A Fast-Learning Methodology for the Cournot Oligopoly Model

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Abstract: - We propose a learning process for a Cournot production model transformed into a leader-2-followers model. We focus our analysis on a learning process in the context of an oligopoly game comprising three players. By applying the training process in line with the essence of the leader-2-followers game, we achieve accelerated convergence to the equilibrium point in a Cournot game involving three players. Notably, the proposed learning process operates effectively under conditions of incomplete market information. An illustrative example underscores the application of the learning process in discerning the rational behavior of players in a Cournot game involving three players.

Key-Words: - Game theory, Oligopoly competition, Cournot model, Stackelberg model, Equilibrium point, Optimal strategies.

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1 Introduction

In the dynamic landscape of modern markets, understanding and effectively navigating the complexities of strategic decision-making is paramount for businesses seeking to maintain competitiveness and maximize profits. Oligopoly models, particularly the Cournot and Stackelberg models, serve as foundational frameworks for studying economic agent behavior and strategic interactions. What sets this study apart is its emphasis on a novel methodology, the learning process, which offers a systematic approach for players to refine their strategies over time. Through theoretical examinations, practical applications in energy markets, and insights into corporate strategy settings, we explore strategic decision-making and propose this innovative methodology as a powerful tool for guiding firms toward sustainable competitive advantage. By illuminating the intersection of game theory, market dynamics, and strategic management, this study offers valuable insights for both academic discourse and practical business application.

Oligopoly models, including the Cournot and Stackelberg models, are pivotal in understanding economic agent behavior in market theory, [1], [2]. Studies comparing welfare and profits underprice and quantity competition in mixed duopolies often prioritize Stackelberg competition, [3]. Recent research has delved deeply into oligopoly models due to their broad applicability.

For instance, [4] explored optimal strategies in a Stackelberg model with linear demand functions and nonlinear cost functions, proposing a linearization approach through Taylor series expansion. Various authors have investigated supply function equilibrium models in electricity markets, drawing on concepts like the Cournot adjustment process, [5], [6], [7], [8], [9], [10].

In today's energy markets, deregulation and competition promotion are prevailing trends. Market system operators and electricity consumers often model their behavior using oligopoly game models, including the Cournot model and dynamic Stackelberg model. These models can be extended to multiple players and different market scenarios, [11], [12], [13], as evidenced by the Networked

Stackelberg Competition (NSC) model [14], which integrates Cournot analysis at various game stages.

Cournot and Stackelberg oligopoly models are theoretically analyzed and investigated. However, this analysis assumes that each player has complete information about the options of the other players in the game. In practice, players do not know the behavior, and particularly the reaction functions of their opponents. The players find themselves in a condition of incomplete information. A theoretical study of the Stackelberg model of determining equilibrium reveals that equilibrium exists under certain conditions, with the follower having an observational advantage during the game. This advantage allows the follower to determine a strategy that yields maximum profit based on the leader's choice. However, the leader has no information about the follower's response function, which hinders strategic decision-making.

The new methodology derived in this paper accelerates the convergence process through a learning process for the leader-2-followers model. One player (chosen at random) is the leader and the other two are followers who announce their strategies simultaneously after the leader's action. The leader's strategy refers to the assumption of naive behavior of the followers at the current stage. Furthermore, the methodology applied to the leader-2-followers model converges to the Cournot equilibrium. This approach speeds up the process of reaching the equilibrium point in the three-player Cournot model.

A similar equilibrium-seeking approach in Cournot energy market models is explored in several articles [7], [8] and [9]. However, our paper differs and is novel because we propose the effective leader’s strategy for achieving the maximal profit based on the predicted responses of the follower. The proposed methodologies are aimed at how to effectively use the Stackelberg model in the conditions of incomplete information about the players. One of the goals of our research is to provide more security and independence for players in oligopoly game models.

