the $US
R(t) is the reserves, denominated in the $US
X(t) is the networth

The change in capital, dK(t), is the rate of investment I(t):

$$dK(t) = I(t)dt \tag{2}$$

The Harrod-Domar growth equation [32] provides the link between the GDP Y(t) and the capital K(t) as:

$$Y(t)dt = K(t)b(t)dt = \left(X(t) + L(t) - R(t)\right)b(t)dt$$

where b(t) is the return on investments.

Also, real consumption C(t), real investments I(t), and real GDP Y(t) are related through the current account equation, [33]:

$$dL(t) - dR(t) = \left[I(t) - S(t)\right]dt \tag{3}$$

In [34], the savings was added to yield:

$$S(t)dt = \left[Y(t) + i_s(t)S(t) + i_r(t)R(t) - C(t) - i(t)L(t)\right]dt \tag{4}$$

where
S(t) domestic savings
C(t) consumption
Y(t) GDP
i_s interest paid on domestic savings
i_r interest paid on foreign reserves

By substituting equation (4) into equation (3), we get:

$$dL(t) - dR(t) = \begin{bmatrix} I(t) - i_s(t)S(t) - i_r(t)R(t) \\ + C(t) - Y(t) + i(t)L(t) \end{bmatrix} dt \tag{5}$$

By rearranging, we get:

$$C(t)dt = \left[Y(t) - I(t) + i_s(t)S(t) + i_r(t)R(t) - i(t)L(t)\right]dt + dL(t) - dR(t) \tag{6}$$

In terms of the nominal values, we get:

$$p(t)C(t)dt = \left[p(t)Y(t) - p(t)I(t) - e(t)i(t)L(t) + p(t)i_s(t)S(t) + e(t)i_r(t)R(t)\right]dt + e(t)dL(t) - e(t)dR(t) \tag{7}$$

where p(t) is the domestic price index, e(t) is the nominal exchange rate (domestic price of foreign exchange or units of domestic currency per unit of $US). Note that the formulation of this equation excludes government expenditures and assumes that the external sector is represented by the change in the external debt, dL(t), the interest rate payments on the debt, i(t)L(t), the change in reserves dR(t), and interest payment on the reserves.

In particular, as we assume that the external debt L(t) is denominated in $US, the term i(t)L(t) stands for the interest payments in $US, at the rate of interest i(t), on US dollar denominated loans and bonds. Further, we assume that the accumulation of debt refers to annual intervals.

By dividing equation (7) by p(t), we obtain the real consumption C(t) (measured in domestic-goods units) as:

$$C(t)dt = \left[Y(t) - I(t) - \frac{e(t)}{p(t)} i(t)L(t) + i_s(t)S(t) + \frac{e(t)}{p(t)} i_r(t)R(t) \right] dt + \frac{e(t)}{p(t)} dL(t) - \frac{e(t)}{p(t)} dR(t) \tag{8}$$

Unlike other models, we do not assume "Purchasing Power Parity" (PPP) or the "Law of One Price", i.e., $e(t)/p(t) \neq 1$. As a matter of fact we estimate the exchange rate.

Further, by rearranging terms in equation (7), we get:

$$dL(t) = dR(t) + i(t)L(t)dt + i_r(t)R(t)dt + \frac{p(t)}{e(t)} \left[C(t) + I(t) - Y(t) - i_s(t)S(t) \right] dt \tag{9}$$

Now, we develop the SDE for the net worth, X(t) , X(t), K(t) in LE, and L(t) and R(t) in \$US:

$$\begin{aligned} dX(t) &= \frac{e(t)}{p(t)} \left(dR(t) - dL(t) \right) + dK(t) \\ &= I(t)dt + \frac{e(t)}{p(t)} \left(dR(t) - dL(t) \right) \\ &= I(t)dt - \frac{e(t)}{p(t)} \left\{ \begin{aligned} & i(t)L(t)dt + i_r(t)R(t)dt \\ & + \frac{p(t)}{e(t)} \left[C(t) + I(t) - Y(t) - i_s(t)S(t) \right] dt \end{aligned} \right\} \\ &= I(t)dt - \frac{e(t)}{p(t)} \left[i(t)L(t)dt + i_r(t)R(t)dt \right] - \left[C(t) + I(t) - Y(t) - i_s(t)S(t) \right] dt \\ &= -\frac{e(t)}{p(t)} \left[i(t)L(t)dt + i_r(t)R(t)dt \right] - \left[C(t) - Y(t) - i_s(t)S(t) \right] dt \end{aligned} \tag{10}$$

By substituting:

$Y(t)dt = K(t)b(t)dt = \left(X(t) + L(t) - R(t) \right) b(t)dt$

in equation (10),

we get:

$$\begin{aligned} dX(t) &= -\frac{e(t)}{p(t)} \left[i(t)L(t)dt + i_r(t)R(t)dt \right] - \left[C(t) - \left(X(t) + L(t) - R(t) \right) b(t) - i_s(t)S(t) \right] dt \\ &= \left[-\frac{e(t)}{p(t)} i(t) + b(t) \right] L(t)dt + \left(-\frac{e(t)}{p(t)} i_r(t) - b(t) \right) R(t)dt - C(t)dt + X(t)b(t)dt + i_s(t)S(t)dt \end{aligned}$$

By dividing by X(t), we get:

$$\frac{dX(t)}{X(t)} = \left[-\frac{e(t)}{p(t)} i(t) + b(t) \right] \frac{L(t)}{X(t)} dt + \left(-\frac{e(t)}{p(t)} i_r(t) - b(t) \right) \frac{R(t)}{X(t)} dt - \frac{C(t)}{X(t)} dt + b(t)dt + i_s(t) \frac{S(t)}{X(t)} dt \tag{11}$$

Define $l(t) = \frac{L(t)}{X(t)}$, $r(t) = \frac{R(t)}{X(t)}$, $c(t) = \frac{C(t)}{X(t)}$,

$$s(t) = \frac{S(t)}{X(t)} \tag{12}$$

then,

$$\begin{aligned} \frac{dX(t)}{X(t)} &= \left[-\frac{e(t)}{p(t)} i(t) + b(t) \right] l(t)dt + \left(-\frac{e(t)}{p(t)} i_r(t) - b(t) \right) r(t)dt - c(t)dt + b(t)dt + i_s(t)s(t)dt \\ &= -c(t)dt + \left[-\frac{e(t)}{p(t)} i(t)l(t) - \frac{e(t)}{p(t)} i_r(t)r(t) + i_s(t)s(t) \right] dt + \left(l(t) - r(t) + 1 \right) b(t)dt \end{aligned} \tag{13}$$

