Some Properties of Quaternion Algebra over the Sets of Real and Complex Numbers

ȚUGUI (BAIAS) ANDREEA-ELENA
Doctoral School of Mathematics,
Ovidius University of Constant,
ROMANIA

Abstract: - In this article, we analyzed complex quaternions and the matrix representations associated with 2x2 complex quaternions. We provided detailed insights into the fundamental properties of quaternions, highlighting essential features of the corresponding matrix representations. Additionally, we examined real quaternions, emphasizing their specific characteristics, and we explored aspects related to matrix representations of quaternions with real coefficients of size 4x4. The central aim of this article is to conduct a detailed comparison between the two types of matrix representations, both in real and complex contexts. In light of the obtained results, the article seeks to make significant contributions to the understanding and application of quaternions in various mathematical domains.

Key-Words: - quaternions, quaternion algebra, complex quaternion, biquaternion, matrix of biquaternions, matrix representations, determinant.

Received: July 19, 2022. Revised: October 11, 2023. Accepted: November 14, 2023. Published: December 18, 2023.

1 Introduction

Quaternion matrices, a complex extension of traditional matrices, have become a focal point in applied mathematics and practical fields.

Recognized for their versatility, they find numerous applications in various real-world domains. This article explores the significant role of quaternion matrices in key sectors such as computer graphics, robotics, electrical engineering, machine learning, medical sciences, communications, and virtual reality.

As powerful mathematical tools, quaternion matrices contribute significantly to technological advancements, enriching our daily lives.

Quaternion matrices, owing to their versatility, bring substantial benefits across different fields.

Within expert systems, they facilitate the representation and manipulation of three-dimensional knowledge, proving useful in spatial data analysis and decision-making in complex situations.

In the realm of deep learning, quaternions can be integrated into neural network architectures to enhance the understanding and modeling of complex spatial information.

This translates into practical applications, such as object recognition in 3D images or three-dimensional simulations, where the efficient representation of orientations and spatial transformations is essential.

Thus, adapting specific technologies and considering the complexity of quaternion matrix calculations can significantly contribute to the efficient management and processing of spatial and three-dimensional information in these domains.

The real quaternion algebra is denoted by $\mathbb{H}$. This algebra has elements respecting the following form: $h = \alpha_0 + \alpha_1 f_1 + \alpha_2 f_2 + \alpha_3 f_3$, where $\alpha_n \in \mathbb{R}, n = \{0, 1, 2, 3\}$,

$$f_1^2 = f_2^2 = f_3^2 = -1$$

and

$$f_1 f_2 = -f_2 f_1 = f_3, f_2 f_3 = -f_3 f_2 = f_1,$$
$$f_3 f_1 = -f_1 f_3 = f_2.$$

The set of quaternions over the real numbers is a division algebra (which means any non-zero quaternion admits an inverse).$\mathbb{H}$ is associative but does not satisfy the commutative property (so they are non-commutative).

One of the bases of real quaternions is $\{1, f_1, f_2, f_3\}$.

Starting from the matrix form of complex numbers I will represent the matrix form of complex quaternions in the following chapter.

More information about the calculation and properties of real quaternions can be found in, [1], [2], [3], [4], and, [5].

2 The Complex Matrix Representations of Quaternions

We consider the field $\mathbb{P}$ that has the form
$\mathbb{P} = \left\{ \begin{pmatrix} z_a & -z_b \\ z_b & z_a \end{pmatrix} | z_a, z_b \in \mathbb{R} \right\}$.
The map

$$\psi: \mathbb{C} \to \mathbb{P}, \psi(z_a + z_b) = \begin{pmatrix} z_a & -z_b \\ z_b & z_a \end{pmatrix},$$

where $i^2 = -1$ is a field morphism and

$$\psi(z) = \begin{pmatrix} z_a & -z_b \\ z_b & z_a \end{pmatrix},$$

is called the matrix representation of the element

$$z = z_a + z_b i \in \mathbb{C}.$$

The complex quaternion algebra is denoted with $\mathbb{H}_{\mathbb{C}}$ and this algebra has the elements of the following form:

$$h_{\mathbb{C}} = r_0 + r_1 f_1 + r_2 f_2 + r_3 f_3,$$

where $r_0, r_1, r_2, r_3 \in \mathbb{C}$, $f_1^2 = f_2^2 = f_3^2 = -1$ and $f_1 f_2 = -f_2 f_1 = f_3$, $f_2 f_3 = -f_3 f_2 = f_1$, $f_3 f_1 = -f_1 f_3 = f_2$.

The form of the complex numbers r_0, r_1, r_2 and r_3 is the next one:
$r_0 = z_{a0} + i z_{b0}$, z_{a0}, $z_{b0} \in \mathbb{R}$,
$r_1 = z_{a1} + i z_{b1}$, z_{a1}, $z_{b1} \in \mathbb{R}$,
$r_2 = z_{a2} + i z_{b2}$, z_{a2}, $z_{b2} \in \mathbb{R}$ and
$r_3 = z_{a3} + i z_{b3}$, z_{a3}, $z_{b3} \in \mathbb{R}$ where $i^2 = -1$.

Biquaternionic algebra (or complex quaternions) is a vector space of fourth dimension over the field of complex numbers.

The canonical basis of biquaternions-$\mathbb{H}_{\mathbb{C}}$ is $\{1, f_1, f_2, f_3\}$, and 1 has the role of a unit element.

Biquaternions belong to a special class of Clifford numbers.

Clifford algebra is an algebra generated by a vector space with a quadratic form and is an unital associative algebra. Also, it generalizes the real numbers, complex numbers, quaternions, and several other hypercomplex number systems.
Clifford numbers are hypercomplex numbers that come from real numbers and complex numbers.

Hypercomplex numbers are obtained by generalizing the construction of complex numbers starting from real numbers.

The important part of the study of complex quaternions is to systematically present their algebraic structures and to determine a complete computational theory.

Biquaternions-$\mathbb{H}_{\mathbb{C}}$ is not part of the algebra with division, because $\exists a_1, a_2 \in \mathbb{H}_{\mathbb{C}}$ such that

$$a_1 \cdot a_2 = 0.$$

A well-known fact about the algebra of complex quaternions is that it is algebraically isomorphic to the 2×2 total matrix algebra $C^{2\times 2}$ through the bijective map $\Gamma: \mathbb{H}_{\mathbb{C}} \to C^{2\times 2}$ satisfying:

$$\Gamma(1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$
$$\Gamma(f_1) = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix},$$
$$\Gamma(f_2) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$
$$\Gamma(f_3) = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}.$$

The matrix representations $\Gamma(1)$, $\Gamma(f_1)$, $\Gamma(f_2)$, $\Gamma(f_3)$ are called the Pauli matrix. Based on this bijective map, we can introduce the following result:

The matrix representation of
$h_{\mathbb{C}} = r_0 + r_1 f_1 + r_2 f_2 + r_3 f_3 \in \mathbb{H}_{\mathbb{C}}$ with
$r_n \in \mathbb{C}, n = \{0,1,2,3\}$ over the set of complex numbers is given by the following expression:

$$\Gamma(h_{\mathbb{C}}) = \begin{pmatrix} r_0 + r_1 i & -(r_2 + r_3 i) \\ r_2 - r_3 i & r_0 - r_1 i \end{pmatrix} \in C^{2\times 2}, [6].$$

Proposition 1.
We consider $\Gamma(1), \Gamma(f_1), \Gamma(f_2), \Gamma(f_3)$, we have:
det($\Gamma(1)$)=det($\Gamma(f_1)$)=det($\Gamma(f_2)$)=det($\Gamma(f_3)$) $= 1$.

Proof.
det($\Gamma(1)$)=1, det($\Gamma(f_1) = -i^2 = 1$, det($\Gamma(f_2)$)=-(-1)=1, det($\Gamma(f_3)) = -i^2 = 1$.