Oligopoly games with three players: We study Cournot and Stackelberg game models with three players. We describe the learning methodology in these cases and show that the learning methodology applying a Stackelberg model at each step leads to the equilibrium of the Cournot model. In such a way, we improve the convergence to the equilibrium point.

2 Methodology

2.1 Oligopoly Games with Three Players

We are considering the learning process for a Cournot model with three players. Three firms (players) produce a homogeneous product and offer it to an oligopoly market at the same time. Depending on the demand for the product, the price p=a - b $(q_0 + q_1 + q_2)$, is determined, where coefficients a and b and the variables q_0, q_1, q_2 represent the quantities that the three producers offer on the market. In this paper, we will consider case b=1. Each player has production costs represented by $C_i(q_i)$, i=0,1,2. This is the classic Cournot model, which has been studied, extended, and commented on by various authors in numerous applications. In practice, the Cournot model is applied in terms of incomplete information, i.e. each player knows his cost function without having information about the other player's cost. In applications, the Cournot model is played successively in several periods under the same conditions in the given market. The consecutive periods of application of the Cournot model are a prerequisite for reaching the desired equilibrium in the given market. Each player uses his reaction function if the other player uses the same strategy from the previous period, i.e. each player has passive behavior. Following this approach, after several steps, the equilibrium strategies for each player are reached, if the equilibrium exists. The implementation of this approach does not depend on the number of players.

The profit functions are (i is the player’s number):
$\pi_i(q_0, q_1, q_2) = p\, q_i - C_i(q_i)$, i=0,1,2.

We suppose that the costs of the players are quadratic functions of the form $C_i(q_i) = 0.5\, c_i(q_i)^2 + d_i q_i$, i=0,1 and c_i, d_i are known constants. The assumption of quadratic player costs and payoffs is often used in-game model applications and analyses. It has been imposed in recent years based on the complexity of the models and their approximation to real processes, [11], [12], [13], [14], [15], [16].

The search-theoretical analysis of the model involves finding the maximum values of the three profit functions. After equating to zero, the first-order condition is reached for each player's response function. The condition $\frac{\partial\, \pi_i(q_0,q_1,q_2)}{\partial\, q_i} = 0$ is recorded as $q_i = R_i(q_j, q_k)$, where i,j,k =0,1,2 and $i \neq j \neq k$.

The reaction function for each player is:

$$q_i = R_i(q_j, q_k) = \frac{a - q_j - q_k - d_i}{2 + c_i}, \text{ i, j, k} = 0,1,2 \text{ and } i \neq j \neq k. \quad (1)$$

To find the equilibrium point one solves a set of three reaction functions. We must solve the set of linear equations:

$$\begin{cases} q_0 = \frac{a - q_1 - q_2 - d_0}{2 + c_0} \\ q_1 = \frac{a - q_0 - q_2 - d_1}{2 + c_1} \\ q_2 = \frac{a - q_0 - q_1 - d_2}{2 + c_2} \end{cases} \quad (2)$$

Set of equations (2) contains three linear equations with three unknowns q_0, q_1, and q_2. The matrix form of the set of equations (2) is A q = v, where

$$A = (a_{ij}) = \begin{pmatrix} 1 & \frac{1}{2 + c_0} & \frac{1}{2 + c_0} \\ \frac{1}{2 + c_1} & 1 & \frac{1}{2 + c_1} \\ \frac{1}{2 + c_2} & \frac{1}{2 + c_2} & 1 \end{pmatrix},$$

$$v = \begin{pmatrix} \frac{a - d_0}{2 + c_0} \\ \frac{a - d_1}{2 + c_1} \\ \frac{a - d_2}{2 + c_2} \end{pmatrix}, q = \begin{pmatrix} q_0 \\ q_1 \\ q_2 \end{pmatrix}.$$

The solution of the set is the Nash equilibrium in a three-player Cournot model. The solution exists because the determinant of coefficients is non-singular.