Assuming that the rate of interest i(t) on US dollar-denominated loans and bonds can be represented by the following process:

$$i(t) = i + \sigma_i \frac{dW_i(t)}{dt}$$

$$\text{i.e., } i(t)dt = idt + \sigma_i dW_i(t) \tag{14}$$

Similarly, for the interest on the reserves and the savings, we get:

$$i_r(t) = i_r + \sigma_{i_r} \frac{dW_{i_r}(t)}{dt}$$

$$\text{i.e., } \quad i_r(t)dt = i_r dt + \sigma_{i_r} dW_{i_r}(t) \tag{15}$$

and $$i_s(t) = i_s + \sigma_{i_s} \frac{dW_{i_s}(t)}{dt}$$

$$\text{i.e., } \quad i_s(t)dt = i_s dt + \sigma_{i_s} dW_{i_s}(t) \tag{16}$$

The other source of uncertainty is the return on investments, b(t), which can be represented by the following process:

$$b(t)dt = bdt + \sigma_b dW_b(t) \tag{17}$$

By substituting equations (14-17) equation (13), we get:

$$\frac{dX(t)}{X(t)} = -c(t)dt + \begin{bmatrix} -\frac{e(t)}{p(t)}\left(idt + \sigma_i dW_i(t)\right)l(t) \\ -\frac{e(t)}{p(t)}\left(i_r dt + \sigma_{i_r} dW_{i_r}(t)\right)r(t) \\ +\left(i_s dt + \sigma_{i_s} dW_{i_s}(t)\right)s(t) \end{bmatrix}$$
$$+\left(l(t) - r(t) + 1\right)\left(bdt + \sigma_b dW_b(t)\right)$$

By collecting terms, we get:

$$\frac{dX(t)}{X(t)} = -c(t)dt + \begin{bmatrix} -\frac{e(t)}{p(t)} il(t) - \frac{e(t)}{p(t)} i_r r(t) \\ +i_s s(t) + \left(l(t) - r(t) + 1\right)b \end{bmatrix} dt$$
$$+\begin{bmatrix} -\frac{e(t)}{p(t)} l(t)\sigma_i dW_i(t) - \frac{e(t)}{p(t)} r(t)\sigma_{i_r} dW_{i_r}(t) \\ +s(t)\sigma_{i_s} dW_{i_s}(t) + \left(l(t) - r(t) + 1\right)\sigma_b dW_b(t) \end{bmatrix} \tag{18}$$

$$\text{Define } \underline{\pi}(t) = \begin{bmatrix} \pi_1(t) \\ \pi_2(t) \\ \pi_3(t) \\ \pi_4(t) \end{bmatrix} = \begin{bmatrix} \left(\frac{e(t)}{p(t)}\right) l(t) \\ \left(\frac{e(t)}{p(t)}\right) r(t) \\ s(t) \\ \left(1 + l(t) - r(t)\right) \end{bmatrix} \tag{19}$$

The new equation for the system dynamics becomes:

$$\frac{dX(t)}{X(t)} = -c(t)dt + \left[-i\pi_1(t) - i_r\pi_2(t) + i_s\pi_3(t) + b\pi_4(t)\right]dt$$
$$+\begin{bmatrix} -\pi_1(t)\sigma_i dW_i(t) - \pi_2(t)\sigma_{i_r} dW_{i_r}(t) + \pi_3(t)\sigma_{i_s} dW_{i_s}(t) \\ +\pi_4(t)\sigma_b dW_b(t) \end{bmatrix} \tag{20}$$

$$\text{Define } \sigma = \begin{bmatrix} -\sigma_i & 0 & 0 & 0 \\ 0 & -\sigma_{i_r} & 0 & 0 \\ 0 & 0 & \sigma_{i_s} & 0 \\ 0 & 0 & 0 & \sigma_b \end{bmatrix},$$

$$d\underline{W}(t) = \begin{bmatrix} dW_i(t) \\ dW_{i_r}(t) \\ dW_{i_s}(t) \\ dW_b(t) \end{bmatrix},$$

$$\underline{\theta}(t) = \sigma^{-1} \begin{bmatrix} -i \\ -i_r \\ i_s \\ b \end{bmatrix} = \begin{bmatrix} i/\sigma_i \\ i_r/\sigma_{i_r} \\ i_s/\sigma_{i_s} \\ b/\sigma_b \end{bmatrix}$$

$$d\underline{W}_Q(t) = d\underline{W}(t) + \underline{\theta}(t)dt,$$

$$Z(t) = \exp\left[-\int_0^t \underline{\theta}^T(s)d\underline{W}(s) - \frac{1}{2}\int_0^t \left\|\underline{\theta}(s)\right\|^2 ds\right] \tag{21a}$$

$$D_s Z(t) = Z(t) D_s \left[-\int_0^t \underline{\theta}^T(s)d\underline{W}(s) - \frac{1}{2}\int_0^t \left\|\underline{\theta}(s)\right\|^2 ds\right]$$
$$= -Z(t)\underline{\theta}^T(s) 1(s)_{s\in[0,t]} \tag{21b}$$

where $D_s Z(t)$ is the Malliavin derivative of Z(t), and

$$Z^\alpha(t) = \exp\left[-\alpha\int_0^t \underline{\theta}^T(s)d\underline{W}(s) - \frac{1}{2}\alpha\int_0^t \left\|\underline{\theta}(s)\right\|^2 ds\right]$$

$$\text{with } D_s Z^\alpha(t) = -\alpha Z(t)\underline{\theta}^T(s) 1(s)_{s\in[0,t]} \tag{21c}$$

where $1(s)_{s\in[0,t]} = \begin{matrix} 1 & s\in[0,t] \\ 0 & \text{elsewhere} \end{matrix}$

Then, $$\frac{dX(t)}{X(t)} = -c(t)dt + \underline{\pi}^T(t)\sigma\, d\underline{W}_Q(t) \quad (22)$$

Note that $\underline{\theta}(t)$ is a deterministic quantity. Thus, the sigma field generated by $\underline{\mathrm{W}}$(t) is the same as that generated by $\underline{W}_Q(t)$.

