An Overview of Generalized Cesàro Sequence Spaces in Geometric Calculus

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Abstract: In this study, we introduce the generalized Cesàro sequence space in geometric calculus and establish a G-modular on this space. The generalized geometric Cesàro sequence space is equipped with the Luxemburg G-norm induced by the G-modular. The connections of the G-modular and Luxemburg G-norm on this space are studied. In addition, we show that the generalized geometric Cesàro sequence space is a G-Banach space under the Luxemburg G-norm and it is G-nonsquare where $p_n > 1$ for all $n \in \mathbb{N}$.

Key-Words: non-Newtonian calculus, geometric calculus, Cesàro sequence spaces, geometric Cesàro sequence spaces, G-modular, Luxemburg G-norm, G-nonsquare

Received: March 7, 2024. Revised: October 9, 2024. Accepted: November 7, 2024. Published: December 2, 2024.

1 Introduction

Grossman and Katz [11] introduced non-Newtonian calculus, which is a novel framework composed of the branches of geometric, bigeometric, harmonic, biharmonic, quadratic, and biquadratic calculus. Non-Newtonian calculus encompasses a diverse range of uses that include subjects like interest rates, the theory of economic elasticity, blood viscosity, biology, and computer science, including image processing and artificial intelligence, functional analysis, probability theory, and differential equations. One of the most well-known classes of non-Newtonian calculus is geometric calculus, which offers a variety of viewpoints that are helpful for applications in the fields of science and engineering. It offers differentiation and integration methods grounded in multiplication rather than addition. In general, geometric calculus is a methodology that allows for a different perspective on problems that can be studied through calculus. Geometric calculus is preferred over a traditional Newtonian one in specific cases, particularly when dealing with issues related to price elasticity and growth. To have a deeper understanding of non-Newtonian calculus, one must be familiar with several forms of arithmetic and their generators. The all, p-absolutely summable, boundedness, convergent and null sequence spaces in the context of non-Newtonian calculus denoted by $\omega(N)$, $l_\infty(N)$, $l_p(N)$, $c(N)$, $c_0(N)$, respectively, are defined and it is shown that these sets constitute a complete metric space by Çakmak and Başar [4]. Güngör [10] investigated some geometric properties of the non-Newtonian geometric sequence spaces $l_p(N)$. Boruah and Hazarika [2] introduced the generalized geometric difference sequence spaces $\ell_\infty^G(\Delta_G^m), c^G(\Delta_G^m), c_0^G(\Delta_G^m)$ with some properties. Mahto et al. [16] introduced bigeometric Cesàro difference sequence spaces and investigated the α-duals of these sequence spaces. More information on the non-Newtonian calculus may be found for the reader in [1, 6-9,12-14,17-20,23,26].

Sequence spaces have applications in a wide variety of disciplines, including economics and engineering. Studies on the geometric and topologic aspects of sequence spaces have been the focus of research in both pure and applied analysis due to the significance of sequence spaces as an example of function spaces and their involvement in the study of the theory of Banach spaces. The Cesàro sequence spaces, ces_p $(1 \leq p < \infty)$ and ces_∞ were established in 1968 as a part of the Dutch Mathematical Society's challenge to find duals. Shiue [24] investigated some properties of these spaces and gave the first norm-based description of them. Leibowitz [15] showed that $ces_1 = \{0\}, ces_p$ are separable reflexive Banach spaces for $1 \leq p < \infty$ and the l_p spaces are in ces_p for $1 < p < \infty$. Sanhan and Suantai [21] defined the generalized Cesàro sequence spaces $ces(p)$. They examined the space for completeness and also discussed its rotundity. Suantai [25] showed that the space $ces(p)$ has property (H) and property (G), and it is rotund. Many mathematicians have extensively researched the Cesàro sequence spaces via geometric and topological properties.

Motivated essentially by the aforementioned publications above, this study considers the geometric calculus concept of generalized Cesàro sequence space a novel and intriguing addition to the current literature in this field. We investigate geometric calculus versions of some concepts and properties given for classical generalized Cesàro sequence spaces. We hope this study will shed fresh light on how to approach solving issues in contexts where the theory of sequence spaces in fields ranging from engineering to economics and the theory of geometric calculus have a wide variety of uses.

Now, we offer a brief introduction to geometric calculus that emphasizes the terminology required for this discussion.

