The Use of Modified Fractional Differential Transform for Multi-term Fractional Order Differential Equations

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Abstract: The differential transform method has been prevalently expedited in the last decades by elucidating the solutions of partial differential equations. In this paper, the multi-term fractional differential equations have been solved by using the modified differential transform method combining the fractional integral operator to omit one term consisting of fractional differential order. Compared to the previous research, the method is effective and approached to approximate solutions that lead to exact solutions.

Key-Words: Multi-term fractional order; multi-term fractional order; dtm; modified differential transform method

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1 Introduction

A plethora of mathematical scientists have considered fractional differential equations to find solutions using different kinds of methods. The differential transform method and its variations have been extended for fractional differential orders and used to gain the solutions of fractional differential equations, [1]. Modifying the fractional reduced differential transform method effectively solves the multi-term fractional order of the fractional differential equations, [2]. The high accuracy of the solution is critical to the method's performance. The differential transform method has been successfully applied to solving fractional differential equations such as the BagleyTorvik, Ricatti, and oscillation equation, [3]. The multi-term fractional differential equations have been scrutinized by using the fractional Taylor vector method to attain the numerical solutions compared to exact solutions, [4]. A series of multi-term fractional differential equations in the Newtonian fluid has been solved using Euler's method, including the trapezoidal quadrature formula and product rectangle rule, [5]. Another effective method, the pseudo-spectral method, solves the fractional differential equations and performs the benefit outputs of the extensive method, [6].

The attained solutions could be expressed as exact or approximate solutions, [7]. The differential transform method is extracted from the Taylor expansion and is important in finding a series of solutions and exact solutions, [8]. The extension of the differential transform method is a reduced differential transform method that is applied to seek analytic solutions and has been expanded to solve fractional differential equations, [9]. The proposed method is for solving one-term fractional differential derivatives and multi-term fractional differential derivatives, [10] has decomposed the fractional differential equation into a system of differential equations, [11]. The output of the differential transform method is usually performed in infinite power series or approximate solutions, [12] and performs solutions using Matlab software, [13].

In this paper, the modified fractional differential transform method is used by solving multi-term fractional derivative equations that have been taken into account in the last few years. The application of the differential transform method combining differential integral operator is the main aim of the paper, [14], [15] that has been depicted in the research, [16]. Some practical related definitions, [17] are also used in this paper to contribute to the performance of finding solutions.

2 Basic definitions and preliminaries

This section will perform some useful definitions applied in fractional differential equations and this paper.

Definition 1. *Gamma function (, [17])*

$$\Gamma(p) = \int_0^\infty e^{-x} x^{p-1} dx$$

Theorem 2.1 (Caputo's Fractional Derivatives). (, [17]) Given a function $y = f(t)$, and an arbitrary order $n-1 < \alpha \leq n$, the Caputo's fractional derivative of order α is given by:

Table 1: Basic results using DTMT method , [18], [19, 20], [21].

Original Functions	DTM for some fundamental functions
$f(t)=ag(t)\pm bh(t)$	$F_k=aG_k\pm bH_k$
$f(t)=g(t)h(t)$	$F_k=\sum_{r=0}^{k}G_rH_{k-r}$
$f(t)=t^p$	$F_k=\delta(k-\alpha p)=\begin{cases}1 & \text{if} \quad k=\alpha p\\ 0 & \text{if} \quad k\neq\alpha p\end{cases}$
$f(t)=\ {}_0^C\mathscr{D}_t^{\alpha}g(t)$	$F_k=\frac{\Gamma(\alpha(k+1)+1)}{\Gamma(1+k\alpha)}G_{k+1}$
$f(t)=\ {}_0^C\mathscr{D}_t^{\beta}g(t),\beta=pq$	$F_k=\frac{\Gamma(q(k+p)+1)}{\Gamma(kq+1)}G_{k+p}$
$f(t)=at^m$	$F_k=a\delta(k-m)$
$f(t)=\ {}_0^C\mathscr{D}_t^{n}g(t)$	$F_k=(k+1)(k+2)\cdots(k+n)G_{k+n}$
$f(t)=\sin\omega t$	$F_k=\frac{\omega^k}{k!}\sin\frac{k\pi}{2}$
$f(t)=\cos\omega t$	$F_k=\frac{\omega^k}{k!}\cos\frac{k\pi}{2}$
$f(t)=e^{\lambda t}$	$F_k=\frac{\lambda!}{k!}$

$$
{}_0^C\mathscr{D}_x^{\alpha}f(x)=\frac{1}{\Gamma(n-\alpha)}\int_a^x(x-s)^{n-\alpha-1}f^{(n)}(s)ds
$$