2.2 Learning Methodology for the Cournot Model

We present the learning methodology for the Cournot model.

Period 0. All players choose the initial strategies q_0^0, q_1^0 and q_2^0 together. Each player assumes that the other players will maintain their strategies in the next period.

Periods s=1, 2, 3, … Players apply their reaction function to choose the strategy within the current period:

$q_0^s = R_0(q_1^{s-1}, q_2^{s-1})$, $q_1^s = R_1(q_0^{s-1}, q_2^{s-1})$ and $q_2^s = R_2(q_0^{s-1}, q_1^{s-1})$, s=1,2,3,….

The behavior of players in period 1 is typical of each subsequent period. If the equilibrium exists after a certain number of periods, it is reached. The learning process for the Cournot model can be presented with the iterative equation

$$q^{(s)} = T\, q^{(s-1)} + v, \text{ with s=1, 2, 3, … ,} \quad (3)$$

where

$$T = \begin{pmatrix} 0 & -\frac{1}{2 + c_0} & -\frac{1}{2 + c_0} \\ -\frac{1}{2 + c_1} & 0 & -\frac{1}{2 + c_1} \\ -\frac{1}{2 + c_2} & -\frac{1}{2 + c_2} & 0 \end{pmatrix}, \; q^{(0)} = \begin{pmatrix} q_0^0 \\ q_1^0 \\ q_2^0 \end{pmatrix} \quad (4)$$

The vector $q^{(0)}$ is defined in period 0 in the learning methodology. Iteration (3) is the Jacobi iteration for solving linear system A q = v. We apply the following well-known convergence theorem.

Theorem 1. (Theorem 3.6, [17]) If A is a strictly diagonally dominant matrix, then the Jacobi iterative method (3) with notations (4) for solving linear system A q = v converges for any initial starting vector $q^{(0)}$.

Proof. We confirm that matrix A is a strictly diagonally dominant one. Matrix A is strictly diagonally dominant if:

$$|a_{ii}| > \sum_{j=1, j \neq i}^{3} |a_{ij}|$$

The above inequalities are equivalent to these numerical inequalities $1 > \frac{2}{2 + c_0}$, $1 > \frac{2}{2 + c_1}$, $1 > \frac{2}{2 + c_2}$. . They are true because c_0, c_1 and c_2 are strictly positive. Thus, matrix A is strictly row diagonally dominant. The proof is presented in [17].

We will present an example of a three-player Cournot model for which we will find the theoretical solution and then apply the described methodology.

2.3 An Example of Three Player Game

The following example helps us to demonstrate the methodology.

Example 1. The cost of the players to produce a quantity q_i is $0.5\, c_i\, q_i^2 + d_i q_i$, i=0,1,2, where c_0 =0.15; d_0 =1.5, c_1 =0.16; d_1 =1.7, and c_2 =0.17; d_2 =1.9. The market price is p(q) = $a - q_0 - q_1 - q_2$ with a=10. We need to find the equilibrium point. We present the theoretical solution. Response functions are given in (1).

The equilibrium point is $\tilde{q}_0 = 2.187$, $\tilde{q}_1 = 1.996$, and $\tilde{q}_2 = 1.802$. The values are rounded to the third decimal place. Table 1 displays the price and profits of players.

Table 1. The Cournot model with three players. Profits of the players

Players	Amounts	Profits
First	2.187	5.142
Second	1.996	4.302
Third	1.802	3.535

Now, we apply the methodology for this example.

Period 0. All players propose the following quantities $q_0^0 = 2.5$, $q_1^0 = 3.0$, and $q_2^0 = 2.7$.

Period s, s=1,2,3, … The three players use their reaction functions and determine the quantities for the current period. The values of their strategies are described in Table 2.

Players may conclude that the equilibrium point is achieved after 70 periods. It shows very low convergence.