The building blocks of every arithmetic system are the four operations on the set $\mathbb{R}$ (addition, subtraction, multiplication and division) and an ordering relation that follows the rules of a completely ordered field. The set $\mathbb{R}$ is referred to as the realm, and the elements of the set $\mathbb{R}$ are termed the numbers of the system. A generator is an injective function whose domain is $\mathbb{R}$ and whose range is a subset of $\mathbb{R}$. The range of the generator η is called non-Newtonian real line and it is demonstrated by $\mathbb{R}_\eta$. $\eta -$ arithmetic operations and ordering relations are described as follows [11]:

$\eta -$ addition $\quad v \dot{+} s = \eta\{\eta^{-1}(v) + \eta^{-1}(s)\}$

$\eta -$ subtraction $\quad v \dot{-} s = \eta\{\eta^{-1}(v) - \eta^{-1}(s)\}$

$\eta -$ multiplication $\quad v \dot{\times} s = \eta\{\eta^{-1}(v) \cdot \eta^{-1}(s)\}$

$\eta -$ division $\quad v \dot{/} s = \eta\{\eta^{-1}(v) \div \eta^{-1}(s)\}$

$\eta -$ order $\quad v \dot{<} s \Leftrightarrow \eta^{-1}(v) < \eta^{-1}(s)$

$\quad (v \dot{\leq} s \Leftrightarrow \eta^{-1}(v) \leq \eta^{-1}(s))$.

Particularly, the identity function generates classical arithmetic.

In $*-$calculus, the paired arithmetics ($\eta -$arithmetic, $\beta -$arithmetic) are utilized for arguments and values, respectively. The subsequent particular calculi are derived when η and β are chosen as either I and exp, representing the identity and exponential functions, respectively [11]:

Calculus	η	β
Classical	I	I
Geometric	I	exp
Anageometric	exp	I
Bigeometric	exp	exp.

The classical arithmetic is derived from the identity function. If the $\eta -$generator is chosen as exponential function defined by $\eta(v) = e^v$ for $v \in \mathbb{R}$, then $\eta^{-1}(v) = \ln v$, η arithmetic turns into geometric arithmetic. The definitions of geometric operations and ordering relation are [2, 11]:

Geometric additon

$$v \oplus s = \eta\{\eta^{-1}(v) + \eta^{-1}(s)\} = e^{(\ln v + \ln s)} = v.s$$

Geometric subtraction

$$v \ominus s = \eta\{\eta^{-1}(v) - \eta^{-1}(s)\} = e^{(\ln v - \ln s)}$$

$$= \frac{v}{s}, s \neq 0$$

Geometric multiplication

$$v \odot s = \eta\{\eta^{-1}(v) \cdot \eta^{-1}(s)\} = e^{(\ln v \times \ln s)} = v^{\ln s}$$

$$= s^{\ln v}$$

Geometric division

$$\frac{v}{s}G \text{ (or } v \oslash s) = \eta\{\eta^{-1}(v) \div \eta^{-1}(s)\} = e^{(\ln v \div \ln s)}$$

$$= v^{\frac{1}{\ln s}}, s \neq 1$$

Geometric order

$$v <_G s \ (v \leq_G s) \Leftrightarrow \ln v < \ln s \ (\ln v \leq \ln s).$$

The set of geometric real numbers which is denoted by $\mathbb{R}_G$, is defined as $\{e^{\upsilon}: \upsilon \in \mathbb{R}\}$. $(\mathbb{R}_G,\oplus,\odot)$ is a field with geometric zero $0_G = 1$ and geometric identity $1_G = e$. The sets of geometric positive real numbers and geometric negative real numbers are defined as $\mathbb{R}_G^+ = \{\upsilon \in \mathbb{R}_G: \upsilon >_G 1\}$ and $\mathbb{R}_G^- = \{\upsilon \in \mathbb{R}_G: \upsilon <_G 1\}$, respectively. The set of all geometric integers are as follows:

$$\mathbb{Z}_G = \{\dots, 0_G \ominus e^2, 0_G \ominus e, 0_G, e, e^2, \dots\}$$

$$= \{\dots, e^{-2}, e^{-1}, 1, e, e^2, \dots\}$$

The geometric absolute value of $\upsilon \in \mathbb{R}_G$ is defined by

$$|\upsilon|_G = \begin{cases} \upsilon, & \upsilon >_G 1 \\ 1, & \upsilon = 1 \\ \dfrac{1}{\upsilon}, & \upsilon <_G 1 \end{cases}$$

and this is equivalent to expression $e^{|\ln \upsilon|}$. For any $\upsilon, s \in \mathbb{R}_G$, the subsequent statements are valid:

i) $|\upsilon \oplus s|_G \leq_G |\upsilon|_G \oplus |s|_G$
ii) $|\upsilon \ominus s|_G \geq_G |\upsilon|_G \ominus |s|_G$
iii) $|\upsilon \odot s|_G = |\upsilon|_G \odot |s|_G$
iv) $|\upsilon \oslash s|_G = |\upsilon|_G \oslash |s|_G$.