Then, the unknowns are: (1) c(t)=C(t)/X(t), (2) l(t)=L(t)/X(t), (3) $r(t)=\frac{R(t)}{X(t)}$, (4) $s(t)=\frac{S(t)}{X(t)}$ and (5) e(t)/p(t).

3 Problem Solution

The objective of this study is to find the optimal values of the normalized consumption, the normalized foreign debt, the normalized reserves, the normalized savings, the domestic saving rate and the real exchange rate. The optimization criterion is rethe expected value of the utility of the networth and consumption at time T (the end of the optimization period). Specifically, we need to find:

$$\max_{c(s),\underline{x}(T)} E\left\{\int_0^T e^{-\rho s}U_c(c(s))ds + U_x\left(X(T)\right)\right\} \quad (23)$$

subject to the dynamic constraints:

$$\frac{dX(t)}{X(t)} = -c(t)dt + \underline{\pi}^T(t)\sigma\, d\underline{W}_Q(t) \quad (22) \quad \text{and}$$

$X(0)=x$

where ρ is the discount rate in the two utility functions, $U_c(c(s))$, the utility function of households, and $U_x(x(T))$, the utility function of the final value of the economy's net worth. The utility functions, $U_c(c(s))$ reflects the importance of increasing the public welfare. The utility function $U_x(x(T))$, emphasizes the importance of the net worth of the society at some future time T.

The optimization methodology is based on the martingale optimality principle. The result of the optimization will be the optimal values of c(t), l(t), r(t), s(t), and e(t)/p(t). To find the optimal values, the basic idea is to find two equations for the networth X(t). The first is obtained from the system dynamics; equation (22). The second equation is obtained through the optimization process. Equating both formulae will yield the unknown controllers $\underline{\pi}(t)$.

In this approach [35], we decouple the problem of determining the optimal terminal wealth profile from the problem of determining the optimal portfolio. Instead of solving the dynamic control problem, we solve the static problem i.e. we find the optimal values for X(T) and C(t) that maximize eqn. (23) subject to the constraint X(0)=x. Given the optimal wealth profile X*(T) and optimal consumption C*(t) , we compute the corresponding generating portfolio $\underline{\pi}(t)$. The last step is achieved using the Clark-Ocone formula. The details of the derivations are given in Appendix A.

After some manipulations (Appendix A), we obtain the optimal exchange rate as:

$$e(t) = \frac{\left(\frac{i\sigma_b^2}{l(t)} + i\sigma_b^2\right)}{b\sigma_i^2}\left(\frac{p(t)}{p^*(t)}\right) = \frac{i/\sigma_i^2}{b/\sigma_b^2}\left(\frac{1+l(t)-r(t)}{l(t)}\right)\left(\frac{p(t)}{p^*(t)}\right) \quad (A.29)$$

As the productivity, b, increases, the exchange rate goes down (appreciates).

By rearranging, we get the optimal normalized value for the foreign loans $l(t) = L(t)/X(t)$ as:

$$l(t) = \frac{i\sigma_b^2(1-r(t))}{\left[b\sigma_i^2\left(\frac{e(t)p^*(t)}{p(t)}\right) - i\sigma_b^2\right]} = \frac{i(1-r(t))}{\left[b\left(\frac{\sigma_i^2}{\sigma_b^2}\right)\left(\frac{e(t)p^*(t)}{p(t)}\right) - i\right]} \quad (A.30)$$

On the limit when the interest rate on foreign loans "*i*" goes very high, l(t) becomes negative, i.e., we are better off lending the money than borrowing it. This is a desired result.

Impact of Savings:

To study the impact of the savings on the foreign loans and reserves, we look at the ratio

$$\frac{\pi_s(t)}{\pi_4(t)} = \frac{i_s / \sigma_s^2}{b / \sigma_b^2} = \frac{i_s \sigma_b^2}{b \sigma_s^2} = \frac{s(t)}{\left(1 + l(t) - r(t)\right)} \qquad \text{(A.31)}$$

when the interest rate on local savings i_s increases, we have the following scenarios: (1) Savings s(t) increases, (2) reserves r(t) increases, (3) foreign loans decrease, and (4) a combination of the previous scenarios.

To study the impact of the savings on the exchange rate, we look at the ratio:

$$\frac{\pi_1(t)}{\pi_3(t)} = \frac{i / \sigma_i^2}{i_s / \sigma_{i_s}^2} = \frac{i \sigma_{i_s}^2}{i_s \sigma_i^2} = \frac{\frac{e(t)}{p(t)} l(t)}{s(t)} \qquad \text{(A.32)}$$

when the interest rate on local savings i_s increases, we have the following scenarios: (1) savings s(t) increases, (2) exchange rate e(t) decreases, (3) inflation increases, (4) foreign loans decrease, and (5) a combination of the previous scenarios.

In this section, we apply the above derived optimal values to the Egyptian economy. The available data was between 1985 and 2016.

Equation (A. 29) could be written as:

$$e(t) = \frac{\left(\frac{i\sigma_b^2}{l(t)} + i\sigma_b^2\right)}{b\sigma_i^2} \left(\frac{p(t)}{p^*(t)}\right) = \frac{i / \sigma_i^2}{b / \sigma_b^2} \left(\frac{1 + l(t) - r(t)}{l(t)}\right) \left(\frac{p(t)}{p^*(t)}\right) \qquad \text{(A.29)}$$

where $p^*(t)$, the US consumer price index, set at 1 in the previous analysis.

By taking the logarithm (ln) of both sides, we get:

$$\ln e(t) = \left[\ln p(t) - \ln p^*(t)\right] + \left[\ln \frac{i}{\sigma_i^2} - \ln \frac{b}{\sigma_b^2}\right] + \ln\left(\frac{1 + l(t) - r(t)}{l(t)}\right) \qquad (24)$$

Equation (24) is a familiar form, with some empirical models being cast in this format, [36], [37], [38].

Below, using available data for Egypt, we present our model's findings about actual, optimal, and PPP exchange rates.