Proposition 2.
We consider $\Gamma(1), \Gamma(f_1), \Gamma(f_2), \Gamma(f_3)$, we have:
$\Gamma^4(f_1) = \Gamma^4(f_2) = \Gamma^4(f_3) = I_2$.

Proof.

$$\Gamma^2(f_1) = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$
$$\Gamma^4(f_1) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$
$$\Gamma^2(f_2) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$
$$\Gamma^4(f_2) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$
$$\Gamma^2(f_3) = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix},$$
$$\Gamma^4(f_3) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Remark.
Here are some known basic properties of complex quaternions listed below.

Theorem 1.
Considering
$h_{1\mathbb{C}} = r_0 + r_1 f_1 + r_2 f_2 + r_3 f_3$,
$h_{2\mathbb{C}} = w_0 + w_1 f_1 + w_2 f_2 + w_3 f_3 \in \mathbb{H}_{\mathbb{C}}$
and $\lambda \in \mathbb{C}$.
Then we have:

1. $h_{1\mathbb{C}} = h_{2\mathbb{C}} \Leftrightarrow \Gamma(h_{1\mathbb{C}}) = \Gamma(h_{2\mathbb{C}})$.
2. $\Gamma(h_{1\mathbb{C}} + h_{2\mathbb{C}}) = \Gamma(h_{1\mathbb{C}}) + \Gamma(h_{2\mathbb{C}})$.
3. $\Gamma(h_{1\mathbb{C}} \cdot h_{2\mathbb{C}}) = \Gamma(h_{1\mathbb{C}}) \cdot \Gamma(h_{1\mathbb{C}})$.
4. $\Gamma(\lambda \cdot h_{1\mathbb{C}}) = \Gamma(h_{1\mathbb{C}} \cdot \lambda) = \lambda\Gamma(h_{1\mathbb{C}})$.
5. $\Gamma(1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_2$[6].

Proof.
1. $r_0 + r_1 f_1 + r_2 f_2 + r_3 f_3 = w_0 + w_1 f_1 + w_2 f_2 + w_3 f_3 \Leftrightarrow$

$$\Leftrightarrow \begin{pmatrix} r_0 + r_1 i & -(r_2 + r_3 i) \\ r_2 - r_3 i & r_0 - r_1 i \end{pmatrix} = \begin{pmatrix} w_0 + w_1 i & -(w_2 + w_3 i) \\ w_2 - w_3 i & w_0 - w_1 i \end{pmatrix}.$$

2. $\Gamma(h_{1\mathbb{C}} + h_{2\mathbb{C}}) =$
$\begin{pmatrix} r_0 + w_0 + (r_1 + w_1)i & -(r_2 + w_2) - (r_3 + w_3)i \\ r_2 + w_2 - (r_3 + w_3)i & r_0 + w_0 - (r_1 + w_1)i \end{pmatrix} \Gamma(h_{1\mathbb{C}}) +$
$\Gamma(h_{2\mathbb{C}}) = \begin{pmatrix} r_0 + r_1 i & -(r_2 + r_3 i) \\ r_2 - r_3 i & r - r_1 i \end{pmatrix} +$
$\begin{pmatrix} w_0 + w_1 i & -(w_2 + w_3 i) \\ w_2 - w_3 i & w_0 - w_1 i \end{pmatrix}$

$$\Gamma(h_{1\mathbb{C}}) + \Gamma(h_{2\mathbb{C}}) = \begin{pmatrix} r_0 + w_0 + (r_1 + w_1)i & -(r_2 + w_2) - (r_3 + w_3)i \\ r_2 + w_2 - (r_3 + w_3)i & r_0 + w_0 - (r_1 + w_1)i \end{pmatrix} = \Gamma(h_{1\mathbb{C}} + h_{2\mathbb{C}})$$

3. $\Gamma(h_{1\mathbb{C}} \cdot h_{2\mathbb{C}}) = \begin{pmatrix} t_0 + t_1 i & -(t_2 + t_3 i) \\ t_2 - t_3 i & t_0 - t_1 i \end{pmatrix} =$
$\begin{pmatrix} r_0 + r_1 i & -(r_2 + r_3 i) \\ r_2 - r_3 i & r_0 - r_1 i \end{pmatrix}$.
$\begin{pmatrix} w_0 + w_1 i & -(wa_2 + w_3 i) \\ w_2 - w_3 i & w_0 - w_1 i \end{pmatrix} = \Gamma(h_{1\mathbb{C}} \cdot \Gamma(h_{2\mathbb{C}})$

4. $\Gamma(\lambda \cdot h_{1\mathbb{C}}) = \begin{pmatrix} \lambda(r_0 + r_1 i) & -\lambda(r_2 + r_3 i) \\ \lambda(r_2 - r_3 i) & \lambda(r_0 - r_1 i) \end{pmatrix}$
$\lambda\Gamma(h_{1\mathbb{C}}) = \lambda \cdot \begin{pmatrix} r_0 + r_1 i & -(r_2 + r_3 i) \\ r_2 - r_3 i & r_0 - r_1 i \end{pmatrix} = \Gamma(\lambda \cdot h_{1\mathbb{C}})$

Remark. In the following example, we will apply properties 2, 3, and 4 to quaternions $h_{1\mathbb{C}}$ and $h_{2\mathbb{C}}$.

Example 1. We consider $h_{1\mathbb{C}} = 2 + 3f_1 - f_2 + 4f_3$, $h_{2\mathbb{C}} = 1 - 2f_1 + f_2 - f_3 \in \mathbb{H}_{\mathbb{C}}$
and $\lambda = 3 + i \in \mathbb{C}$.
Then we have:
3. $h_{1\mathbb{C}} + h_{2\mathbb{C}} = 3 + f_1 + 3f_3$
$\Gamma(h_{1\mathbb{C}} + h_{2\mathbb{C}}) = \begin{pmatrix} 3 + i & -3i \\ 3i & 3 - i \end{pmatrix}$ (1);
$\Gamma(h_{1\mathbb{C}}) + \Gamma(h_{2\mathbb{C}}) = \begin{pmatrix} 2 + 3i & 1 - 4i \\ -1 - 4i & 2 - 3i \end{pmatrix}$
$+\begin{pmatrix} 1 - 2i & -1 + i \\ 1 + i & 1 + 2i \end{pmatrix} = \begin{pmatrix} 3 + i & -3i \\ 3i & 3 - i \end{pmatrix}$ (2);
(1)=(2).
4. $h_{1\mathbb{C}} \cdot h_{2\mathbb{C}} = 13 - 4f_1 - 4f_2 + 3f_3$
$\Gamma(h_{1\mathbb{C}} \cdot h_{2\mathbb{C}}) = \begin{pmatrix} 13 - 4i & 4 - 3i \\ -4 - 3i & 13 - 4i \end{pmatrix}$ (1);
$\Gamma(h_{1\mathbb{C}} \cdot \Gamma(h_{2\mathbb{C}}) =$
$\begin{pmatrix} 2 + 3i & 1 - 4i \\ -1 - 4i & 2 - 3i \end{pmatrix} . \begin{pmatrix} 1 - 2i & -1 + i \\ 1 + i & 1 + 2i \end{pmatrix} =$
$\begin{pmatrix} 13 - 4i & 4 - 3i \\ -4 - 3i & 13 - 4i \end{pmatrix}$ (2);
(1)=(2).
5. $(\lambda \cdot h_{1\mathbb{C}})$ =(3+i)($2 + 3f_1 - f_2 + 4f_3$) =
=(6+2i)+(9+3i) f_1- (3+i) f_2+(12+4i) f_3.
$\Gamma(\lambda \cdot h_{1\mathbb{C}}) = \begin{pmatrix} 3 + 11i & 7 - 11i \\ 1 + 13i & 9 - 7i \end{pmatrix}$ (1);
$\lambda\Gamma(h_{1\mathbb{C}}) = (3 + i) \cdot \begin{pmatrix} 2 + 3i & 1 - 4i \\ -1 - 4i & 2 - 3i \end{pmatrix} =$
$=\begin{pmatrix} 3 + 11i & 7 - 11i \\ 1 + 13i & 9 - 7i \end{pmatrix}$ (2);
(1)=(2).