For any $\upsilon \in \mathbb{R}_G$, $\upsilon^{p_G} = e^{(\ln \upsilon)^p} = \upsilon^{\ln^{p-1}\upsilon}$ and $\sqrt[p]{\upsilon}^G = e^{(\ln \upsilon)^{\frac{1}{p}}}$ [2, 11].

Definition 1.1. [3, 5] A geometric vector space (G-vector space) over a geometric field $\mathbb{R}_G$ is a non-empty set V equipped with two operations $\oplus$ and $\odot$, called geometric vector addition and geometric scalar multiplication, respectively, which satisfy the following properties:

GV1) *Closure:* If $\upsilon, s \in V$, then $\upsilon \oplus s$ belong to V.

GV2) *Associative law:* $(\upsilon \oplus s) \oplus t = \upsilon \oplus (s \oplus t)$ for all $\upsilon, s, t \in V$.

GV3) *Additive identity:* V contains an additive identity element denoted by θ_G, such that $\upsilon \oplus \theta_G = \upsilon$ for all $\upsilon \in V$.

GV4) *Additive inverse:* For all $\upsilon \in V$, there is a vector $\theta_G \ominus \upsilon \in V$ with $\upsilon \oplus (\theta_G \ominus \upsilon) = \theta_G$ and $(\theta_G \ominus \upsilon) \oplus \upsilon = \theta_G$.

GV5) *Commutative law:* $\upsilon \oplus s = s \oplus \upsilon$ for all $\upsilon, s \in V$.

GV6) *Closure:* If $\upsilon \in V$ and $\lambda \in \mathbb{R}_G$, then $\lambda \odot \upsilon$ belong to V.

GV7) *Distributive laws:*

$\lambda \odot (\upsilon \oplus s) = \lambda \odot \upsilon \oplus \lambda \odot s$, for all $\upsilon, s \in V$ and $\lambda \in \mathbb{R}_G$.

$(\lambda \oplus \beta) \odot \upsilon = \lambda \odot \upsilon \oplus \beta \odot \upsilon$, for all $\upsilon \in V$ and $\lambda, \beta \in \mathbb{R}_G$

GV8) *Associative law:* $\lambda \odot (\beta \odot \upsilon) = (\lambda \odot \beta) \odot \upsilon$ for all $\upsilon \in V$ and $\lambda, \beta \in \mathbb{R}_G$.

GV9) *Unitary law:* $e \odot \upsilon = \upsilon$ for all $\upsilon \in V$.

The set of all sequences of the geometric real numbers demonstrated by ω_G, i.e., $\omega_G = \{\nu = (\nu_k) |\ \nu : \mathbb{N} \to \mathbb{R}_G\}$. Based on the algebraic operations $\oplus$ addition and $\odot$ multiplication, ω_G is a G-vector space over $\mathbb{R}_G$ [2,4].

Definition 1.2. [4] Let V be a G-vector space. If the function $\|\cdot\|_G : V \to \mathbb{R}_G$ holds the following properties for all $\nu, y \in V$ and $\lambda \in \mathbb{R}_G$,

GN1) $\|\nu\|_G = 1 \Leftrightarrow \nu = \theta_G$
GN2) $\|\lambda \odot \nu\|_G = |\lambda|_G \odot \|\nu\|_G$
GN3) $\|\nu \oplus y\|_G \leq_G \|\nu\|_G \oplus \|y\|_G$

then $(V, \|\cdot\|_G)$ is said to be a G-normed space.

Definition 1.3. [4, 27] Let $(V, \|.\|_G)$ be G-normed space and (ν_k) be a sequence in V. If for every given $\varepsilon >_G 1$, there exist $k_0 = k_0(\varepsilon) \in \mathbb{N}$ and $\nu \in V$ such that $\|\nu_k \ominus \nu\|_G <_G \varepsilon$ for all $k \geq k_0$, then (ν_k) is said to be G-convergent and it is denoted by $\nu_k \xrightarrow{G} \nu$ as $k \to \infty$.

Definition 1.4. [27] Let $(V, \|.\|_G)$ be G-normed space and (ν_k) be a sequence in V. If for every given $\varepsilon >_G 1$, there is $k_0 = k_0(\varepsilon) \in \mathbb{N}$ such that $\|\nu_k \ominus \nu_m\|_G <_G \varepsilon$ for all $k, m \geq k_0$, then (ν_k) is said to be G-Cauchy sequence.

If every G-Cauchy sequence in V converges, then it is said that V is a G-Banach. For example, $(\mathbb{R}_G, |.|_G)$ is a G-Banach space.

Definition 1.5. [10] Let $f: I \subseteq \mathbb{R}_G \to \mathbb{R}_G$ be a function. The function f is said to be G-convex, if for every $r, s \in I$ it satisfies

$$f(\gamma \odot r \oplus (e \ominus \gamma) \odot s)$$
$$\leq_G \gamma \odot f(r) \oplus (e \ominus \gamma) \odot f(s)$$

with $\gamma \in [1, e]$.