Table 2. The quantitative values of players at each period. Author's calculations

	t_0	t_1		t_{69}	t_{70}	t_{71}
q_0	2.5	1.3023	…	2.182	2.189	2.182
q_1	3.0	1.4351	…	1.990	1.998	1.991
q_2	2.7	1.1981	….	1.803	1.809	1.803

3 The Improved Learning Methodology based on the Leader-2-Followers Model

We propose the following methodology to explain the behavior of players under conditions of the Leader-2-followers model. We assume that two followers choose their strategies simultaneously. We could say that the two followers are playing a Cournot game with each other. This assumption helps in finding their response functions as it is done in (1).

Period 0. The leader proposes the quantity of his choice q_0^0 on the first step. The followers observe the leader's choice, but they cannot use their reaction functions because they do not know the choice of another follower. Follower 1 proposes quantity q_1^0, and follower 2 proposes quantity q_2^0. Note that the game is a relationship between the two followers and is modeled as a Cournot game.

Period 1. The leader has information about the quantities offered by the followers. The leader chooses his strategy in stage 1 applying $q_0^1 = R_0(q_1^0, q_2^0)$. The followers must make their decisions based on the leader's quantity and the assumption of passive behavior of the other followers. Follower 1 calculates the quantity q_1^1 through its response function $q_1^1 = R_1(q_0^1, q_2^0)$. Follower 2 calculates the quantity q_2^1 through its response function $q_2^1 = R_2(q_0^1, q_1^0)$. The price of the product is determined, depending on the quantity on the market p = a – q_0^1 - $q_1^1 - q_2^1$. Each player gets his winnings. To the end of Period 1, the quantities are q_0^1 , q_1^1 and q_2^1.

Periods s=2, 3, … Players apply their reaction function to choose the strategy within the current period:
$q_0^s = R_0(q_1^{s-1}, q_2^{s-1})$, $q_1^s = R_1(q_0^s, q_2^{s-1})$ and
$q_2^s = R_2(q_0^s, q_1^{s-1})$, s=2,3, 4, ,….

The behavior of players in period 2 is typical of each subsequent period. If the equilibrium exists after a certain number of periods, it is reached. This learning process can be presented with the iterative equation:
$q^{(s)} = T_1\, q^{(s)} + T_2\, q^{(s-1)} + v$, with s= 2, 3,…, (5)
where

$$T_1 = \begin{pmatrix} 0 & 0 & 0 \\ -\frac{1}{2+c_1} & 0 & 0 \\ -\frac{1}{2+c_2} & 0 & 0 \end{pmatrix},$$

$$T_2 = \begin{pmatrix} 0 & -\dfrac{1}{2+c_0} & -\dfrac{1}{2+c_0} \\ 0 & 0 & -\dfrac{1}{2+c_1} \\ 0 & -\dfrac{1}{2+c_2} & 0 \end{pmatrix},$$

$$v = \begin{pmatrix} \frac{a-d_0}{2+c_0} \\ \frac{a-d_1}{2+c_1} \\ \frac{a-d_2}{2+c_2} \end{pmatrix}, q^{(0)} = \begin{pmatrix} q_0^0 \\ q_1^0 \\ q_2^0 \end{pmatrix}. \qquad (6)$$

The vector $q^{(0)}$ is defined on Period 0 in the learning methodology. Iteration (5) is a modification of the Gauss-Seidel iterative method for solving the linear set of equations (2). We apply the following well-known convergence theorem.

Theorem 2. (Theorem 3.7, [17]) The Gauss-Seidel iterative method for solving linear system (2) converges for any initial vector $q^{(0)}$, if the matrix A of coefficients of (2) is strictly diagonally dominant.

Proof. The proof is presented in [17].

As a corollary of Theorem 2 it follows that the iterative method (5) with notations (6) for solving linear system (2) converges for any initial vector $q^{(0)}$.

We will present an example of a three-player Cournot model for which we will find the theoretical solution and then apply the described methodology.