Figure 1 (Appendix) shows the actual and approximate optimal external debt ratios, l(t)=L(t)/X(t), for Egypt. We display both conservative (γ=higher) and risky (γ=lower) estimates for the approximate optimal external debt. As shown in Figure 1 (Appendix), the actual external debt-to-net worth ratio is lower than the risky approximate optimal external debt-to-net worth ratio for almost the entire period analyzed, 1985-2017 (with the exception of 1991 and 2011). This marked a stable period during the tenure of president Mubarak. The actual external debt-to-net worth ratio is higher than the conservative approximate optimal external debt-to-net worth ratio for the period 1985-1995; is about equal between 1996-2010; and is higher again during 2011-2017 after the revolution of 2011 where Mubarak was toppled. These results indicate that Egypt's actual external debt-to-net worth ratio was higher than the conservative approximate optimal external debt-to-net worth ratio for 18 out of the 33 years in the sample, implying that the country had contracted more debt than it should during these years. Egyptian balance-of-payments and sovereign debt developments in subsequent years, after 2017, justify these empirical findings.

In Figure 2 (Appendix) we present the estimated approximate optimal exchange rate for Egypt, the nominal (official) exchange rate and the purchasing power parity (PPP) exchange rate (using WDI data for annual inflation P(t), the PPP and the nominal (official) exchange rates).

As shown in Figure 2 (Appendix), Egypt's nominal (official) exchange rate (LE/$US) was much less than its optimal values during 1985-1994. This is a reflection of government intervention to keep low prices. The exchange rate then stayed consistently above its optimal values during 1998-2008. Two devaluations occurred in 1991 and 2002. Starting in 2012, the nominal exchange rate is above its optimal values, and converging in 2015-2016.

4 Conclusions

This paper studies the interaction between the current account, real exchange rate, foreign reserves, domestic savings, domestic interest rate, inflation, and external debt in an environment where the return on capital, the domestic interest rate, the foreign interest rate, and the return on reserves are stochastic variables. The calibrated model of such dynamic interaction reveals that an "overvalued" exchange rate leads to a steady rise in the external debt. In turn, the accumulation of debt due to ensuing trade account deficits and the interest rate payments on the debt exert downward pressure on the exchange rate, which may lead to a currency (balance of payments) crisis. Specifically, a significant depreciation of the currency increases the debt burden in terms of local currency and increases the probability of a debt crisis.

It should be noted that the determination of the optimal real exchange rate stems from the maximization of a utility function as opposed to the equilibrium real exchange rate. We find that the optimal real exchange rate is proportional to the return on domestic investments and inversely related to the U.S. interest rate. Further, we analyze the link between the real (and nominal) exchange rate, productivity, and external debt. These interlinks are crucial for the economic and financial stability of a country and, as far we know, it is the first time that are studied. Since both exchange rates and external debt are dynamic concepts, representation of their interlinks (i.e., the dynamic interaction between these variables analyzed by utilizing a dynamic approach) is presented in optimal value terms. The developed framework can be appropriately adapted for specific country conditions and serve as a policy tool for informing government authorities in assessing the state of their external debt and drawing conclusions on followed exchange rate policies.

The results of our model, as of any model, to be used for policy analysis, need refining and expansion of model parameters. The production function for example could be changed. The relationship between investments and GDP could also have other elements. Future research could add more variables such as government expenditure, money supply and others.

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APPENDIX

In this appendix, we present the martingale optimality principle, [7], [29], [31], [39], [40] and how to used to find the optimal value of the controllers $\underline{\pi}(t)=\begin{bmatrix}\pi_1(t)\\ \pi_2(t)\\ \pi_3(t)\\ \pi_4(t)\end{bmatrix}=\begin{bmatrix}\left(\dfrac{e(t)}{p(t)}\right)l(t)\\ \left(\dfrac{e(t)}{p(t)}\right)r(t)\\ s(t)\\ (1+l(t)-r(t))\end{bmatrix}$. The basic idea is to find two equations for the networth X(t). The first is obtained from the system dynamics, while the second equation is obtained through the optimization process. Equating both formulae will yield the unknown controllers $\underline{\pi}(t)$.

First, we state the SDE of the net worth, X(t), as given by equation (22):

$$\frac{dX(t)}{X(t)}=-c(t)dt+\underline{\pi}^T(t)\sigma\, d\underline{W}_Q(t) \quad (22)$$

which could be written as:

$$dX(t)=-C(t)dt+X(t)\underline{\pi}^T(t)\sigma\, d\underline{W}_Q(t) \quad \text{(A. 1)}$$

i.e., $dX(t)+C(t)dt=X(t)\underline{\pi}^T(t)\sigma\, d\underline{W}_Q(t)$ (A.2)

By integrating, we get:

$$X(t)+\int_0^t C(s)ds=X(0)+\int_0^t X(s)\underline{\pi}^T(s)\sigma\, d\underline{W}_Q(s) \quad \text{(A.3)}$$

This quantity "$X(t)+\int_0^t C(s)ds$" is a martingale under the measure Q.

Thus,

$$E_Q\left\{X(T)+\int_0^T C(s)ds/\mathcal{F}_t\right\}=X(t)+\int_0^t C(s)ds \quad \text{(A.4)}$$

and satisfies the equation:

$$E_Q\left\{X(T)+\int_0^T C(s)ds/\mathcal{F}_t\right\}=X(0)+\int_0^t X(s)\underline{\pi}^T(s)\sigma\, d\underline{W}_Q(s) \quad \text{(A.5)}$$

Equation (A.5) is also valid for the optimal values X*(T) and C*(t).

This is the first equation in the unknowns $\underline{\pi}(t)$. We need another equation for

$$E_Q\left\{X^*(T)+\int_0^T C^*(s)ds/\mathcal{F}_t\right\}.$$

We use the generalized Clark-Ocone formula, [39]:

For a random variable F(T) and under general conditions:

$$E_Q\{F(T)/\mathcal{F}_t\}=E\{F(T)Z(T)\}+\int_0^t\left[E_Q\left\{\begin{matrix}D_sF(T)\\ -F(T)\int_s^T D_s\underline{\theta}(u)d\underline{W}_Q(u)/\mathcal{F}_s\end{matrix}\right\}\right]^T d\underline{W}_Q(s) \quad \text{(A.6)}$$

Note that $D_s\underline{\theta}(u)$, the Malliavin derivative of $\underline{\theta}(u)$, is a matrix.