Proposition 1.6. [22] Let $r, s \geq_G 1$, then $(r \oplus s)^{p_G} \leq_G r^{p_G} \oplus s^{p_G}$ for $0 < p < 1$.

2 Main Results

This section introduces the idea of generalized geometric Cesàro sequence space with a new perspective on the concept of Cesàro sequence space and it provides a basic explanation of the theory behind this sequence space.

First, we will present the concepts of modular, modular spaces and Luxemburg norm according to geometric calculation style.

Definition 2.1. Let V be a G-vector space on $\mathbb{R}_G$ field. The function $\varrho : V \to [1,\infty]$ is called G-modular provided that it satisfies the given requirements:

i) $\varrho(\nu) = 1$ if and only if $\nu = \theta_G$,

ii) $\varrho(\alpha \odot \nu) = \varrho(\nu)$ for all scalar $\alpha \in \mathbb{R}_G$ with $|\alpha|_G = e$,

iii) $\varrho(\alpha \odot \nu \oplus \beta \odot y) \leq_G \varrho(\nu) \oplus \varrho(y)$ for all $\nu, y \in V$ and $\alpha, \beta \geq_G 1$ with $\alpha \oplus \beta = e$.

Moreover, the G-modular ϱ is called G-convex (G-convex modular) if

iv) $\varrho(\alpha \odot \nu \oplus \beta \odot y) \leq_G \alpha \odot \varrho(\nu) \oplus \beta \odot \varrho(y)$ for all $\nu, y \in X$ and $\alpha, \beta \geq_G 1$ with $\alpha \oplus \beta = e$.

Definition 2.2. If ϱ is a G-modular in V, we define

$$V_\varrho = \left\{ \nu \in V : \varrho\left(e^\lambda \odot \nu\right) \overset{G}{\to} 1, \lambda \longrightarrow 0^+ \right\}$$

and it is called G-modular spaces.

Theorem 2.3. If ϱ is a G-convex modular, then the function $\|.\|_G : V_\varrho \longrightarrow \mathbb{R}_G$ defined as

$$\|\nu\|_G = {}_G\inf\left\{ e^\lambda >_G 1 : \varrho\left(\frac{\nu}{e^\lambda} G\right) \leq_G e \right\}$$

is G-norm on V_ϱ.

Proof.

GN1) Let $\nu = \theta_G$. Since $\varrho\left(\frac{\theta_G}{e^\lambda} G\right) = 1$ for every $e^\lambda \in \mathbb{R}_G$, then we find $\|\nu\|_G = 1$.

Conversely, let $\|\nu\|_G = 1$. Since ϱ is a G-convex modular

$$\begin{aligned} \varrho(\alpha \odot \nu) &= \varrho(\alpha \odot \nu \oplus (e \ominus \alpha) \odot \theta_G) \\ &\leq_G \alpha \odot \varrho(\nu) \oplus (e \ominus \alpha) \odot \varrho(\theta_G) \\ &= \alpha \odot \varrho(\nu) \end{aligned}$$

for all $\alpha \in \mathbb{R}_G$ with $1 \leq_G \alpha \leq_G e$. Therefore, we can write

$$\begin{aligned} \varrho(\nu) &= \varrho\left(e^n \odot \frac{1}{e^n} G \odot \nu\right) \\ &\leq_G \frac{1}{e^n} G \odot \varrho(e^n \odot \nu) \\ &= \frac{1}{e^n} G \odot \varrho\left(\frac{\nu}{\frac{1}{e^n} G} G\right) \end{aligned}$$

for every $n \in \mathbb{N}$. Hence, it is obtained that $\varrho(\nu) = 1$ which implies $\nu = \theta_G$.

GN2) Take any $\nu \in V_\varrho$ and $\alpha \in \mathbb{R}_G$. If $\alpha = 1$, then clearly $\|\alpha \odot \nu\|_G = |\alpha|_G \odot \|\nu\|_G$. Assume that $\alpha \in \mathbb{R}_G - \{1\}$. Based on the provided definition of $\|.\|_G$, we can write

$$\begin{aligned} &\left\| \frac{\alpha \odot \nu}{|\alpha|_G} G \right\|_G \\ &= {}_G\inf\left\{ e^\lambda >_G 1 : \varrho\left(\frac{\frac{\alpha \odot \nu}{|\alpha|_G} G}{e^\lambda} G \right) \leq_G e \right\} \\ &= {}_G\inf\left\{ e^\lambda >_G 1 : \varrho\left(\frac{\nu}{\frac{e^\lambda \odot |\alpha|_G}{\alpha} G} G \right) \leq_G e \right\}. \end{aligned}$$