Sequentially playing out the Cournot model leads to a learning process [1] in which each player assumes passive behavior to his opponent and substitutes the opponent's strategy in his response function. In effect, this means that each player substitutes the opponent's strategy from the previous period into his profit function and seeks a maximum. This is implemented at every stage of the learning process. The Stackelberg model differs from the Cournot model in that the leader replaces the follower's response function in his own profit function. If the leader violates this condition and uses a quantity instead of a reaction function to find his maximum payoff, then the model transforms to a Cournot model regardless of whether the game is leader-follower. This is exactly what happens in the proposed methodology in the second period and every subsequent one. The leader-two-follower game turns into a Cournot game because the leader implies the passive behavior of the followers.

After a few periods, the players achieve the equilibrium point $(\tilde{q}_0, \tilde{q}_1, \tilde{q}_2)$. Then the price and profits are defined and computed.

4 An Academic Application

We apply the above learning process to find the equilibrium point of the introduced example.

Period 0.

Stage 1. The leader offers a quantity $q_0^0 = 3.0$.
Stage 2. The followers offer quantities q_1^0 , and q_2^0 (half of the monopoly quantity):

$q_1^0 = 0.5\, R_1(q_0^0, 0) = 1.227,$

$q_2^0 = 0.5\, R_2(q_0^0, 0) = 1.175.$

Period s=1,2,3….

Following the improved methodology, we compute the values of players at each period, and we present them in Table 3. We can conclude that after the period t_4 the learning process approximates the equilibrium point. We can define the equilibrium as $(\tilde{q}_0, \tilde{q}_1, \tilde{q}_2) = (2.186, 1.993, 1.81)$. However, this equilibrium point differs from the Nash equilibrium in the Stackelberg model. Furthermore, that is the equilibrium point in the Cournot model. Compared with the values in Table 2, we conclude that the improved methodology achieved equilibrium faster than the learning methodology for the Cournot model. That is a way to quickly compute the equilibrium in the Cournot model for more than two players.

Table 3. The quantitative values of players at each period. The author's calculations are based on the improved methodology

	t_0	t_1	t_2	t_3	t_4	t_5
q_0	3.0	2.8363	2.1648	2.1864	2.1857	2.1857
q_1	1.227	1.9855	1.9792	1.9877	1.9913	1.9930
q_2	1.175	1.8602	1.8201	1.8131	1.8095	1.8078

5 Conclusion

Our proposed methodology, the learning process, offers an effective framework for all players to iteratively refine their strategies. This iterative approach persists until market conditions stabilize, necessitating adjustments only when changes occur in the players' cost functions or market prices. In such instances, the methodology is reactivated to adapt to the evolving landscape.

While particularly tailored for energy markets due to their dynamic nature, this methodology can be seamlessly applied to various corporate strategy settings. Notably, industries characterized by frequent market fluctuations and intense competition stand to benefit greatly. For instance, in telecommunications, where technological advancements and regulatory shifts often reshape market dynamics, this methodology can guide firms in crafting responsive strategies. Similarly, in the fast-paced tech sector, where innovation drives market trends, this approach can aid companies in maintaining competitiveness amid constant disruption.

Moreover, in sectors like retail and consumer goods, where consumer preferences and market trends evolve swiftly, the ability to continually adjust strategies based on competitor behavior and market shifts is invaluable. By employing this methodology, firms can stay agile and strategically positioned to capitalize on emerging opportunities and mitigate risks.

Overall, the adaptability and effectiveness of this methodology make it a versatile tool for navigating the complexities of corporate strategy settings across diverse industries, enabling firms to thrive in dynamic and competitive environments.

Declaration of Generative AI and AI-assisted Technologies in the Writing Process

The authors wrote, reviewed and edited the content as needed and they have not utilised artificial intelligence (AI) tools. The authors take full responsibility for the content of the publication.

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The authors have no conflicts of interest to declare.