Based on the Theorem 2.1, we have derivative of function $f(x)=C$, (C is a constant):

$$
{}_0^C\mathscr{D}_x^{\alpha}f(x)=0
$$

The inverse operator ${}_0^C\mathscr{D}_x^{\alpha}$, called J_x^{α} is fractional integral operator of order α is given as the following

$$
J^{\alpha}f(x)=\frac{1}{\Gamma(\alpha)}\int_0^x(x-w)^{\alpha-1}f(w)dw
$$

We have the Caputo fractional derivative property

$$
J^{\alpha}({}_0^C\mathscr{D}_x^{\alpha}f(x))=f(x)-\sum_{k=0}^{n-1}f^{(k)}(0)\frac{x^k}{k!},x>0 \qquad (1)
$$

Suppose the

$$
f(t)=\sum_{k=0}^{\infty}F_kt^{kq}
$$

where $\beta=pq$ is the fractional derivative order, p is integer part, q is fractional part, and F_k,G_k,H_k is the transformation of $f(t),g(t),h(t)$ respectively. For the initial integer-order derivative condition with $k=0,1,...,(pq-1)$, we could transform as follows:

$$
F_k=\begin{cases}0 & \text{if} \quad kq\notin\mathbb{Z}^+\\ \frac{1}{(kq)!}\left(\frac{d^{kq}}{dt^{kq}}f(t)\right)_{t=t_0} & \text{if} \quad kq\in\mathbb{Z}^+\end{cases}
$$

3 Methodology

Step 1: Integrating the equation (1) to transform one of the fractional differential terms.

Step 2: Using differential transform method in Table 1 to transform the term of fractional order in the form defined by

$$
F_k=\sum_{k=0}^{\infty}H_k
$$

Step 3: Applying the inductive method to calculate the terms of the transform and gain the solution satisfies the condition:

$$
f(t)=\lim_{k\to\infty}F_k
$$

4 Applications

Example 4.1. Consider the nonlinear fractional equation , [5]

$$
{}_0^C\mathscr{D}_t^{\alpha}y(t)+y(t)=\frac{2t^{2-\alpha}}{\Gamma(3-\alpha)}+t^2-t,1<\alpha<2 \qquad (2)
$$

subjected to the condition $y(0)=0,y'(0)=-1$.

Step 1:

Taking transform both sides Eq. (4.1) using Eq. (1):

$$
y(t)-y(0)-ty'(0)+J^{\alpha}[y(t)]=J^{\alpha}\left[\frac{2t^{2-\alpha}}{\Gamma(3-\alpha)}+t^2-t\right]
$$

Step 2:

Taking DTM transform on both sides, we have

$$
H_k+\delta(k-1)+J^{\alpha}[H_k]=J^{\alpha}\left[\begin{array}{c}\frac{2}{\Gamma(3-\alpha)}\delta(k-2+\alpha)\\ +\delta(k-2)-\delta(k-1)\end{array}\right]
$$

Step 3:

Using the initial conditions, we have:

$H_0=0,H_1=-1$.

We choose $k=2$ then $H_2=1$.

Using the inductive method, we have the following terms:

$H_3=0,H_4=0,\cdots$

The exact solution collected is $y(t)=t^2-t$.

The solution is compared in Table 3 and depicted in Figure 2 (APPENDIX).