Let $\alpha >_G 1$. Considering $e^{\lambda_1} = \frac{e^\lambda \odot |\alpha|_G}{\alpha} G$, then we obtain

$$\begin{aligned} &\left\| \frac{\alpha \odot \nu}{|\alpha|_G} G \right\|_G \\ &= {}_G\inf\left\{ \frac{e^{\lambda_1} \odot \alpha}{|\alpha|_G} G >_G 1 : \varrho\left(\frac{\nu}{e^{\lambda_1}} G\right) \leq_G e \right\} \\ &= \frac{\alpha}{|\alpha|_G} G \odot {}_G\inf\left\{ e^{\lambda_1} >_G 1 : \varrho\left(\frac{\nu}{e^{\lambda_1}} G\right) \leq_G e \right\} \\ &= \|\nu\|_G. \end{aligned} \tag{1}$$

Let $\alpha <_G 1$. If we take it as $e^{\lambda_2} = \frac{e^\lambda \odot |\alpha|_G}{1 \ominus \alpha} G$, we get

$$\begin{aligned} &\left\| \frac{\alpha \odot \nu}{|\alpha|_G} G \right\|_G \\ &= {}_G\inf\left\{ \frac{e^{\lambda_2} \odot (1 \ominus \alpha)}{|\alpha|_G} G >_G 1 : \varrho\left(\frac{(1 \ominus e) \odot \nu}{e^{\lambda_2}} G\right) \leq_G e \right\} \\ &= \frac{1 \ominus \alpha}{|\alpha|_G} G \odot {}_G\inf\left\{ e^{\lambda_2} >_G 1 : \varrho\left(\frac{\nu}{e^{\lambda_2}} G\right) \leq_G e \right\} \\ &= \frac{1 \ominus \alpha}{|\alpha|_G} \odot \|\nu\|_G \\ &= \|\nu\|_G. \end{aligned} \tag{2}$$

By using (1) and (2), we obtain

$$\left\| \frac{\alpha \odot \nu}{|\alpha|_G} G \right\|_G = \|\nu\|_G \tag{3}$$

for all $\alpha \in \mathbb{R}_G - \{1\}$. If we take $e^{\lambda'} = e^\lambda \odot |\alpha|_G$, then we can see that

$$\begin{aligned} &\left\| \frac{\alpha \odot \nu}{|\alpha|_G} G \right\|_G \\ &= {}_G\inf\left\{ e^\lambda >_G 1 : \varrho\left(\frac{\alpha \odot \nu}{e^\lambda \odot |\alpha|_G} G\right) \leq_G e \right\} \\ &= {}_G\inf\left\{ \frac{e^{\lambda'}}{|\alpha|_G} G \odot \alpha >_G 1 : \varrho\left(\frac{\alpha \odot \nu}{e^{\lambda'}} G\right) \leq_G e \right\} \\ &= \frac{1}{|\alpha|_G} G \odot {}_G\inf\left\{ e^{\lambda'} >_G 1 : \varrho\left(\frac{\alpha \odot \nu}{e^{\lambda'}} G\right) \leq_G e \right\} \\ &= \frac{1}{|\alpha|_G} G \odot \|\alpha \odot \nu\|_G. \end{aligned} \tag{4}$$

Using the expressions (3) and (4), we obtain

$$\|\alpha \odot \nu\|_G = |\alpha|_G \odot \|\nu\|_G$$

for all $\alpha \in \mathbb{R}_G - \{1\}$.

GN3) Let any $\nu, y \in V_\varrho$ be given. Let

$$A = \left\{ e^{\lambda_1} >_G 1 : \varrho\left(\frac{\nu}{e^{\lambda_1}} G\right) \leq_G e \right\}$$

$$B = \left\{ e^{\lambda_2} >_G 1 : \varrho\left(\frac{y}{e^{\lambda_2}} G\right) \leq_G e \right\}$$

and

$$C = \left\{ e^{\lambda_3} >_G 1 : \varrho\left(\frac{\nu \oplus y}{e^{\lambda_3}} G \right) \leq_G e \right\}.$$

Since ϱ is a G-convex modular, we find

$$\varrho\left(\frac{\nu \oplus y}{e^{\lambda_1} \oplus e^{\lambda_2}} G\right) = \varrho\left(\frac{\nu}{e^{\lambda_1} \oplus e^{\lambda_2}} G \oplus \frac{y}{e^{\lambda_1} \oplus e^{\lambda_2}} G\right)$$

$$= \varrho\left(\frac{e^{\lambda_1}}{e^{\lambda_1} \oplus e^{\lambda_2}} G \odot \frac{\nu}{e^{\lambda_1}} G \oplus \frac{e^{\lambda_2}}{e^{\lambda_1} \oplus e^{\lambda_2}} G \odot \frac{y}{e^{\lambda_2}} G\right)$$