Remark. In the following, I will introduce some information about complex quaternions.
For $h_{\mathbb{C}} = r_0 + r_1 f_1 + r_2 f_2 + r_3 f_3 \in \mathbb{H}_{\mathbb{C}}$, the **dual quaternion** of $h_{\mathbb{C}}$ is
$$\overline{h_{\mathbb{C}}} = r_0 - r_1 f_1 - r_2 f_2 - r_3 f_3;$$
the **complex conjugate** of $h_{\mathbb{C}}$ is
$$h_{\mathbb{C}}^* = \overline{r_0} + \overline{r_1} f_1 + \overline{r_2} f_2 + \overline{r_3} f_3;$$

the **Hermitian conjugate** of $h_{\mathbb{C}}$ is

$$\overline{h_{\mathbb{C}}}^* = \overline{r_0} - \overline{r_1} f_1 - \overline{r_2} f_2 - \overline{r_3} f_3;$$

the **semi-norm** of $h_{\mathbb{C}}$ is

$$n(h_{\mathbb{C}}) = r_0^2 + r_1^2 + r_2^2 + r_3^2.\,[6]$$

Example 2.
For $h_{\mathbb{C}} = 2 + 3f_1 - if_2 + 4f_3 \in \mathbb{H}_{\mathbb{C}}$,
the **dual quaternion** of $h_{\mathbb{C}}$ is

$$\overline{h_{\mathbb{C}}} = 2 - 3f_1 + \mathrm{i} f_2 - 4f_3;$$

the **complex conjugate** of $h_{\mathbb{C}}$ is

$$h_{\mathbb{C}}^* = \overline{2} + \overline{3} f_1 + \overline{(-\mathrm{i})} f_2 + \overline{4} f_3;$$

the **Hermitian conjugate** of $h_{\mathbb{C}}$ is

$$\overline{h_{\mathbb{C}}}^* = \overline{2} - \overline{3} f_1 - \overline{(-\mathrm{i})} f_2 - \overline{4} f_3;$$

the **semi-norm** of $h_{\mathbb{C}}$ is

$$n(h_{\mathbb{C}}) = 2^2 + 3^2 + (-\mathrm{i})^2 + 4^2 = 28$$

Theorem 2.
Considering
$h_{1\mathbb{C}} = r_0 + r_1 f_1 + r_2 f_2 + r_3 f_3 \in \mathbb{H}_{\mathbb{C}}$. *Then,*

1. $\Gamma\left(\overline{h_{1\mathbb{C}}}\right) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \Gamma^T(h_{1\mathbb{C}}) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ *where* $\overline{h_{1\mathbb{C}}} = r_0 - r_1 f_1 - r_2 f_2 - r_3 f_3$.
2. $\Gamma(h_{1\mathbb{C}}^*) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \cdot \overline{\Gamma(h_{1\mathbb{C}})} \cdot \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ *where* $h_{1\mathbb{C}}^* = h_{1\mathbb{C}}$.
3. $\Gamma\left(\overline{(h_{1\mathbb{C}})}^*\right) = \overline{\Gamma(h_{1\mathbb{C}})}^T = \Gamma(h_{1\mathbb{C}})^*$, *the conjugate transpose of the complex matrix* $\Gamma(h_{1\mathbb{C}})$.
4. $det\left(\Gamma(h_{1\mathbb{C}})\right) = r_0^2 + r_1^2 + r_2^2 + r_3^2$, *is the semi-norm of* $h_{1\mathbb{C}}$.
5. $h_{1\mathbb{C}} = \frac{1}{4} \cdot E_2 \cdot \Gamma(h_{1\mathbb{C}}) \cdot \overline{E_2}$ *where* $E_2 = (1 - if_1, f_2 + if_3)$.
6. $h_{1\mathbb{C}}$ *is invertible if and only if* $\Gamma(h_{1\mathbb{C}})$ *is invertible. In this case, we know* $\Gamma^{-1}(h_{1\mathbb{C}}) = \Gamma[(h_{1\mathbb{C}})^{-1}]$ *and* $(h_{1\mathbb{C}})^{-1} = \frac{1}{4} \cdot E_2 \cdot \Gamma^{-1}(h_{1\mathbb{C}}) \cdot \overline{E_2}$ *.[6]*

Proof.
1. $\Gamma\left(\overline{h_{1\mathbb{C}}}\right) = \begin{pmatrix} r_0 - r_1 i & r_2 + r_3 i \\ -r_2 + r_3 i & r_0 + r_1 i \end{pmatrix}$ and

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \cdot \Gamma^T(h_{1\mathbb{C}}) \cdot \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} r_0 + r_1 i & r_2 - r_3 i \\ -r_2 - r_3 i & r_0 - r_1 i \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \cdot \Gamma^T(h_{1\mathbb{C}}) \cdot \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -r_2 - r_3 & r_0 - r_1 i \\ -(r_0 + r_1 i) & -(r_2 - r_3 i) \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \cdot \Gamma^T(h_{1\mathbb{C}}) \cdot \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} r_0 - r_1 i & r_2 + r_3 i \\ -r_2 + r_3 i & r_0 + r_1 i \end{pmatrix} = \Gamma\left(\overline{h_{1\mathbb{C}}}\right).$$

2. $\Gamma(h_{1\mathbb{C}}^*) = \begin{pmatrix} r_0 + r_1 i & -(r_2 + r_3 i) \\ r_2 - r_3 i & r_0 - r_1 i \end{pmatrix}$ and

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \cdot \overline{\Gamma(h_{1\mathbb{C}})} \cdot \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} r_0 - r_1 i & -r_2 + r_3 i \\ r_2 + r_3 i & r_0 + r_1 i \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \cdot \overline{\Gamma(h_{1\mathbb{C}})} \cdot \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} r_0 + r_3 i & r_0 + r_1 i \\ -r_0 + r_1 i & r_2 - r_3 i \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \cdot \overline{\Gamma(h_{1\mathbb{C}})} \cdot \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} r_0 + r_1 i & -(r_2 + r_3 i) \\ r_2 - r_3 i & r_0 - r_1 i \end{pmatrix} = \Gamma(h_{1\mathbb{C}}^*).$$

3. If follows from a direct verification.

4. $det\left(\Gamma(h_{1\mathbb{C}})\right) = \begin{vmatrix} r_0 + r_1 i & -(r_2 + r_3 i) \\ r_2 - r_3 i & r_0 - r_1 i \end{vmatrix} =$
$= r_0^2 - r_0 r_1 i + r_0 r_1 i + r_1^2 + r_2^2 - r_2 r_3 i + r_2 r_3 i + r_3^2$

$$det\left(\Gamma(h_{1\mathbb{C}})\right) = r_0^2 + r_1^2 + r_2^2 + r_3^2.$$

5. It follows from a direct verification.

6. It follows from a direct verification.

3 The Real Matrix Representations of Quaternions

The set of quaternions over the real numbers is a division algebra (which means any non-zero quaternion admits an inverse).

Definition 1.
The conjugate of $h = \alpha_0 + \alpha_1 f_1 + \alpha_2 f_2 + \alpha_3 f_3 = Re(h) + Im(h) \in \mathbb{H}$, *where* $\alpha_0, \alpha_1, \alpha_3, \alpha_4 \in \mathbb{R}$ *is defined as*
$\overline{h} = \alpha_0 - \alpha_1 f_1 - \alpha_2 f_2 - \alpha_3 f_3 = Re(h) - Im(h)$
with $\alpha_0, \alpha_1, \alpha_3, \alpha_4 \in \mathbb{R}$.