Example 4.2. Cogitate the fractional differential equation, [4]

$$D^2y(t) = -y(t) + 2y'(t) - D^{\frac{1}{2}}y(t) + t^7 + \frac{2048}{429\sqrt{\pi}}t^{6.5} - 14t^6 + 42t^5 - t^2 - \frac{8}{3\sqrt{\pi}}t^{1.5} + 4t - 2. \quad (3)$$

satisfy the condition $y(0) = y'(0) = 0, \beta = \frac{1}{2}$.
Step 1:
Taking transform both sides Eq. (4.2) using Eq. (1):

$$J^{\beta}\left[D^2y(t)\right] = -y(t) + J^{\beta}\left[2y'(t) - y(t) + t^7 + \frac{2048}{429\sqrt{\pi}}t^{6.5} - 14t^6 + 42t^5 - t^2 - \frac{8}{3\sqrt{\pi}}t^{1.5} + 4t - 2\right]$$

Step 2:
Taking DTM transform on both sides, we have

$$J^{\beta}\left[(k+1)(k+2)H_{k+2}\right] = -H_k + J^{\beta}\left[\begin{array}{c} 2(k+1)H_{k+1} - H_k + \delta(k-7) \\ +\frac{2048}{429\sqrt{\pi}}\delta(k-6.5) - 14\delta(k-6) \\ +42\delta(k-5) - \delta(k-2) \\ -\frac{8}{3\sqrt{\pi}}\delta(k-1.5) + 4\delta(k-1) - 2\delta(k) \end{array}\right]$$

Step 3:
Using the initial condition, we have $H_0 = H_1 = 0$. We will calculate the terms of the DTM transform as the following:

$$\begin{aligned} k = 0 &: H_2 = -1 \\ k = 1 &: H_3 = 0 \\ k = 2 &: H_4 = 0 \\ k = 3 &: H_5 = 0 \\ k = 4 &: H_6 = 0 \\ k = 5 &: H_7 = 1 \\ k = 6 &: H_8 = 0 \\ k = 7 &: H_9 = 0 \\ k = 8 &: H_{10} = 0 \\ &\cdots \end{aligned}$$

The exact solution obtained is $y(t) = t^7 - t$.

The solution is compared in Table 5 and depicted in Figure 3 (APPENDIX).

Example 4.3. Taking into account the Bagley-Torvik equation, [3]

$$aD^2y(t) + b{}_0^C\mathscr{D}_t^{\alpha}y(t) + cy(t) = c(t+1) \quad (4)$$

satisfy the condition $y(0) = y'(0) = 1, \alpha = \frac{3}{2}$.
Step 1:
Taking transform both sides Eq. (4.3) using Eq. (1):

$$aJ^{\alpha}\left[D^2y(t)\right] + b\left[y(t) - y(0) - ty'(0)\right] + cJ^{\alpha}\left[y(t)\right] = cJ^{\alpha}\left[t+1\right]$$

Step 2:
Taking the DTM transform of the equation, we have

$$aJ^{\alpha}\left[(k+1)(k+2)H_{k+2}\right] + b\left[H_k - \delta(k) - \delta(k-1)\right] + cJ^{\alpha}\left[H_k\right] = cJ^{\alpha}\left[\delta(k-1) + \delta(k)\right].$$

Step 3:
We calculate the terms of the DTM transform:
$H_0 = 1; H_1 = 1$; We choose $k = 0$: then $H_2 = 0$.
$k = 1$: then $H_3 = 1$.
$k = 2$: then $H_4 = 0$;
Using the inductive method, we have the following terms of the transform:
$H_5 = 0; \cdots$
The exact solution is $y(t) = 1 + t$.

The solution is compared in Table 4 and depicted in Figure 4 (APPENDIX).

Example 4.4. Consider the two-term fractional order initial value problem, [5]

$${}_0^C\mathscr{D}_t^{\alpha}y(t) + 1.3{}_0^C\mathscr{D}_t^{\beta}y(t) + 2.6y(t) = \sin(2t) \quad (5)$$

satisfy the condition $y(0) = y'(0) = y''(0) = 0; \alpha = 2.2, \beta = 1.5$.
Step 1:
Taking transform both sides Eq. (4.4) using Eq. (1):

$$y(t) - y(0) - ty'(0) - t^2y''(0) + 1.3J^{\alpha}\left[D^{\beta}y(t)\right] + J^{\alpha}\left[2.6y(t)\right] = J^{\alpha}\left[\sin(2t)\right]$$

Using initial conditions, we simplify the equation as follows

$$y(t) + 1.3J^{\alpha}\left[{}_0^C\mathscr{D}_t^{\beta}y(t)\right] = J^{\alpha}\left[\sin(2t) - 2.6y(t)\right]$$