$$\leq_G \frac{e^{\lambda_1}}{e^{\lambda_1} \oplus e^{\lambda_2}} G \odot \varrho\left(\frac{\nu}{e^{\lambda_1}} G\right) \oplus \frac{e^{\lambda_2}}{e^{\lambda_1} \oplus e^{\lambda_2}} G \odot \varrho\left(\frac{y}{e^{\lambda_2}} G\right)$$

$$\leq_G \frac{e^{\lambda_1}}{e^{\lambda_1} \oplus e^{\lambda_2}} G \oplus \frac{e^{\lambda_2}}{e^{\lambda_1} \oplus e^{\lambda_2}} G$$

$$= e$$

for arbitrary $e^{\lambda_1} \oplus e^{\lambda_2}$. Selecting $D = \left\{ e^{\lambda_1} \oplus e^{\lambda_2} : e^{\lambda_1} \in A \wedge e^{\lambda_2} \in B \right\}$, then we can see that $D \subset C$. Because of $D \subset C$, ${}_G\inf C \leq_G {}_G\inf D = {}_G\inf A \oplus {}_G\inf B$ holds. As a result, we get

$$\|\nu \oplus y\|_G \leq_G \|\nu\|_G \oplus \|y\|_G.$$

Now, the concept of generalized geometric Cesàro sequence space is given, which forms the basis of this paper:

Definition 2.4. Let $p = (p_n)$ represent a bounded sequence of positive real numbers with $p_n \geq 1$ for all $n \in \mathbb{N}$. The generalized geometric Cesàro sequence space is defined as

$$Ces^G(p) = \{\nu = (\nu_k) \in \omega_G : \sigma(\nu) <_G \infty\}$$

whenever

$$\sigma(\nu) = G\sum_{n=1}^{\infty}\left(\frac{e}{e^n} G \odot G\sum_{k=1}^{n} |\nu_k|_G\right)^{(p_n)_G}.$$

In the present study, we make the assumption that $p = (p_n)$ be a sequence of positive real numbers where $p_n \geq 1$ for every $n \in \mathbb{N}$ and $M = \sup_{n\in\mathbb{N}} p_n$.

Proposition 2.5. Let $q \in [1, \infty)$ and $r, s \in \mathbb{R}_G$, then the following statement holds:

$$|r \oplus s|_G^{q_G} \leq_G (e^2)^{q_G} \odot \left(|r|_G^{q_G} \oplus |s|_G^{q_G}\right).$$

Proof. We can write

$$|r \oplus s|_G \leq_G |r|_G \oplus |s|_G$$

$$\leq_G e^2 \odot {}_G\text{maks}\{|r|_G, |s|_G\}$$

for any $r, s \in \mathbb{R}_G$. Therefore, we get

$$|r \oplus s|_G^{q_G} \leq_G (e^2)^{q_G} \odot \left({}_G\text{maks}\{|r|_G, |s|_G\}\right)^{q_G}$$

$$= (e^2)^{q_G} \odot {}_G\text{maks}\left\{|r|_G^{q_G}, |s|_G^{q_G}\right\}$$

$$\leq_G (e^2)^{q_G} \odot \left(|r|_G^{q_G} + |s|_G^{q_G}\right).$$

Theorem 2.6. $Ces^G(p)$ is a G-vector spaces under geometric addition and geometric scalar multiplication operations for geometric real sequences.

Proof. Given any $\nu = (\nu_k),\ y = (y_k) \in Ces^G(p)$ and $\lambda \in \mathbb{R}_G$. Let $M = \sup_{n\in\mathbb{N}} p_n$. If it is choosen $\alpha_n = \frac{p_n}{M}$ for all $n \in \mathbb{N}$, we have

$$\sigma(\nu \oplus y)$$

$$= G\sum_{n=1}^{\infty}\left(\frac{e}{e^n} G \odot G\sum_{k=1}^{n} |\nu_k \oplus y_k|_G\right)^{(p_n)_G}$$

$$= G\sum_{n=1}^{\infty}\left(\frac{e}{e^n} G \odot G\sum_{k=1}^{n} |\nu_k \oplus y_k|_G\right)^{(M\alpha_n)_G}$$

$$\leq_G G\sum_{n=1}^{\infty}\left(\frac{e}{e^n} G \odot G\sum_{k=1}^{n} |\nu_k|_G \oplus \frac{e}{e^n} G \odot G\sum_{k=1}^{n} |y_k|_G\right)^{(M\alpha_n)_G}$$

$$\leq_G G\sum_{n=1}^{\infty}\left(\left(\frac{e}{e^n} G \odot G\sum_{k=1}^{n} |\nu_k|_G\right)^{(\alpha_n)_G} \oplus \left(\frac{e}{e^n} G \odot G\sum_{k=1}^{n} |y_k|_G\right)^{(\alpha_n)_G}\right)^{M_G}.$$