Definition 2. *Let's consider a quaternion* $h \in \mathbb{H}$ *with the form* $h = \alpha_0 + \alpha_1 f_1 + \alpha_2 f_2 + \alpha_3 f_3$. *The real number* $|h| = \sqrt{\alpha_0^2 + \alpha_1^2 + \alpha_2^2 + \alpha_3^2}$ *represents the norm of the quaternion.*

The semi-norm is then defined as $||h_1|| = \alpha_0^2 + \alpha_1^2 + \alpha_2^2 + \alpha_3^2$.

Definition 3. *Any quaternion* $h \neq 0$, $h = \alpha_0 + \alpha_1 f_1 + \alpha_2 f_2 + \alpha_3 f_3 \in \mathbb{H}$ *is invertible and* h^{-1} can be written as $\frac{\overline{h}}{|h|^2}$, *because*

$$h \cdot \frac{\overline{h}}{|h|^2} = \frac{h \cdot \overline{h}}{|h|^2} = \frac{|h|^2}{|h|^2} = 1.$$

Definition 4. *In, [7], the map for the quaternion algebra* $\mathbb{H}$ *is defined as follows:* $\phi: h = \alpha_0 + \alpha_1 f_1 + \alpha_2 f_2 + \alpha_3 f_3 \in \mathbb{H} \to \mathcal{M}_4(\mathbb{R})$, *where* $\phi(h)$ *has the form presented below:*

$$\phi(h) = \begin{pmatrix} \alpha_0 & -\alpha_1 & -\alpha_2 & -\alpha_3 \\ \alpha_1 & \alpha_0 & -\alpha_3 & \alpha_2 \\ \alpha_2 & \alpha_3 & \alpha_0 & -\alpha_1 \\ \alpha_3 & -\alpha_2 & \alpha_1 & \alpha_0 \end{pmatrix}.$$

However, $h = \alpha_0 + \alpha_1 f_1 + \alpha_2 f_2 + \alpha_3 f_3 \in \mathbb{H}$ *is an isomorphism between* $\mathbb{H}$ *and the algebra of the matrices*

$$\mathrm{B} = \left\{ \begin{pmatrix} \alpha_0 & -\alpha_1 & -\alpha_2 & -\alpha_3 \\ \alpha_1 & \alpha_0 & -\alpha_3 & \alpha_2 \\ \alpha_2 & \alpha_3 & \alpha_0 & -\alpha_1 \\ \alpha_3 & -\alpha_2 & \alpha_1 & \alpha_0 \end{pmatrix} \\ \alpha_0, \alpha_1, \alpha_2, \alpha_3 \in \mathbb{R} \right\}.$$

We can remark that the matrix $\phi(h) \in \mathcal{M}_4(\mathbb{R})$ has as columns the coefficients in $\mathbb{R}$ of the basis

$$\{1, f_1, f_2, f_3\}$$

for the elements

$$\{\alpha_0, \alpha_1 f_1, \alpha_2 f_2, \alpha_3 f_3\}.$$

The matrix $\phi(h)$ is called the left matrix representation of the element $h \in \mathbb{H}$.

Definition 5. *Analogously with the left matrix representation, for the element t* $h = \alpha_0 + \alpha_1 f_1 + \alpha_2 f_2 + \alpha_3 f_3 \in \mathbb{H}$, *inm [7], the right matrix representation was defined as follows:* $\gamma: \mathbb{H} \to \mathcal{M}_4(\mathbb{R})$, *where* $\gamma(h)$ *is given by:* $\gamma(h) = \begin{pmatrix} \alpha_0 & -\alpha_1 & -\alpha_2 & -\alpha_3 \\ \alpha_1 & \alpha_0 & \alpha_3 & -\alpha_2 \\ \alpha_2 & -\alpha_3 & \alpha_0 & \alpha_1 \\ \alpha_3 & \alpha_2 & -\alpha_1 & \alpha_0 \end{pmatrix}$.

However, γ is an isomorphism between $\mathbb{H}$ and the algebra of the matrices

$$\mathcal{C} = \left\{ \begin{pmatrix} \alpha_0 & -\alpha_1 & -\alpha_2 & -\alpha_3 \\ \alpha_1 & \alpha_0 & \alpha_3 & -\alpha_2 \\ \alpha_2 & -\alpha_3 & \alpha_0 & \alpha_1 \\ \alpha_3 & \alpha_2 & -\alpha_1 & \alpha_0 \end{pmatrix} \\ , \alpha_0, \alpha_1, \alpha_2, \alpha_3 \in \mathbb{R} \right\}.$$

The matrix $\gamma(h)$ is called the right matrix representation of the quaternion $h \in \mathbb{H}$.

To define the matrix representations of imaginary units, consider the following expressions: For the first imaginary part f_1 of quaternions h=f_1, the matrix representation ϕ (f_1) is given by:

$$\phi(f_1) = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

where f_1 is the initial imaginary component of the quaternion h=f_1, and ϕ (f_1) represents this part in matrix form.

For the second imaginary part $f_2$of the quaternion h $= f_2$, the matrix representation $\phi(f_2)$ is defined as:

$$\phi(f_2) = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix},$$

where $\phi(f_2)$ serves as the matrix representation of f_2, the secondary imaginary component of the quaternion h=f_2.

The matrix representation $\phi(f_3)$ for the third imaginary part of the quaternion h=f_3 is given by:

$$\phi(f_3) = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

This matrix, $\phi(f_3)$, represents f_3, the third imaginary part of the quaternion h=f_3.

The unit matrix is considered

$$I_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Remark. In what follows, I will employ the matrix forms presented in Proposition 1. Let's examine the matrix

$$\Pi = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} = \phi(f_1).$$

Next, consider the matrix

$$E_4 = (1, -f_1, -f_2, -f_3)^T = \begin{pmatrix} 1 \\ -f_1 \\ -f_2 \\ -f_3 \end{pmatrix}.$$

Proposition 3.
Let's consider $y = y_0 + y_1 f_1 + y_2 f_2 + y_3 f_3 \in \mathbb{H}$. *The vector representation of* y *is denoted by* $\vec{y} = [y_0, y_1, y_2, y_3]^T$. *So for all*
$h_1 = \alpha_0 + \alpha_1 f_1 + \alpha_2 f_2 + \alpha_3 f_3 \in \mathbb{H}$,
$h_2 = \beta_0 + \beta_1 f_1 + \beta_2 f_2 + \beta_3 f_3 \in \mathbb{H}$ *we have*

$$\overrightarrow{h_1 \cdot y} = \phi(h_1) \cdot \vec{y};$$

$$\overrightarrow{y \cdot h_2} = \gamma(h_2) \cdot \vec{y};$$

$$\overrightarrow{h_1 \cdot y \cdot h_2} = \phi(h_1) \cdot \gamma(h_2) \cdot \vec{y} = \gamma(h_2) \cdot \phi(h_1) \cdot \vec{y};$$

$\phi(h_1) \cdot \gamma(h_2) = \gamma(h_2) \cdot \phi(h_1)$, [8].