Step 2:
Using DTM to simplify the equation

$$J^{\alpha}\left[H_{k+p}\right] = \frac{\Gamma(kq+1)}{1.3\Gamma(q(k+p)+1)}\left(J^{\alpha}\left[\frac{2^k}{k!}\sin\left(\frac{k\pi}{2}\right) - 2.6H_k\right] - 1.3H_k\right)$$

where $\beta = pq; p = 3, q = 0.5$;
Step 3:
Using the initial conditions:
$H_0 = 0; H_1 = 0; H_2 = 0$;

We calculate the terms of DTM up to $k = 17$ as follows:

$$\begin{aligned}
k=0 &: H_3 = 0;\\
k=1 &: H_4 = 0.68171;\\
k=2 &: H_5 = 0;\\
k=3 &: H_6 = -0.22724;\\
k=4 &: H_7 = -0.23443;\\
k=5 &: H_8 = 2.8405\times 10^{-2};\\
k=6 &: H_9 = 5.2097\times 10^{-2};\\
k=7 &: H_{10} = 4.3553\times 10^{-2};\\
k=8 &: H_{11} = -4.7361\times 10^{-3};\\
k=9 &: H_{12} = -7.4958\times 10^{-3};\\
k=10 &: H_{13} = -5.5859\times 10^{-3};\\
k=11 &: H_{14} = 6.049\times 10^{-4};\\
k=12 &: H_{15} = 7.6911\times 10^{-4};\\
k=13 &: H_{16} = 5.1853\times 10^{-4};\\
k=14 &: H_{17} = -5.1113\times 10^{-5};\\
k=15 &: H_{18} = -5.9492\times 10^{-5};\\
k=16 &: H_{19} = -3.6897\times 10^{-5};\\
k=17 &: H_{20} = 3.3606\times 10^{-6},\cdots
\end{aligned}$$

Solution is expressed in the form of $y = \sum_{k=0}^{\infty} H_k t^{kq}$

$$\begin{aligned}
y(t) = {}& 0.68171t^2 - 0.22724t^3 - 0.23443t^{3.5}\\
&+ 0.028405t^4 + 0.052097t^{4.5} + 0.043553t^5\\
&- 0.0047361t^{5.5} - 0.0074958t^6 - 5.5859t^{6.5}\\
&+ 0.0006049t^7 + 0.00076911t^{7.5}\\
&+ 0.00051853t^8 - 0.000051113t^{8.5}\\
&- 0.000059492t^9 - 0.000036897t^{9.5}\\
&+ 0.0000033606t^{10}\cdots
\end{aligned}$$

The solution is close to the solutions gained in, [4] using the fractional Taylor method (FTM) performed in Table 1 and depicted in Figure 5 (APPENDIX).

Example 4.5. Consider the general two-term fractional order initial value problem

$$ {}_0^C\mathscr{D}_t^{\alpha} y(t) + 1.3\,{}_0^C\mathscr{D}_t^{\beta} y(t) + 2.6y(t) = \sin(2t) \qquad (6)$$

satisfy the condition $y(0) = y'(0) = y''(0) = 0$. and $2 < \alpha < 3, 0 < \beta < 1$.

Step 1:

Taking transform both sides Eq. (4.5) using Eq. (1):

$$\begin{aligned}
&y(t) - y(0) - ty'(0) - t^2y''(0)\\
&+ 1.3J^{\alpha}\left[{}_0^C\mathscr{D}_t^{\beta} y(t)\right] + J^{\alpha}[2.6y(t)] = J^{\alpha}[\sin(2t)]
\end{aligned}$$

Using initial conditions and simplifying

$$y(t) + 1.3J^{\alpha}\left[{}_0^C\mathscr{D}_t^{\alpha} y(t)\right] = J^{\alpha}[\sin(2t) - 2.6y(t)]$$

Step 2:

Taking DTM on both sides, we have

$$\begin{aligned}
J^{\alpha}[H_{k+1}] = \frac{\Gamma(k\beta+1)}{1.3\Gamma(\beta(k+1)+1)}\Bigg[J^{\alpha}\Bigg[\frac{2^k}{k!}\sin\left(\frac{k\pi}{2}\right)\\
- 2.6H_k\Bigg] - 1.3H_k\Bigg]
\end{aligned}$$

Step 3:

We have $H_0 = 0$ using initial conditions. We calculate the terms of DTM: Taking

$$\begin{aligned}
k=0 &: H_1 = 0\\
k=1 &: H_2 = H_2 = \frac{1.5385\Gamma(\beta+1)}{\Gamma(2\beta+1)}\\
k=2 &: H_3 = -\frac{3.077\Gamma(\beta+1)}{\Gamma(3\beta+1)}\\
k=3 &: H_4 = -\frac{6.1540\Gamma(\beta+1)}{\Gamma(4\beta+1)} - 1.0256\frac{\Gamma(3\beta+1)}{\Gamma(4\beta+1)}
\end{aligned}$$

$k = 4$:

$$H_5 = \frac{12.308\Gamma(\beta+1)}{\Gamma(5\beta+1)} + \frac{2.0512\Gamma(3\beta+1)}{\Gamma(5\beta+1)}.$$

$k = 5$:

$$\begin{aligned}
H_6 = \frac{\Gamma(5\beta+1)}{1.3\Gamma(6\beta+1)}\Bigg(\frac{4}{15} - \frac{32.001\Gamma(\beta+1)}{\Gamma(5\beta+1)}\\
- 5.3331\frac{\Gamma(3\beta+1)}{\Gamma(5\beta+1)}\Bigg)
\end{aligned}$$

$k = 6$:

$$\begin{aligned}
H_7 = 1.5385\frac{\Gamma(5\beta+1)}{\Gamma(7\beta+1)}\Bigg(\frac{32.001}{\Gamma(5\beta+1)}\Gamma(\beta+1)\\
+ 5.3331\frac{\Gamma(3\beta+1)}{\Gamma(5\beta+1)} - \frac{4}{15}\Bigg)\\
\ldots
\end{aligned}$$

The solution collected in the form as follows

$$\begin{aligned} y(t) &= \frac{1.5385\Gamma(\beta+1)}{\Gamma(2\beta+1)}t^2 - \frac{3.077\Gamma(\beta+1)}{\Gamma(3\beta+1)}t^3 \\ &- \left(\frac{6.1540\Gamma(\beta+1)}{\Gamma(4\beta+1)} + \frac{1.0256\Gamma(3\beta+1)}{\Gamma(4\beta+1)}\right)t^4 \\ &+ \left(\frac{12.308\Gamma(\beta+1)}{\Gamma(5\beta+1)} + \frac{2.0512\Gamma(3\beta+1)}{\Gamma(5\beta+1)}\right)t^5 \\ &+ \frac{\Gamma(5\beta+1)}{1.3\Gamma(6\beta+1)}\left(\frac{4}{15} - \frac{32.001\Gamma(\beta+1)}{\Gamma(5\beta+1)} \right. \\ &\left. - \frac{5.3331\Gamma(3\beta+1)}{\Gamma(5\beta+1)}\right)t^6 \\ &+ \frac{1.5385\Gamma(5\beta+1)}{\Gamma(7\beta+1)}\left(\frac{32.001\Gamma(\beta+1)}{\Gamma(5\beta+1)} \right. \\ &\left. + \frac{5.3331\Gamma(3\beta+1)}{\Gamma(5\beta+1)} - \frac{4}{15}\right)t^7 \\ &+ \cdots \end{aligned}$$

The solution is depicted in Figure 5 and Figure 6 (APPENDIX).

5 Conclusion

The modified differential transform method is the combination of the fractional integral operator and the differential transform method. The method has been successfully applied to seek the solutions of multi-fractional differential equations. The main purpose of the method is to eliminate one of the fractional differential orders and then use the differential transform method for the other term. The solutions have been compared to other methods by using the Matlab application to illustrate the numerical results and two-dimensional graph performance. The solutions found could lead to exact solutions or be close to the ones using other methods.