from Proposition 1.6. We get

$$\sigma(\nu \oplus y)$$

$$\leq_G G\sum_{n=1}^{\infty}(e^2)^{M_G} \odot \left[\left(\left(\frac{e}{e^n} G \odot G\sum_{k=1}^{n} |\nu_k|_G\right)^{(\alpha_n)_G}\right)^{M_G} \oplus \left(\left(\frac{e}{e^n} G \odot G\sum_{k=1}^{n} |y_k|_G\right)^{(\alpha_n)_G}\right)^{M_G}\right]$$

$$= (e^2)^{M_G} \odot \left[G\sum_{n=1}^{\infty}\left(\frac{e}{e^n} G \odot G\sum_{k=1}^{n} |\nu_k|_G\right)^{(p_n)_G} \oplus G\sum_{n=1}^{\infty}\left(\frac{e}{e^n} G \odot G\sum_{k=1}^{n} |y_k|_G\right)^{(p_n)_G}\right]$$

$$= (e^2)^{M_G} \odot [\sigma(\nu) \oplus \sigma(y)] <_G \infty$$

from Proposition 2.5. This demonstrates that $\sigma(\nu \oplus y) <_G \infty$, i.e., $\nu \oplus y \in Ces^G(p)$.

Taken $L = {}_G\text{maks}\left\{e, |\lambda|_G^{M_G}\right\}$, it is obtained that

$$\sigma(\lambda \odot \nu) = G\sum_{n=1}^{\infty}\left(\frac{e}{e^n} G \odot G\sum_{k=1}^{n} |\lambda \odot \nu_k|_G\right)^{(p_n)_G}$$

$$= G\sum_{n=1}^{\infty}\left(\frac{e}{e^n} G \odot G\sum_{k=1}^{n} |\lambda|_G \odot |\nu_k|_G\right)^{(p_n)_G}$$

$$\leq_G |\lambda|_G^{(p_n)_G} \odot G\sum_{n=1}^{\infty}\left(\frac{e}{e^n}G \odot G\sum_{k=1}^{n}|v_k|_G\right)^{(p_n)_G}$$
$$\leq_G L \odot \sigma(v) <_G \infty.$$

This shows that $\lambda \odot v \in Ces^G(p)$. Consequently, $Ces^G(p)$ is G-vector space.

Proposition 2.7. The function $f: \mathbb{R}_G \to \mathbb{R}_G,\ t \to |t|_G^{(p_n)_G}$ is G-convex where $p_n \geq 1$ for all $n \in \mathbb{N}$.

Proof. Let's taken $\gamma \in [1, e]$. We have

$$f(\gamma \odot v \oplus (e \ominus \gamma) \odot y)\text{t}$$
$$= |\gamma \odot v \oplus (e \ominus \gamma) \odot y|_G^{(p_n)_G}$$
$$= \left[\exp\{\left|\ln\left(v^{\ln\gamma} \cdot y^{(1-\ln\gamma)}\right)\right|\}\right]^{(p_n)_G}$$
$$= \exp\left\{\left(\ln\left(e^{|\ln\gamma \cdot \ln v + (1-\ln\gamma)\cdot \ln y|}\right)\right)^{p_n}\right\}$$
$$= \exp\{|\ln\gamma \cdot \ln v + (1-\ln\gamma)\cdot \ln y|^{p_n}\} \qquad (5)$$

for all $v, y \in \mathbb{R}_G$. Since $\gamma \in [1, e]$ implies $\ln\gamma \in [0,1]$, we get

$$|\ln\gamma \cdot \ln v + (1-\ln\gamma)\cdot \ln y|^{p_n}$$
$$\leq \ln\gamma \cdot |\ln v|^{p_n} + (1-\ln\gamma)\cdot |\ln y|^{p_n} \qquad (6)$$

by aid of the convexity of the function $t \to |t|^{p_n}$ in classical calculus. From (5) and (6), we find

$$f(\gamma \odot v \oplus (e \ominus \gamma) \odot y)$$
$$\leq_G \exp\{\ln\gamma \cdot |\ln v|^{p_n} + (1-\ln\gamma)\cdot |\ln y|^{p_n}\}$$
$$= \exp\{\ln\gamma \cdot \ln e^{|\ln v|^{p_n}} + (1-\ln\gamma)\cdot \ln e^{|\ln y|^{p_n}}\}$$
$$= \exp\{\ln\gamma \cdot \ln|v|_G^{p_n} + (1-\ln\gamma)\cdot \ln|y|_G^{p_n}\}$$
$$= \exp\left\{\ln\gamma \cdot \ln e^{\ln|v|_G^{p_n}} + (1-\ln\gamma)\cdot \ln e^{\ln|y|_G^{p_n}}\right\}$$
$$= \exp\left\{\ln\gamma \cdot \ln|v|_G^{(p_n)_G} + (1-\ln\gamma)\cdot \ln|y|_G^{(p_n)_G}\right\}$$
$$= \exp\left\{\ln\left(e^{\ln\gamma\cdot\ln|v|_G^{(p_n)_G}}\right) + \ln\left(e^{\ln e^{(\ln e - \ln\lambda)}\cdot\ln|y|_G^{(p_n)_G}}\right)\right\}$$
$$= \gamma \odot |v|_G^{(p_n)_G} \oplus (e \ominus \gamma) \odot |y|_G^{(p_n)_G}.$$