Proof.
We can observe that $\vec{y} = \phi(y)\beta_4^T$, and
$\vec{y} = \gamma(y)\beta_4^T$ where $\beta_4 = (1, 0, 0, 0)$

$$\overrightarrow{h_1 y} = \phi(h_1 y)\beta_4^T = \phi(h_1)\phi(y)\beta_4^T = \phi(h_1)\vec{y}$$
$$\overrightarrow{y h_2} = \gamma(y h_2)\beta_4^T = \gamma(y)\gamma(h_2)\beta_4^T = \gamma(h_2)\vec{y}$$
$$\overrightarrow{h_1 y h_2} = \overrightarrow{h_1 (y h_2)} = \phi(h_1)\overrightarrow{(y h_2)} = \phi(h_1)\gamma(h_1)\vec{y}$$
$$\overrightarrow{h_1 y h_2} = \overrightarrow{(h_1 y) h_2} = \gamma(h_2)\overrightarrow{(h_1 y)} = \gamma(h_2)\phi(h_1)\vec{y};$$

Lemma 1
Let's consider $h = Re(h) + Im(h) \in \mathbb{H}$ *given.*
Moreover, let $\alpha_0, \alpha_1, \alpha_2, \alpha_3$ *be coefficients in the field of real numbers.*
Then, we can assert that the quaternion's diagonal is $(\alpha_0, \alpha_0, \alpha_0, \alpha_0)$, *and it satisfies the following unitary similarity factorization equality:*

$$S \begin{pmatrix} \alpha_0 & 0 & 0 & 0 \\ 0 & \alpha_0 & 0 & 0 \\ 0 & 0 & \alpha_0 & 0 \\ 0 & 0 & 0 & \alpha_0 \end{pmatrix} S^* = \begin{pmatrix} \alpha_0 & -\alpha_1 & -\alpha_2 & -\alpha_3 \\ \alpha_1 & \alpha_0 & -\alpha_3 & \alpha_2 \\ \alpha_2 & \alpha_3 & \alpha_0 & -\alpha_1 \\ \alpha_3 & -\alpha_2 & \alpha_1 & \alpha_0 \end{pmatrix} \in \mathbb{R}^{4x4},$$

where the matrix S is written as:

$$S = S^* = \frac{1}{2}\begin{pmatrix} 1 & f_1 & f_2 & f_3 \\ -f_1 & 1 & f_3 & -f_2 \\ -f_2 & -f_3 & 1 & f_1 \\ -f_3 & f_2 & -f_1 & 1 \end{pmatrix}$$ *which is a unitary matrix over* $\mathbb{H}$.*[8]*

Remark. Two quaternion matrices $\phi(h_1)$ and $\phi(h_2)$ of the same order $m \times n$ are considered equal if all of their components are equal:
$\alpha_{i,j} = \beta_{i,j}$, for all $i \in \{1, \cdots, m\}$, $j \in \{1, \cdots, n\}$.

Lemma 2.
Either $h_1 = \alpha_0 + \alpha_1 f_1 + \alpha_2 f_2 + \alpha_3 f_3 \in \mathbb{H}$, $h_2 = \beta_0 + \beta_1 f_1 + \beta_2 f_2 + \beta_3 f_3 \in \mathbb{H}$ *and* $\eta \in \mathbb{R}$ *(a real scalar number). Then*

(1) $h_1 = h_2$ *if and only if* $\phi(h_1) = \phi(h_2)$, *where*

$$\phi(h_1) = \begin{pmatrix} \alpha_0 & -\alpha_1 & -\alpha_2 & -\alpha_3 \\ \alpha_1 & \alpha_0 & -\alpha_3 & \alpha_2 \\ \alpha_2 & \alpha_3 & \alpha_0 & -\alpha_1 \\ \alpha_3 & -\alpha_2 & \alpha_1 & \alpha_0 \end{pmatrix},$$

$$\phi(h_2) = \begin{pmatrix} \beta_0 & -\beta_1 & -\beta_2 & -\beta_3 \\ \beta_1 & \beta_0 & -\beta_3 & \beta_2 \\ \beta_2 & \beta_3 & \beta_0 & -\beta_1 \\ \beta_3 & -\beta_2 & \beta_1 & \beta_0 \end{pmatrix};$$

(2) $\phi\,(h_1 + h_2) = \phi(h_1) + \phi(h_2)$,
$\phi(h_1 h_2) = \phi(h_1)\phi(h_2)$,
$\phi(\eta h_1) = \eta\phi(h_1)$;

(3) $\phi(1) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = I_4$.

(4) $h_1 = \frac{1}{4} A_4 \phi(h_1) A_4^*$; $A_4 = (1, f_1, f_2, f_3)$ *and* $A_4^* = (1, -f_1, -f_2, -f_3)^T$.

(7) $\phi(\overline{h_1}) = \phi^T(h_1)$.

$$(5)\phi^T(h_1)\phi(h_1) = \phi(h_1)\phi^T(h_1) = |h_1|^2 I_4.$$

(6) $\phi(h_1^{-1}) = \phi^{-1}(h_1), \Longrightarrow h_1$ *is non-zero.*

(7) $det[\phi(h_1)] =$

$$\begin{vmatrix} \alpha_0 & -\alpha_1 & -\alpha_2 & -\alpha_3 \\ \alpha_1 & \alpha_0 & -\alpha_3 & \alpha_2 \\ \alpha_2 & \alpha_3 & \alpha_0 & -\alpha_1 \\ \alpha_3 & -\alpha_2 & \alpha_1 & \alpha_0 \end{vmatrix} = |h_1|^4 \text{[8]}.$$

Proof.

(1) $\alpha_0 + \alpha_1 f_1 + \alpha_2 f_2 + \alpha_3 f_3 = \beta_0 + \beta_1 f_1 + \beta_2 f_2 + \beta_3 f_3$
if and only if

$$\begin{pmatrix} \alpha_0 & -\alpha_1 & -\alpha_2 & -\alpha_3 \\ \alpha_1 & \alpha_0 & -\alpha_3 & \alpha_2 \\ \alpha_2 & \alpha_3 & \alpha_0 & -\alpha_1 \\ \alpha_3 & -\alpha_2 & \alpha_1 & \alpha_0 \end{pmatrix} = \begin{pmatrix} \beta_0 & -\beta_1 & -\beta_2 & -\beta_3 \\ \beta_1 & \beta_0 & -\beta_3 & \beta_2 \\ \beta_2 & \beta_3 & \beta_0 & -\beta_1 \\ \beta_3 & -\beta_2 & \beta_1 & \beta_0 \end{pmatrix}$$

(2) $h_1 + h_2 = (\alpha_0 + \beta_0) + (\alpha_1 f_1 + \alpha_2 f_2 + \alpha_3 f_3) + (\beta_1 f_1 + \beta_2 f_2 + \beta_3 f_3)$ $h_1 + h_2 = (\alpha_0 + \beta_0) + (\alpha_1 + \beta_1) f_1 + (\alpha_2 + \beta_2) f_2 + (\alpha_3 + \beta_3) f_3$

$\phi(h_1 + h_2)$=

$$\begin{pmatrix} \alpha_0 + \beta_0 & -(\alpha_1 + \beta_1) & -(\alpha_2 + \beta_2) & -(\alpha_3 + \beta_3) \\ \alpha_1 + \beta_1 & \alpha_0 + \beta_0 & -(\alpha_3 + \beta_3) & \alpha_2 + \beta_2 \\ \alpha_2 + \beta_2 & \alpha_3 + \beta_3 & \alpha_0 + \beta_0 & -(\alpha_1 + \beta_1) \\ \alpha_3 + \beta_3 & -\alpha_2 + \beta_2 & \alpha_1 + \beta_1 & \alpha_0 + \beta_0 \end{pmatrix}$$

$$= \begin{pmatrix} \alpha_0 & -\alpha_1 & -\alpha_2 & -\alpha_3 \\ \alpha_1 & \alpha_0 & -\alpha_3 & \alpha_2 \\ \alpha_2 & \alpha_3 & \alpha_0 & -\alpha_1 \\ \alpha_3 & -\alpha_2 & \alpha_1 & \alpha_0 \end{pmatrix} + \begin{pmatrix} \beta_0 & -\beta_1 & -\beta_2 & -\beta_3 \\ \beta_1 & \beta_0 & -\beta_3 & \beta_2 \\ \beta_2 & \beta_3 & \beta_0 & -\beta_1 \\ \beta_3 & -\beta_2 & \beta_1 & \beta_0 \end{pmatrix} = \phi(h_1) + \phi(h_2).$$

Remark.