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References:

[1] Abuasad, S., Hashim, I., & Abdul Karim, S. A. (2019). Modified Fractional Reduced Differential Transform Method for the Solution of Multiterm Time-Fractional Diffusion Equations. Advances in Mathematical Physics, 2019, 114. https://doi.org/10.1155/2019/5703916

[2] Al-rabtah, A., & Abuasad, S. (2023). Effective Modified Fractional Reduced Differential Transform Method for Solving Multi-Term Time-Fractional Wave-Diffusion Equations. Symmetry, 15(9), 1721. https://doi.org/10.3390/sym15091721

[3] Arikoglu, A., & Ozkol, I. (2007). Solution of fractional differential equations by using differential transform method. Chaos, Solitons & Fractals, 34(5), 14731481. https://doi.org/10.1016/j.chaos.2006.09.004

[4] Avc, ., & Mahmudov, N. I. (2020). Numerical Solutions for Multi-Term Fractional Order Differential Equations with Fractional Taylor Operational Matrix of Fractional Integration. Mathematics, 8(1), 96. https://doi.org/10.3390/math8010096

[5] El-Mesiry, A. E. M., El-Sayed, A. M. A., & El-Saka, H. A. A. (2005). Numerical methods for multi-term fractional (arbitrary) orders differential equations. Applied Mathematics and Computation, 160(3), 683699. https://doi.org/10.1016/j.amc.2003.11.026

[6] Esmaeili, S., & Shamsi, M. (2011). A pseudo-spectral scheme for the approximate solution of a family of fractional differential equations. Communications in Nonlinear Science and Numerical Simulation, 16(9), 36463654. https://doi.org/10.1016/j.cnsns.2010.12.008

[7] Tuan, N. M., Kooprasert, S., Sirisubtawee, S., & Meesad, P. (2024). The bilinear neural network method for solving Benney-Luke equation. Partial Differential Equations in Applied Mathematics, 10, 100682. https://doi.org/10.1016/j.padiff.2024.100682

[8] Idrees, M., & Muhammad, F. (2012). Application of Differential Transform Method Stiff Systems. 6.

[9] Keskin, Y., & Oturanç, G. (2009). Reduced Differential Transform Method for Partial Differential Equations. International Journal of Nonlinear Sciences and Numerical Simulation, 10(6). https://doi.org/10.1515/IJNSNS.2009.10.6.741

[10] Leszczyski, J., & Ciesielski, M. (2002). A Numerical Method for Solution of Ordinary Differential Equations of Fractional Order. In R. Wyrzykowski, J. Dongarra, M. Paprzycki, & J. Waniewski (Eds.), Parallel Processing and Applied Mathematics (Vol. 2328, pp. 695702). Springer Berlin Heidelberg. https://doi.org/10.1007/3-540-48086-2_77

[11] Noori, S. R. M., & Taghizadeh, N. (2020). Study of convergence of reduced differential transform method for different classes of nonlinear differential equations [Preprint]. Preprints. https://doi.org/10.22541/au.159285253.32751770

[12] Sohail, M., & Mohyud-Din, S. T. (2012). Reduced Differential Transform Method for Solving a System of Fractional PDEs. 10.

[13] Sunday, O. A., & Lois, J. F. (2021). Construction of Functions for Fractional Derivatives using Matlab. Journal of Advances in Mathematics and Computer Science, 110. https://doi.org/10.9734/jamcs/2021/v36i630368

[14] Tuan, N. M., Koonprasert, S., Sirisubtawee, S., Meesad, P., & Khansai, N. (2024). New Solutions of Benney-Luke Equation Using The (G'/G,1/G) Method. Wseas Transactions on mathematics, 23, 267-275. https://doi.org/10.37394/23206.2024.23.29

[15] Tuan, N. M. (2023). A Study of Applied Reduced Differential Transform Method Using Volterra Integral Equations in Solving Partial Differential Equations. EQUATIONS, 3, 93103. https://doi.org/10.37394/232021.2023.3.11

[16] Wang, Y. (2019). Positive solutions for a class of two-term fractional differential equations with multipoint boundary value conditions. Advances in Difference Equations, 2019(1), 304. https://doi.org/10.1186/s13662-019-2250-x

[17] Das, S. (2011). Functional Fractional Calculus. Springer Berlin Heidelberg. https://doi.org/10.1007/978-3-642-20545-3

[18] Nazari, D., & Shahmorad, S. (2010). Application of the fractional differential transform method to fractional-order integro-differential equations with nonlocal boundary conditions. Journal of Computational and Applied Mathematics, 234(3), 883891. https://doi.org/10.1016/j.cam.2010.01.053

[19] Benhammouda, B., Vazquez-Leal, H., & Sarmiento-Reyes, A. (2014). Modified Reduced Differential Transform Method for Partial Differential-Algebraic Equations. Journal of Applied Mathematics, 2014, 19. https://doi.org/10.1155/2014/279481