By setting $F(T)=X^*(T)+\int_0^T C^*(s)ds$, and equating (A.5) and (A.6), we get:

$$X^*(s)\underline{\pi}^T(s)\sigma=\left[E_Q\left\{\begin{matrix}D_sF(T)\\ -F(T)\int_s^T D_s\underline{\theta}(u)d\underline{W}_Q(u)/\mathcal{F}_s\end{matrix}\right\}\right]^T$$

By taking the transpose of both sides, we get:

$$X^*(s)\sigma^T\underline{\pi}(s)=E_Q\left\{\begin{matrix}D_sF(T)\\ -F(T)\int_s^T D_s\underline{\theta}(u)d\underline{W}_Q(u)/\mathcal{F}_s\end{matrix}\right\} \quad \text{(A.7)}$$

For deterministic $\underline{\theta}(t)$, as in our case, $D_s\underline{\theta}(u)=0$, we get:

$$X^*(s)\sigma^T \underline{\pi}(s) = E_Q\left\{D_s F(T)/\mathcal{F}_s\right\}$$
$$= E_Q\left\{D_s\left[X^*(T) + \int_0^T C^*(u)du\right]/\mathcal{F}_s\right\}$$

i.e.,

$$X^*(s)\sigma^T \underline{\pi}(s)$$
$$= E_Q\left\{D_s X^*(T)/\mathcal{F}_s\right\} + E_Q\left\{D_s\left[\int_0^T C^*(u)du\right]/\mathcal{F}_s\right\} \quad \text{(A.8)}$$

We need to find an expression for the optimal networth X*(T) and the optimal consumption C*(t) that satisfy equation (A.8). This is obtained through the maximization of the utility function. We then find their Malliavin derivatives and substitute into equation (A.8) to get the second expression.By taking the expectation of both sides of (A.3), we get:

$$E_Q\left\{X(t) + \int_0^t C(s)ds\right\} = X(0)$$
$$+ E_Q\left\{\int_0^t X(s)\underline{\pi}^T(s)\sigma\, d\underline{W}_Q(s)\right\} = X(0) \quad \text{(A.9)}$$

The optimization:

For the purposes of this analysis, we define the utility function of the consumption as:

$$U(C(t)) = \frac{C^{1-\gamma}}{1-\gamma},\ \gamma > 0, \gamma \neq 1 \quad \text{(A.10a)}$$

And the utility function of the final wealth:

$$U(X(T)) = \frac{X(T)^{1-\gamma_x}}{1-\gamma_x},\ \gamma_x > 0, \gamma_x \neq 1 \quad \text{(A.10b)}$$

Also, we define the objective function as:

$$V(X(0)) = \max_{C(s),X(T)} E\left\{\begin{array}{l}\int_0^T e^{-\rho s}U(C(s))ds \\ +U_x(X(T))\end{array}\right\} \quad \text{(A.11)}$$

Thus, we have two utility functions, $U(C(t))$ and $U_x(X(T))$. The first reflects the desire to increase the public welfare or consumption, while the second one reflects the desire to increase the net worth of the society at some future time T.

The optimization problem could now be stated as follows:

Find C(t) and X(T) that maximize $V(X(0))$,

$$V(X(0)) = \max_{C(s),x(T)} E\left\{\begin{array}{l}\int_0^T e^{-\rho s}U(C(s))ds \\ +U_x(X(T))\end{array}\right\} \quad \text{(A.11)}$$

subject to the constraint:

$$E_Q\left\{X(t) + \int_0^T C(s)ds\right\} = E\left\{Z(T)X(T) + \int_0^T Z(s)C(s)ds\right\}$$
$$= X(0) \quad \text{(A.9)}$$

Using the method of the Lagrange multiplier, we need to find:

$$\max_{C(s),X(T)} E\left\{\int_0^T e^{-\rho s}\frac{C(s)^{1-\gamma}}{1-\gamma}ds + U_x(X(T))\right\}$$
$$-\Lambda\left[E\left\{Z(T)X(T) + \int_0^T Z(s)C(s)ds\right\} - X(0)\right] \quad \text{(A.12)}$$

which has the form:

$$\max_{C(s),X(T)} E\left\{\begin{array}{l}\int_0^T\left[e^{-\rho s}\frac{C(s)^{1-\gamma}}{1-\gamma} - \Lambda Z(s)C(s)\right]ds \\ +U_x(X(T)) \\ -\Lambda\left[Z(T)X(T) - X(0)\right]\end{array}\right\} \quad \text{(A.13)}$$

where Λ is the Lagrange multiplier. Assuming that the conditions for the exchange of derivative and expectation are satisfied, taking the derivative for C(t) we get:

$$\frac{\partial}{\partial C(s)} E\left\{\int_0^T\left[e^{-\rho s}\frac{C(s)^{1-\gamma}}{1-\gamma} - \Lambda Z(s)C(s)\right]ds\right\} = 0 \quad \text{(A.14)}$$

i.e., $$\frac{\partial}{\partial C(s)}\left[e^{-\rho s}\frac{C(s)^{1-\gamma}}{1-\gamma}-\Lambda Z(s)C(s)\right]=0 \tag{A.15}$$

which yields:

$$e^{-\rho s}C^*(s)^{-\gamma}-\Lambda Z(s)=0$$

i.e., $$C^*(s)^{-\gamma}=e^{\rho s}\Lambda Z(s) \tag{A.16}$$

By taking the natural log of both sides, we get:

$$-\gamma \ln C^*(s)=\rho s+\ln \Lambda Z(s)$$

and $$\ln C^*(s)=-\frac{\rho s}{\gamma}+\left(\frac{-1}{\gamma}\right)\ln \Lambda Z(s)$$