Now we are trying to prove this expression:

$$\phi(h_1 \cdot h_2) = \phi(h_1) \cdot \phi(h_2).$$

I will start by multiplying two quaternions and then I will write the corresponding matrix representations.

$$\begin{aligned} h_1 \cdot h_2 = & (\alpha_0\beta_0 - \alpha_1\beta_1 - \alpha_2\beta_2 - \alpha_3\beta_3) \\ & +(\alpha_0\beta_1 + \beta_0\alpha_1 + \alpha_2\beta_3 - \beta_2\alpha_3) f_1 \\ & +(\alpha_0\beta_2 + \beta_0\alpha_2 + \alpha_3\beta_2 - \beta_3\alpha_1) f_2 \\ & +(\alpha_0\beta_3 + \beta_0\alpha_3 + \alpha_1\beta_2 - \beta_1\alpha_2) f_3, \end{aligned}$$

We will mark the brackets above with x, y, z, t, as follows:
x=$(\alpha_0\beta_0 - \alpha_1\beta_1 - \alpha_2\beta_2 - \alpha_3\beta_3)$
y=$(\alpha_0\beta_1 + \beta_0\alpha_1 + \alpha_2\beta_3 - \beta_2\alpha_3)$
z=$(\alpha_0\beta_2 + \beta_0\alpha_2 + \alpha_3\beta_2 - \beta_3\alpha_1)$

t=$(\alpha_0\beta_3 + \beta_0\alpha_3 + \alpha_1\beta_2 - \beta_1\alpha_2)$.

Then the final form of the multiplication of the two quaternions will be:

$$h_1 \cdot h_2 = x + y f_1 + z f_2 + t f_3.$$

$$\phi(h_1 \cdot h_2) = \begin{pmatrix} x & -y & -z & -t \\ y & x & -t & z \\ z & t & x & -y \\ t & -z & y & x \end{pmatrix} = \begin{pmatrix} \alpha_0 & -\alpha_1 & -\alpha_2 & -\alpha_3 \\ \alpha_1 & \alpha_0 & -\alpha_3 & \alpha_2 \\ \alpha_2 & \alpha_3 & \alpha_0 & -\alpha_1 \\ \alpha_3 & -\alpha_2 & \alpha_1 & \alpha_0 \end{pmatrix} \cdot \begin{pmatrix} \beta_0 & -\beta_1 & -\beta_2 & -\beta_3 \\ \beta_1 & \beta_0 & -\beta_3 & \beta_2 \\ \beta_2 & \beta_3 & \beta_0 & -\beta_1 \\ \beta_3 & -\beta_2 & \beta_1 & \beta_0 \end{pmatrix} = \phi(h_1) \cdot \phi(h_2).$$

To prove this equality $\phi(\eta \cdot h_1) = \eta \cdot \phi(h_1)$ we will calculate $\eta \cdot h_1$ as follows: $\eta \cdot h_1 = \eta\alpha_0 + \eta\alpha_1 f_1 + \eta\alpha_2 f_2 + \eta\alpha_3 f_3$.
The matrix representation of the quaternion $\eta \cdot h_1$ is:

$$\phi(\eta \cdot h_1) = \begin{pmatrix} \eta\alpha_0 & -\eta\alpha_1 & -\eta\alpha_2 & -\eta\alpha_3 \\ \eta\alpha_1 & \eta\alpha_0 & -\eta\alpha_3 & \eta\alpha_2 \\ \eta\alpha_2 & \eta\alpha_3 & \eta\alpha_0 & -\eta\alpha_1 \\ \eta\alpha_3 & -\eta\alpha_2 & \eta\alpha_1 & \eta\alpha_0 \end{pmatrix}.$$

$$\phi(\eta \cdot h_1) = \eta \cdot \begin{pmatrix} \alpha_0 & -\alpha_1 & -\alpha_2 & -\alpha_3 \\ \alpha_1 & \alpha_0 & -\alpha_3 & \alpha_2 \\ \alpha_2 & \alpha_3 & \alpha_0 & -\alpha_1 \\ \alpha_3 & -\alpha_2 & \alpha_1 & \alpha_0 \end{pmatrix} = \eta \cdot \phi(h_1).$$

The relations from $(3), (4), (5), (6), (7)$ are obvious.

Lemma 3.

Left matrix representation for the quaternion $h = \alpha_0 + \alpha_1 f_1 + \alpha_2 f_2 + \alpha_3 f_3 \in \mathbb{H}$ *is given by:*

$$\gamma(h) := R\phi^T(h)R = \begin{pmatrix} \alpha_0 & -\alpha_1 & -\alpha_2 & -\alpha_3 \\ \alpha_1 & \alpha_0 & \alpha_3 & -\alpha_2 \\ \alpha_2 & -\alpha_3 & \alpha_0 & \alpha_1 \\ \alpha_3 & \alpha_2 & -\alpha_1 & \alpha_0 \end{pmatrix},$$

where

$$R = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

Here are some basic properties for $\gamma(h)$:

$$\gamma(h_1 + h_2) = \gamma(h_1) + \gamma(h_2),$$
$$\gamma(h_1 \cdot h_2) = \gamma(h_1) \cdot \gamma(h_2),$$
$$\gamma(\overline{h_1}) = \gamma^T(h_1).$$

The determinant of γ(h) is given by:

$$det[\gamma(h)] = \begin{vmatrix} \alpha_0 & -\alpha_1 & -\alpha_2 & -\alpha_3 \\ \alpha_1 & \alpha_0 & \alpha_3 & -\alpha_2 \\ \alpha_2 & -\alpha_3 & \alpha_0 & \alpha_1 \\ \alpha_3 & \alpha_2 & -\alpha_1 & \alpha_0 \end{vmatrix}$$

$$det[\gamma(h)] = |h|^4, [9]$$

Remark. As follows, I will present two matrix representations of a quaternion h, which has the form $h = \alpha_0 + \alpha_1 f_1 + \alpha_2 f_2 + \alpha_3 f_3$, where $\alpha_i \in \mathbb{R}$.

We denote with $\phi(h) = H_h^l$ the left matrix representation and with $\gamma(h) = H_h^r$ the right matrix representation.

$$\phi(h) = H_h^l = \begin{pmatrix} \alpha_0 & -\alpha_1 & -\alpha_2 & -\alpha_3 \\ \alpha_1 & \alpha_0 & -\alpha_3 & \alpha_2 \\ \alpha_2 & \alpha_3 & \alpha_0 & -\alpha_1 \\ \alpha_3 & -\alpha_2 & \alpha_1 & \alpha_0 \end{pmatrix},$$

$$\gamma(h) = H_h^r = \begin{pmatrix} \alpha_0 & -\alpha_1 & -\alpha_2 & -\alpha_3 \\ \alpha_1 & \alpha_0 & \alpha_3 & -\alpha_2 \\ \alpha_2 & -\alpha_3 & \alpha_0 & \alpha_1 \\ \alpha_3 & \alpha_2 & -\alpha_1 & \alpha_0 \end{pmatrix}.$$

Theorem 3.

Consider the quaternion $h = \alpha_0 + \alpha_1 f_1 + \alpha_2 f_2 + \alpha_3 f_3 \in \mathbb{H}$.

The two matrix representations of a quaternion satisfy the following three properties:

1. $\left(H_h^l\right)^2 = \phi(h^2)$ *;*
2. $(H_h^r)^2 = \gamma(h^2)$ *;*
3. $\left(H_h^l\right)(H_h^r) = (H_h^r)\left(H_h^l\right)$*[9]* .

Proposition 4.