[20] Chen, C.-K., & Ho, S.-H. (1996). Application of differential transformation to eigenvalue problems. Applied Mathematics and Computation, 79(23), 173188. https://doi.org/10.1016/0096-3003(95)00253-7

[21] Abuteen, E., Freihat, A., Al-Smadi, M., Khalil, H., & Khan, R. A. (2016). Approximate Series Solution of Nonlinear, Fractional Klein-Gordon Equations Using Fractional Reduced Differential Transform Method. Journal of Mathematics and Statistics, 12(1), 2333. https://doi.org/10.3844/jmssp.2016.23.33

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APPENDIX

Table 2: Solutions of example 4.1

Values	DTM transform	Exact solution	errors
0.0	0.0000	0.0000	0.0000
0.1	-0.0900	-0.0900	0.0000
0.2	-0.1600	-0.1600	0.0000
0.3	-0.2100	-0.2100	0.0000
0.4	-0.2400	-0.2400	0.0000
0.5	-0.2500	-0.2500	0.0000
0.6	-0.2400	-0.2400	0.0000
0.7	-0.2100	-0.2100	0.0000
0.8	-0.1600	-0.1600	0.0000
0.9	-0.0900	-0.0900	0.0000
1.0	0.0000	0.0000	0.0000

Table 3: Solutions of example 4.2

Values	DTM transform	Exact solution	errors
0.0	0.0000	0.0000	0.0000
0.1	-0.1000	-0.1000	0.0000
0.2	-0.2000	-0.2000	0.0000
0.3	-0.2998	-0.2998	0.0000
0.4	-0.3984	-0.3984	0.0000
0.5	-0.4922	-0.4922	0.0000
0.6	-0.5720	-0.5720	0.0000
0.7	-0.6176	-0.6176	0.0000
0.8	-0.5903	-0.5903	0.0000
0.9	-0.4217	-0.4217	0.0000
1.0	0.0000	0.0000	0.0000

Table 4: Solutions of example 4.4

Values	Fractional Taylor , [4]	DTM	errors
0.0	0.0000	0.0000	0.0000
0.1	-0.0000	0.0000	-0.0000
0.2	0.0001	0.0001	0.0000
0.3	0.0021	0.0014	0.0007
0.4	0.0137	0.0100	0.0037
0.5	0.0569	0.0473	0.0096
0.6	0.1791	0.1693	0.0098
0.7	0.4682	0.4978	-0.0296
0.8	1.0708	1.2675	-0.1967
0.9	2.2137	2.8907	-0.6770
1.0	4.2294	6.0437	-1.8144

Table 5: Solutions of example 4.3

Values	DTM transform	Exact solution	errors
0.0	1.0000	1.0000	0.0000
0.1	1.1000	1.1000	0.0000
0.2	1.2000	1.2000	0.0000
0.3	1.3000	1.3000	0.0000
0.4	1.4000	1.4000	0.0000
0.5	1.5000	1.5000	0.0000
0.6	1.6000	1.6000	0.0000
0.7	1.7000	1.7000	0.0000
0.8	1.8000	1.8000	0.0000
0.9	1.9000	1.9000	0.0000
1.0	2.0000	2.0000	0.0000