Thus,

$$C^*(t)=e^{-t\rho/\gamma}\left(\Lambda\right)^{(-1/\gamma)}\left(Z(t)\right)^{(-1/\gamma)},\ 0<t\le T\,, \tag{A. 17}$$

with Λ being a constant deterministic value. We can now find an expression for the optimal X(T):

$$\frac{\partial}{\partial X(T)}E\left\{\begin{array}{l}\int_0^T\left[e^{-\rho s}\frac{C(s)^{1-\gamma}}{1-\gamma}-\Lambda Z(s)C(s)\right]ds\\ +U_x\left(X(T)\right)\\ -\Lambda\left[Z(T)X(T)-X(0)\right]\end{array}\right\}=0 \tag{A.18}$$

i.e.,

$$\frac{\partial}{\partial X(T)}E\left\{U_x\left(X(T)\right)-\Lambda\left[Z(T)X(T)-X(0)\right]\right\}=0$$

By interchanging derivative and expectations, we get:

$$\frac{\partial}{\partial X(T)}U_x\left(X(T)\right)-\frac{\partial}{\partial X(T)}\Lambda\left[Z(T)X(T)-X(0)\right]=0$$

By setting $U_x\left(X(T)\right)=\dfrac{X(T)^{1-\gamma_x}}{1-\gamma_x}$, we get:

$$\frac{\partial}{\partial X(T)}\frac{X(T)^{1-\gamma_x}}{1-\gamma_x}-\frac{\partial}{\partial X(T)}\Lambda\left[Z(T)X(T)-X(0)\right]=0$$

which yields:

$$\left(X^*(T)\right)^{-\gamma_x}-\Lambda Z(T)=0$$

i.e., $$X^*(T)^{-\gamma_x}=\Lambda Z(T) \tag{A.19}$$

By taking the natural log of both sides, we get:

$$-\gamma_x \ln X^*(T)=\ln \Lambda Z(T)$$

and $$\ln X^*(T)=\left(\frac{-1}{\gamma_x}\right)\ln \Lambda Z(T)$$

Thus, $$X^*(T)=\left(\Lambda\right)^{(-1/\gamma_x)}\left(Z(T)\right)^{(-1/\gamma_x)} \tag{A.20}$$

and $$Z(T)X^*(T)=\left(\Lambda\right)^{(-1/\gamma_x)}\left(Z(T)\right)^{(1-1/\gamma_x)} \tag{A.21}$$

Obtaining the Second Equation in $\underline{\pi}(t)$:

We need to find the elements of equation (A.8). First we find $E_Q\left\{D_s X(T)/\mathcal{F}_s\right\}$ as:

$$E_Q\left\{D_t X^*(T)/\mathcal{F}_t\right\}=E_Q\left\{D_t\left(\left(\Lambda\right)^{(-1/\gamma_x)}Z(T)\right)^{(-1/\gamma_x)}/\mathcal{F}_t\right\}$$
$$=\left(\Lambda\right)^{(-1/\gamma_x)}E_Q\left\{D_t\left(Z(T)\right)^{(-1/\gamma_x)}/\mathcal{F}_t\right\}$$

Remember that

$$D_t\left(Z^\alpha(T)\right)=-\left(Z^\alpha(T)\right)\left[\begin{array}{l}\alpha\underline{\theta}(t)+\alpha\int_t^T D_t\underline{\theta}(s)d\underline{W}(s)\\ +\alpha\int_t^T\underline{\theta}(s)D_t\underline{\theta}(s)ds\end{array}\right]$$

For deterministic $\underline{\theta}(t)$,

$$D_t\left(Z^\alpha(T)\right)=-\alpha\left(Z^\alpha(T)\right)\underline{\theta}(t)1(t)_{t\in[0,T]}$$

$$E_Q\left\{D_t X^*(T)/\mathcal{F}_t\right\}=\left(\Lambda\right)^{(-1/\gamma_x)}\left(\frac{1}{\gamma_x}\right)\underline{\theta}(t)$$
$$E_Q\left\{\left(Z(T)\right)^{(-1/\gamma_x)}1(t)_{t\in[0,T]}/\mathcal{F}_t\right\}$$

$$=\left(\frac{1}{\gamma_x}\right)\underline{\theta}(t)E_Q\left\{X^*(T)1(t)_{t\in[0,T]}/\mathcal{F}_t\right\} \tag{A.22}$$

Recall that

$$Z^\alpha(t)=\exp\left[-\alpha\int_0^t\underline{\theta}^T(s)d\underline{W}(s)-\frac{1}{2}\alpha\int_0^t\left\|\underline{\theta}(s)\right\|^2ds\right],$$

with $D_s Z^{\alpha}(t) = -\alpha Z(t)\underline{\theta}^T(s)1(s)_{s\in[0,t]}$

$$\text{where } 1(s)_{s\in[0,t]} = \begin{cases} 1 & s \in [0,t] \\ 0 & \text{elsewhere} \end{cases}$$

Also, remember that

$$X(s)\sigma^T \underline{\pi}(s) = E_Q\{D_s F(T)/\mathcal{F}_s\}$$
$$= E_Q\left\{D_s\left[X(T) + \int_0^T C(u)du\right]/\mathcal{F}_s\right\}$$
$$= E_Q\{D_s X(T)/\mathcal{F}_s\} + E_Q\left\{D_s\left[\int_0^T C(u)du\right]/\mathcal{F}_s\right\} \tag{A.8}$$

We now need to find the second component of equation (A.8), namely $E_Q\left\{D_s\left[\int_0^T C^*(u)du\right]/\mathcal{F}_s\right\}$.

By substituting for the optimal C, we get:

$$E_Q\left\{D_t\left[\int_0^T C^*(u)du\right]/\mathcal{F}_t\right\}$$
$$= E_Q\left\{D_t\left[\int_0^T e^{-u\rho/\gamma}(\Lambda)^{(-1/\gamma)}(Z(u))^{(-1/\gamma)}du\right]/\mathcal{F}_t\right\} \tag{A.23}$$

$$= E_Q\left\{\int_0^T e^{-u\rho/\gamma}(\Lambda)^{(-1/\gamma)} D_t(Z(u))^{(-1/\gamma)}du/\mathcal{F}_t\right\}$$

$$= \left(\frac{1}{\gamma}\right)E_Q\left\{\int_0^T e^{-u\rho/\gamma}(\Lambda)^{(-1/\gamma)}\underline{\theta}(t)(Z(u))^{(-1/\gamma)}1(t)_{t\in[0,u]}du/\mathcal{F}_t\right\}$$

$$= \left(\frac{1}{\gamma}\right)(\Lambda)^{(-1/\gamma)}\underline{\theta}(t)E_Q\left\{\int_t^T e^{-u\rho/\gamma}(Z(u))^{(-1/\gamma)}du/\mathcal{F}_t\right\}$$
$$= \left(\frac{1}{\gamma}\right)\underline{\theta}(s)E_Q\left\{\int_s^T C^*(u)du/\mathcal{F}_s\right\}$$