For each quaternion $h = \alpha_0 + \alpha_1 f_1 + \alpha_2 f_2 + \alpha_3 f_3 \in \mathbb{H}$, *we have :*

$$\det H_h^l = \det H_h^r.[9]$$

Proof.

$$detH_h^l = \begin{vmatrix} \alpha_0 & -\alpha_1 & -\alpha_2 & -\alpha_3 \\ \alpha_1 & \alpha_0 & -\alpha_3 & \alpha_2 \\ \alpha_2 & \alpha_3 & \alpha_0 & -\alpha_1 \\ \alpha_3 & -\alpha_2 & \alpha_1 & \alpha_0 \end{vmatrix}$$

$$\det H_h^l = (\alpha_0^2 + \alpha_1^2 + \alpha_2^2 + \alpha_3^2)^2 = |h|^4.$$

$$detH_h^r = \begin{vmatrix} \alpha_0 & -\alpha_1 & -\alpha_2 & -\alpha_3 \\ \alpha_1 & \alpha_0 & \alpha_3 & -\alpha_2 \\ \alpha_2 & -\alpha_3 & \alpha_0 & \alpha_1 \\ \alpha_3 & \alpha_2 & -\alpha_1 & \alpha_0 \end{vmatrix}$$

$$\det H_h^r = (\alpha_0^2 + \alpha_1^2 + \alpha_2^2 + \alpha_3^2)^2 = |h|^4.$$

Preposition 5.

Let's consider the matrix N_8 *where* $N_8 = \begin{pmatrix} \Pi E_4 \\ -E_4 \end{pmatrix}$, *therefore we have* $-\frac{1}{4} N_8^T \cdot N_8 = 1$, [9].

Proof.

To demonstrate this relation we must determine the extended form of the matrix N_8 as follows:

$$N_8 = \begin{pmatrix} f_1 \\ 1 \\ f_3 \\ -f_2 \\ -1 \\ f_1 \\ f_2 \\ f_3 \end{pmatrix}.$$

Now we will determine the product of N_8^T and N_8.

$$N_8^T N_8 =$$

$$(f_1 \quad 1 \quad f_3 \quad -f_2 \quad -1 \quad f_1 \quad f_2 \quad f_3)\cdot\begin{pmatrix} f_1\\ 1\\ f_3\\ -f_2\\ -1\\ f_1\\ f_2\\ f_3\end{pmatrix}$$

$$= -4.$$

$$-\frac{1}{4}N_8^T\cdot N_8 = -\frac{1}{4}(-4) = 1.$$

Proposition 6.
Let's take $\epsilon \in \mathcal{M}_4(\mathbb{R})$ *and* $h = \alpha_0 + \alpha_1 f_1 + \alpha_2 f_2 + \alpha_3 f_3 \in \mathbb{H}$. *If we consider* ϵ *a matrix that has this form:*

$$\epsilon = \begin{pmatrix} 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & -1 & 0\\ 0 & 0 & 0 & -1\end{pmatrix}$$

then we can say:
1. $\epsilon\phi(h)\epsilon = \phi(h^*)$,
where $h^* = \alpha_0 + \alpha_1 f_1 - \alpha_2 f_2 - \alpha_3 f_3$ 2. $\epsilon\gamma(h)\epsilon = \gamma(h^*)$,
where $h^* = \alpha_0 + \alpha_1 f_1 - \alpha_2 f_2 - \alpha_3 f_3$.[9]

Proof. 1. $\epsilon\phi(h)\epsilon =$

$$= \begin{pmatrix} 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & -1 & 0\\ 0 & 0 & 0 & -1\end{pmatrix}\begin{pmatrix} \alpha_0 & -\alpha_1 & -\alpha_2 & -\alpha_3\\ \alpha_1 & \alpha_0 & -\alpha_3 & \alpha_2\\ \alpha_2 & \alpha_3 & \alpha_0 & -\alpha_1\\ \alpha_3 & -\alpha_2 & \alpha_1 & \alpha_0\end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & -1 & 0\\ 0 & 0 & 0 & -1\end{pmatrix},$$

$\epsilon\phi(h)\epsilon=$

$$= \begin{pmatrix} \alpha_0 & -\alpha_1 & -\alpha_2 & -\alpha_3\\ \alpha_1 & \alpha_0 & -\alpha_3 & \alpha_2\\ -\alpha_2 & -\alpha_3 & -\alpha_0 & \alpha_1\\ -\alpha_3 & \alpha_2 & -\alpha_1 & -\alpha_0\end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & -1 & 0\\ 0 & 0 & 0 & -1\end{pmatrix},$$

$$\epsilon\phi(h)\epsilon = \begin{pmatrix} \alpha_0 & -\alpha_1 & \alpha_2 & \alpha_3\\ \alpha_1 & \alpha_0 & \alpha_3 & -\alpha_2\\ -\alpha_2 & -\alpha_3 & \alpha_0 & -\alpha_1\\ -\alpha_3 & \alpha_2 & \alpha_1 & \alpha_0\end{pmatrix}$$

and the matrix representation of
$h^* = \alpha_0 + \alpha_1 f_1 - \alpha_2 f_2 - \alpha_3 f_3$ is

$$\phi(h^*) = \begin{pmatrix} \alpha_0 & -\alpha_1 & \alpha_2 & \alpha_3\\ \alpha_1 & \alpha_0 & \alpha_3 & -\alpha_2\\ -\alpha_2 & -\alpha_3 & \alpha_0 & -\alpha_1\\ -\alpha_3 & \alpha_2 & \alpha_1 & \alpha_0\end{pmatrix} =$$

$$= \epsilon\phi(h)\epsilon.$$

Proceeding in the same way as for property 1, I will also determine the other two equalities.
2. $\epsilon\gamma(h)\epsilon =$

$$= \begin{pmatrix} 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & -1 & 0\\ 0 & 0 & 0 & -1\end{pmatrix}\cdot\begin{pmatrix} \alpha_0 & -\alpha_1 & -\alpha_2 & -\alpha_3\\ \alpha_1 & \alpha_0 & \alpha_3 & -\alpha_2\\ \alpha_2 & -\alpha_3 & \alpha_0 & \alpha_1\\ \alpha_3 & \alpha_2 & -\alpha_1 & \alpha_0\end{pmatrix}$$

$$\cdot\begin{pmatrix} 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & -1 & 0\\ 0 & 0 & 0 & -1\end{pmatrix};$$

$$\epsilon\gamma(h)\epsilon =$$

$$= \begin{pmatrix} \alpha_0 & -\alpha_1 & -\alpha_2 & -\alpha_3\\ \alpha_1 & \alpha_0 & \alpha_3 & -\alpha_2\\ -\alpha_2 & \alpha_3 & -\alpha_0 & -\alpha_1\\ -\alpha_3 & -\alpha_2 & \alpha_1 & -\alpha_0\end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & -1 & 0\\ 0 & 0 & 0 & -1\end{pmatrix},$$

$$\epsilon\gamma(h)\epsilon = \begin{pmatrix} \alpha_0 & -\alpha_1 & \alpha_2 & \alpha_3\\ \alpha_1 & \alpha_0 & -\alpha_3 & \alpha_2\\ \alpha_2 & -\alpha_3 & \alpha_0 & \alpha_1\\ -\alpha_3 & -\alpha_2 & -\alpha_1 & \alpha_0\end{pmatrix} = \gamma(h^*).$$

Proposition 7.
If $h = \alpha_0 + \alpha_1 f_1 + \alpha_2 f_2 + \alpha_3 f_3$, *where* $\alpha_i \in \mathbb{R}$, *then we have:*

i. $\phi(h)\cdot E_4 = E_4\cdot h$.
ii. $\Pi\cdot E_4 = E_4\cdot f_1$.
iii. $\phi(f_1\cdot h) = \Pi\cdot\phi(h)$ *and*
$\phi(h\cdot f_1) = \phi(h)\cdot\Pi$.*[9]*