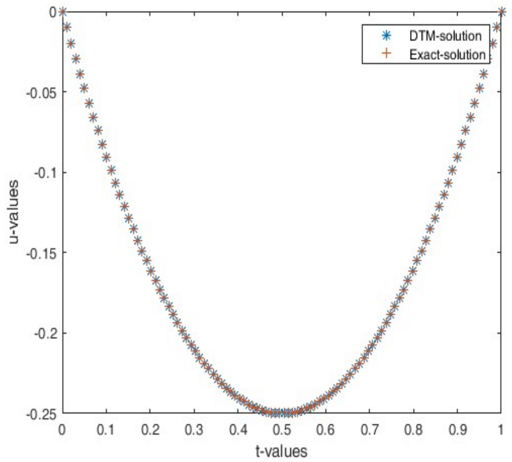

Figure 1: 2D solution performance of example 4.1

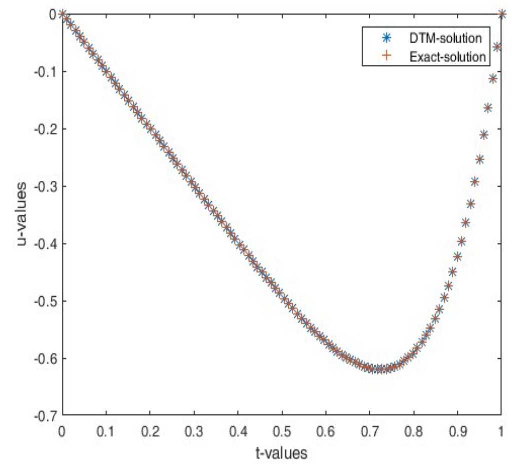

Figure 2: 2D solution of example 4.2

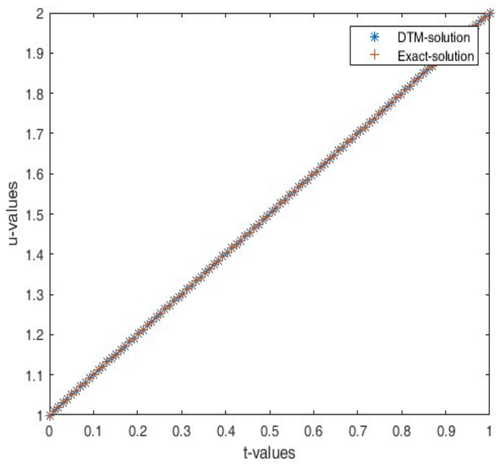

Figure 3: 2D solution performance of example 4.3

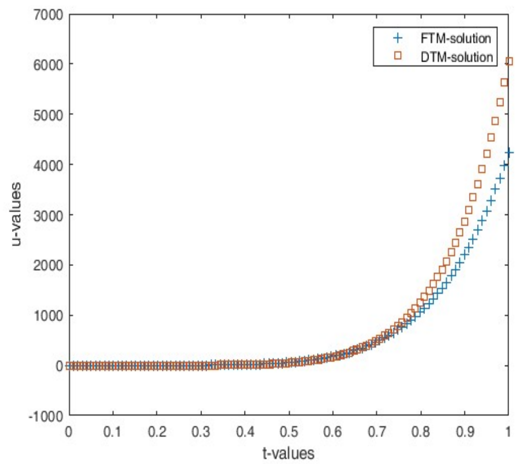

Figure 4: 2D solution of example 4.4,$\alpha = 1.5, \beta = 2.2$

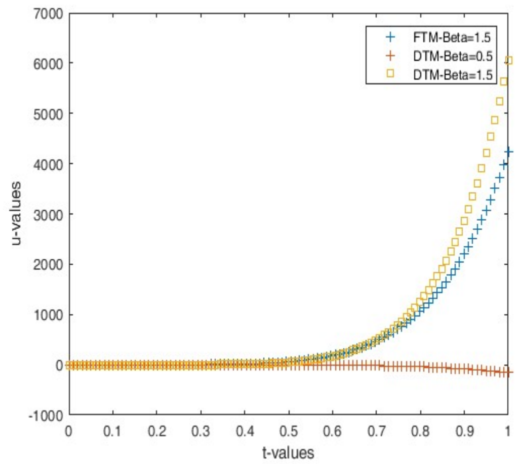

Figure 5: 2D solution of example 4.5,$\alpha = 2.2, \beta = 1.5, \beta = 0.5$

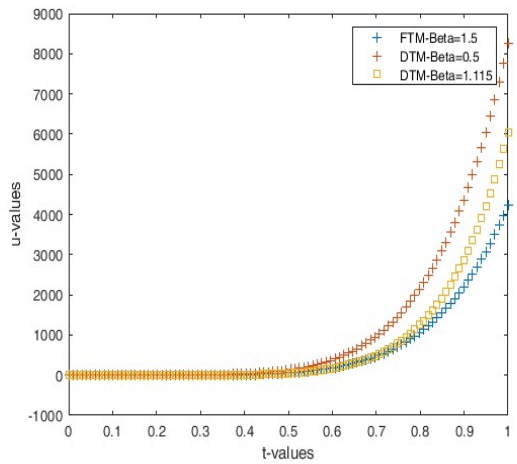

Figure 6: 2D solution of example 4.5,$\alpha = 2.2, \beta = 1.5, \beta = 1.115$