By substituting into equation (A.8), where we use the optimal values, we get:

$$X^*(t)\sigma^T\underline{\pi}(t) = (\Lambda)^{(-1/\gamma_x)}\left(\frac{1}{\gamma_x}\right)\underline{\theta}(t)E_Q\left\{(Z(T))^{(-1/\gamma_x)}/\mathcal{F}_t\right\}$$
$$+\left(\frac{1}{\gamma}\right)(\Lambda)^{(-1/\gamma)}\underline{\theta}(t)E_Q\left\{\int_t^T e^{-u\rho/\gamma}(Z(u))^{(-1/\gamma)}du/\mathcal{F}_t\right\} \tag{A.24}$$

This is the second equation in the unknown vector $\underline{\pi}(t)$.

If we set $\gamma_x = \gamma$, then:

$$X^*(t)\sigma^T\underline{\pi}(t) = \left(\frac{1}{\gamma}\right)\underline{\theta}(t)\begin{bmatrix} E_Q\{X^*(T)/\mathcal{F}_t\} \\ +E_Q\left\{\int_t^T C^*(u)du/\mathcal{F}_t\right\}\end{bmatrix}$$
$$= \left(\frac{1}{\gamma}\right)\underline{\theta}(t)E_Q\left\{X^*(T) + \int_t^T C^*(u)du/\mathcal{F}_t\right\}$$

By dividing both sides by $X^*(t)$ and moving the (σ^T) to the right-hand side, we get:

$$\underline{\pi}(t) = \left(\frac{1}{\gamma}\right)(\sigma^T)^{-1}\underline{\theta}(t)\frac{1}{X^*(t)}E_Q\left\{\begin{matrix} X^*(T) \\ +\int_t^T C^*(u)du/\mathcal{F}_t\end{matrix}\right\} \tag{A.25}$$

This is the second equation into the unknowns $\underline{\pi}(t)$ as function of the optimal wealth and optimal consumption. Remember that:

$$\underline{\pi}(t) = \begin{bmatrix}\pi_1(t) \\ \pi_2(t) \\ \pi_3(t) \\ \pi_4(t)\end{bmatrix} = \begin{bmatrix}\left(\frac{e(t)}{p(t)}\right)l(t) \\ \left(\frac{e(t)}{p(t)}\right)r(t) \\ s(t) \\ (1 + l(t) - r(t))\end{bmatrix} \tag{19}$$

$$\sigma = \begin{bmatrix} -\sigma_i & 0 & 0 & 0 \\ 0 & -\sigma_{i_r} & 0 & 0 \\ 0 & 0 & \sigma_{i_s} & 0 \\ 0 & 0 & 0 & \sigma_b\end{bmatrix},\ d\underline{W}(t) = \begin{bmatrix} dW_i(t) \\ dW_{i_r}(t) \\ dW_{i_s}(t) \\ dW_b(t)\end{bmatrix},$$
$$d\underline{W}_Q(t) = d\underline{W}(t) + \underline{\theta}(t)dt,$$

Notice that the sign of the elements of the variance σ is of no value in the analysis since multiplying the Wiener process by +1 or -1 yields the same process.

$$\underline{\theta}(t) = \sigma^{-1}\begin{bmatrix} -i \\ -i_r \\ i_s \\ b \end{bmatrix} = \begin{bmatrix} i/\sigma_i \\ i_r/\sigma_{i_r} \\ i_s/\sigma_{i_s} \\ b/\sigma_b \end{bmatrix}$$

Thus, $$\sigma^{-1}\underline{\theta}(t) = \begin{bmatrix} i/\sigma_i^2 \\ i_r/\sigma_{i_r}^2 \\ i_s/\sigma_{i_s}^2 \\ b/\sigma_b^2 \end{bmatrix}$$

Recall that

$$dX(t) = -C(t)dt + X(t)\underline{\pi}^T(t)\sigma\, d\underline{W}_Q(t) \quad (22)$$

In terms of the language of control theory, $\underline{\pi}(t)$ is the unknown controllers, equation (22) is the system dynamics and equation (A.25) is the estimate of the control in a feedback form.

A Feedback Solution:

By rearranging equation (A.25), we get:

$$\underline{\pi}(t) = \left(\frac{1}{\gamma}\right)\left(\sigma^T\right)^{-1}\underline{\theta}(t)\frac{1}{X^*(t)}E_Q\left\{\begin{matrix} X^*(T) \\ +\int_t^T C^*(u)du/\mathcal{F}_t \end{matrix}\right\}$$

$$= \begin{bmatrix} \left(\frac{e(t)}{p(t)}\right)l(t) \\ \left(\frac{e(t)}{p(t)}\right)r(t) \\ s(t) \\ (1+l(t)-r(t)) \end{bmatrix}$$

$$= \left(\frac{1}{\gamma}\right)\begin{bmatrix} i/\sigma_i^2 \\ i_r/\sigma_{i_r}^2 \\ i_s/\sigma_{i_s}^2 \\ b/\sigma_b^2 \end{bmatrix}\frac{1}{X^*(t)}E_Q\left\{X^*(T) + \int_t^T C^*(u)du/\mathcal{F}_t\right\}$$

(A.26)

To get rid of the scalar quantity "$\frac{1}{X^*(t)}E_Q\left\{X^*(T) + \int_t^T C^*(u)du/\mathcal{F}_t\right\}$", and to obtain a unique solution for $l(t)$ and r(t), we use the ratios $\frac{\pi_1(t)}{\pi_2(t)}$, $\frac{\pi_1(t)}{\pi_4(t)}$ … and so on.

$$\frac{\pi_1(t)}{\pi_2(t)} = \frac{i/\sigma_i^2}{i_r/\sigma_{i_r}^2} = \frac{i\sigma_{i_r}^2}{i_r\sigma_i^2} = \frac{l(t)}{r(t)} \quad (A.27)$$

$$\frac{\pi_1(t)}{\pi_4(t)} = \frac{i/\sigma_i^2}{b/\sigma_b^2} = \frac{i\sigma_b^2}{b\sigma_i^2} = \frac{\left(\frac{e(t)p^*(t)}{p(t)}\right)l(t)}{\left(1+l(t)-r(t)\right)} \quad (A.28)$$