Proof.

i. $$\phi(h)\cdot E_4 = \begin{pmatrix} \alpha_0 & -\alpha_1 & -\alpha_2 & -\alpha_3\\ \alpha_1 & \alpha_0 & -\alpha_3 & \alpha_2\\ \alpha_2 & \alpha_3 & \alpha_0 & -\alpha_1\\ \alpha_3 & -\alpha_2 & \alpha_1 & \alpha_0\end{pmatrix}\cdot$$

$$\begin{pmatrix} 1\\ -f_1\\ -f_2\\ -f_3\end{pmatrix}$$

$$\phi(h)\cdot E_4 = \begin{pmatrix} \alpha_0 + \alpha_1 f_1 + \alpha_2 f_2 + \alpha_3 f_3\\ \alpha_1 - \alpha_0 f_1 + \alpha_3 f_2 - \alpha_2 f_3\\ \alpha_2 - \alpha_3 f_1 - \alpha_0 f_2 + \alpha_1 f_3\\ \alpha_3 + \alpha_2 f_1 - \alpha_1 f_2 - \alpha_0 f_3\end{pmatrix}$$

$$\phi(h)\cdot E_4=\begin{pmatrix}\alpha_0+\alpha_1 f_1+\alpha_2 f_2+\alpha_3 f_3\\ -f_1(\alpha_0+\alpha_1 f_1+\alpha_2 f_2+\alpha_3 f_3)\\ -f_2(\alpha_0+\alpha_1 f_1+\alpha_2 f_2+\alpha_3 f_3)\\ -f_3(\alpha_0+\alpha_1 f_1+\alpha_2 f_2+\alpha_3 f_3)\end{pmatrix}$$

$$=\begin{pmatrix}1\\-f_1\\-f_2\\-f_3\end{pmatrix}\cdot(\alpha_0+\alpha_1 f_1+\alpha_2 f_2+\alpha_3 f_3)$$

$$ii.\ \Pi\cdot E_4=\begin{pmatrix}0&-1&0&0\\1&0&0&0\\0&0&0&-1\\0&0&1&0\end{pmatrix}\cdot\begin{pmatrix}1\\-f_1\\-f_2\\-f_3\end{pmatrix}=\begin{pmatrix}f_1\\1\\f_3\\-f_2\end{pmatrix}$$

$$\Pi\cdot E_4=\begin{pmatrix}1\\-f_1\\-f_2\\-f_3\end{pmatrix}\cdot f_1=E_4\cdot f_1.$$

iii. For $h=\alpha_0+\alpha_1 f_1+\alpha_2 f_2+\alpha_3 f_3\in\mathbb{H}$, we have $f_1 h=\alpha_1+\alpha_0 f_1-\alpha_3 f_2+\alpha_2 f_3$. It results that

$$\phi(f_1\cdot h)=\begin{pmatrix}-\alpha_1&-\alpha_0&\alpha_3&-\alpha_2\\ \alpha_0&-\alpha_1&-\alpha_2&-\alpha_3\\ -\alpha_3&\alpha_2&-\alpha_1&-\alpha_0\\ \alpha_2&\alpha_3&\alpha_0&-\alpha_1\end{pmatrix}.$$

$$\Pi\cdot\phi(h)=\begin{pmatrix}0&-1&0&0\\1&0&0&0\\0&0&0&-1\\0&0&1&0\end{pmatrix}$$

$$\begin{pmatrix}\alpha_0&-\alpha_1&-\alpha_2&-\alpha_3\\ \alpha_1&\alpha_0&-\alpha_3&\alpha_2\\ \alpha_2&\alpha_3&\alpha_0&-\alpha_1\\ \alpha_3&-\alpha_2&\alpha_1&\alpha_0\end{pmatrix}$$

$$\Pi\cdot\phi(h)=\begin{pmatrix}-\alpha_1&-\alpha_0&\alpha_3&-\alpha_2\\ \alpha_0&-\alpha_1&-\alpha_2&-\alpha_3\\ -\alpha_3&\alpha_2&-\alpha_1&-\alpha_0\\ \alpha_2&\alpha_3&\alpha_0&-\alpha_1\end{pmatrix}.$$

So the required relationships are obtained.

Proposition 8.
For $h=f_1$, $h=f_2$, $h=f_3\in\mathbb{H}$, we have:

$$det(\phi\ (f_1))=det(\phi(f_2))=det(\phi\ (f_3))=1.$$

Proof.

$$det(\phi(f_1))=\begin{vmatrix}0&-1&0&0\\1&0&0&0\\0&0&0&-1\\0&0&1&0\end{vmatrix}=1,$$

$$det(\phi(f_2))=\begin{vmatrix}0&0&-1&0\\0&0&0&1\\1&0&0&0\\0&-1&0&0\end{vmatrix}=1,$$

$$det(\phi\ (f_3))=\begin{vmatrix}0&0&0&-1\\0&0&-1&0\\0&1&0&0\\1&0&0&0\end{vmatrix}=1.$$

4 Conclusion

This article is structured into two distinct chapters.

In the first chapter, it explores the algebra of complex quaternions and its connections with 2x2 matrices. Complex quaternions are represented through a matrix formulation and the theorems and properties presented contribute to the development of a fundamental understanding.

The isomorphism with 2x2 matrices and details related to conjugation and norms provide a robust framework for the efficient manipulation of these complex mathematical entities..

In the second chapter, we conducted a study on the matrix forms of quaternions on the set of real numbers and introduced two real matrix representations: the right matrix representation and the left matrix representation.

Based on these matrix representations, we detailed several theorems and propositions with their corresponding proofs.

This article represents a significant starting point for further research on matrices of biquaternions and real quaternion matrices.

References:

[1] J. Vince, Quaternions for Computer Graphics, *Springer London Dordrecht Heidelberg*, New York, (2021).

[2] G.M. Dixon, Division algebras, *Springer-Science Business Media*, B, (1994).

[3] J. Voight, Quaternion algebras, *Springer*, March 27-2021.

[4] H. Aslaksen, Quaternion Determinants, *The Mathematical Intelligencer*, June 1996.

[5] C. Flaut, Some remarks regarding quaternions and octonions, *Bulletin mathématique de la Société des Sciences Mathématiques de Roumanie*, vol. 52 (100), No.4, 2009, pp.427-439.

[6] Y. Tian, Matrix Theory over the Complex Quaternion Algebra, *Mathematics, Rings and Algebras*, 1 April 2000, https://doi.org/10.48550/arXiv.math/0004005.

[7] C. Flaut, M. Stefanescu, Some equations over generalized quaternion and octonion division algebras, Bull. Math. Soc. Sci. Math. Roumanie, 52(4), (100), 2009, pp.427-439.

[8] Y. Tian, Matrix representations of octonions and their applications, *Adv.Appl. Clifford Algebras*, vol.10(1), 2000, pp.61-90.

[9] C. Flaut, V. Shpakivskyi Real matrix representations for the complex quaternions, *Adv. Appl. Clifford Algebras*, vol.23(3), 2013, pp.657-671.

Contribution of Individual Authors to the Creation of a Scientific Article (Ghostwriting Policy)

I am the sole author of this article. The results obtained in this article comprise Proposition 1, Proposition 2, Proposition 8, and examples supporting the presented theory.

Sources of Funding for Research Presented in a Scientific Article or Scientific Article Itself

I do not benefit from any funding source.

Conflict of Interest

As a doctoral student at Ovidius University, this paper has been prepared with the purpose of fulfilling the minimum requirements for the completion of doctoral studies. The publication of this article contributes to achieving this academic goal, and as an author, I confirm that the information and results presented are handled with integrity and transparency. There are no conflicts of interest to declare.