By rearranging, we get:

$$b\sigma_i^2\left(\frac{e(t)p^*(t)}{p(t)}\right)l(t) = i\sigma_b^2\left(1+l(t)-r(t)\right)$$

i.e.,

$$\left[b\sigma_i^2\left(\frac{e(t)p^*(t)}{p(t)}\right) - i\sigma_b^2\right]l(t) = i\sigma_b^2\left(1-r(t)\right)$$

The optimal exchange rate is given as:

$$e(t) = \frac{\left(\frac{i\sigma_b^2}{l(t)} + i\sigma_b^2\right)}{b\sigma_i^2}\left(\frac{p(t)}{p^*(t)}\right) = \frac{i/\sigma_i^2}{b/\sigma_b^2}\left(\frac{1+l(t)-r(t)}{l(t)}\right)\left(\frac{p(t)}{p^*(t)}\right) \quad (A.29)$$

As the productivity, b, increases the exchange rate goes down.

By rearranging, we get the optimal normalized value for the foreign loans $l(t) = L(t)/X(t)$ as:

$$l(t) = \frac{i\sigma_b^2\left(1-r(t)\right)}{\left[b\sigma_i^2\left(\frac{e(t)p^*(t)}{p(t)}\right) - i\sigma_b^2\right]} = \frac{i\left(1-r(t)\right)}{\left[b\left(\frac{\sigma_i^2}{\sigma_b^2}\right)\left(\frac{e(t)p^*(t)}{p(t)}\right) - i\right]} \quad (A.30)$$

This is the desired result.

To study the impact of the savings on the foreign loans and reserves, we look at the ratio:

$$\frac{\pi_s(t)}{\pi_4(t)} = \frac{i_s/\sigma_s^2}{b/\sigma_b^2} = \frac{i_s\sigma_b^2}{b\sigma_s^2} = \frac{s(t)}{(1+l(t)-r(t))} \tag{A.31}$$

When the interest rate on local savings i_s increases, we have the following scenarios: (1) Savings s(t) increases, (2) reserves r(t) increases, (3) foreign loans decrease, (4) a combination of the previous scenarios.

To study the impact of the savings on the exchange rate, we look at the ratio:

$$\frac{\pi_1(t)}{\pi_3(t)} = \frac{i/\sigma_i^2}{i_s/\sigma_{i_s}^2} = \frac{i\sigma_{i_s}^2}{i_s\sigma_i^2} = \frac{\left(\frac{e(t)p^*(t)}{p(t)}\right)l(t)}{s(t)} \tag{A.32}$$

When the interest rate on local savings i_s increases, we have the following scenarios: (1) Savings s(t) increases, (2) exchange rate e(t) decreases, (3) inflation increases, (4) foreign loans decrease, (5) a combination of the previous scenarios.

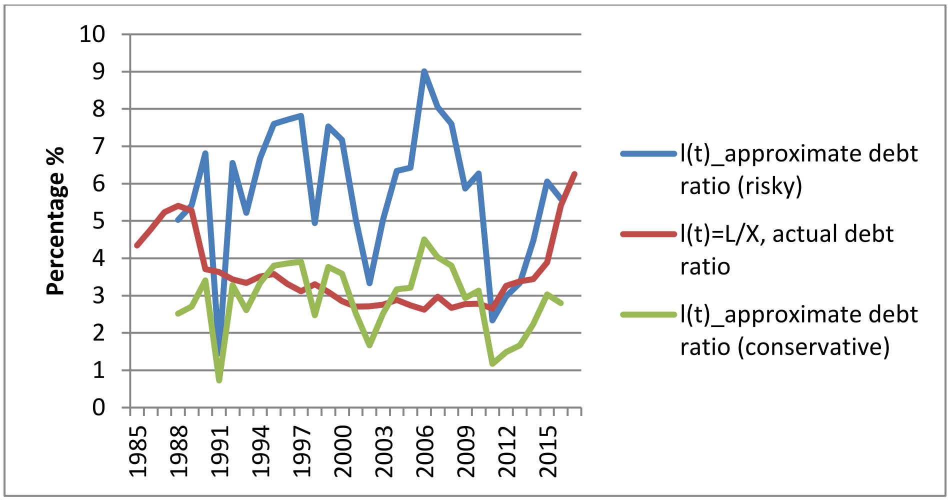

Fig. 1: Actual, Optimal (conservative), and Optimal (risky) Debt Ratios l(t)=L(t)/X(t)

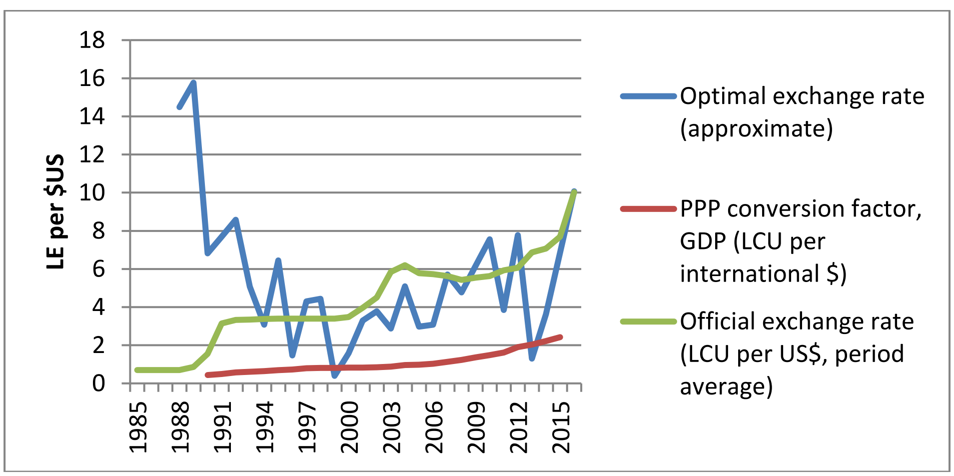

Fig. 2: Optimal, Official, and PPP Exchange Rates