Hence, the proof is completed.

Theorem 2.8. σ is a G-convex modular on $Ces^G(p)$.

Proof. Let $v, y \in Ces^G(p)$.

i) It is obvious that $\sigma(v) = 1 \Leftrightarrow x = 1$.

ii) For $\alpha \in \mathbb{R}_G$ with $|\alpha|_G = e$, we write

$$\sigma(\alpha \odot v) = G\sum_{n=1}^{\infty}\left(\frac{e}{e^n}G \odot G\sum_{k=1}^{n}|\alpha \odot v_k|_G\right)^{(p_n)_G}$$
$$= G\sum_{n=1}^{\infty}|\alpha|_G^{(p_n)_G} \odot \left(\frac{e}{e^n}G \odot G\sum_{k=1}^{n}|v_k|_G\right)^{(p_n)_G}$$
$$= \sigma(v).$$

iii) Let $\alpha, \beta \in \mathbb{R}_G$ with $\alpha \oplus \beta = e,\ \alpha, \beta \geq_G 1$. Since $t \longrightarrow |t|_G^{(p_n)_G}$ is G-convex function for all $n \in \mathbb{N}$, we have

$$\sigma(\alpha \odot v \oplus \beta \odot y)$$
$$= G\sum_{n=1}^{\infty}\left(\frac{e}{e^n}G \odot G\sum_{k=1}^{n}|\alpha \odot v_k \oplus \beta \odot y_k|_G\right)^{(p_n)_G}$$
$$\leq_G G\sum_{n=1}^{\infty}\left(\alpha \odot \left(\frac{e}{e^n}G \odot G\sum_{k=1}^{n}|v_k|_G\right)\right)^{(p_n)_G}$$
$$\oplus \left(\beta \odot \left(\frac{e}{e^n}G \odot G\sum_{k=1}^{n}|y_k|_G\right)\right)^{(p_n)_G}$$
$$= \alpha \odot G\sum_{n=1}^{\infty}\left(\frac{e}{e^n}G \odot G\sum_{k=1}^{n}|v_k|_G\right)^{(p_n)_G}$$
$$\oplus \beta \odot G\sum_{n=1}^{\infty}\left(\frac{e}{e^n}G \odot G\sum_{k=1}^{n}|y_k|_G\right)^{(p_n)_G}$$
$$= \alpha \odot \sigma(v) \oplus \beta \odot \sigma(y).$$

Proposition 2.9. The G-modular σ on $Ces^G(p)$ has the subsequent properties:

i) If $1 <_G a <_G e$, then $a^{M_G} \odot \sigma\left(\frac{v}{a}G\right) \leq_G \sigma(v)$ and $\sigma(a \odot v) \leq_G a \odot \sigma(v)$.

ii) If $a >_G e$, then $\sigma(v) \leq_G a^{M_G} \odot \sigma\left(\frac{v}{a}G\right)$.

iii) If $a \geq_G e$, then $\sigma(v) \leq_G a \odot \sigma(v) \leq_G \sigma(a \odot v)$.

Proof.

i) Let $1 <_G a <_G e$. Hence we find

$$\sigma(v) = G\sum_{n=1}^{\infty}\left(\frac{e}{e^n}G \odot G\sum_{k=1}^{n}\left|\frac{a \odot v_k}{a}G\right|_G\right)^{(p_n)_G}$$
$$= G\sum_{n=1}^{\infty}|a|_G^{(p_n)_G} \odot \left(\frac{e}{e^n}G \odot G\sum_{k=1}^{n}\left|\frac{v_k}{a}G\right|_G\right)^{(p_n)_G}$$
$$\geq_G a^{M_G} \odot G\sum_{n=1}^{\infty}\left(\frac{e}{e^n}G \odot G\sum_{k=1}^{n}\left|\frac{v_k}{a}G\right|_G\right)^{(p_n)_G}$$
$$= a^{M_G} \odot \sigma\left(\frac{v}{a}G\right).$$

Since $a^{(p_n)_G} \leq_G a$ for all $n \in \mathbb{N}$, we have

$$\sigma(a \odot \nu) = G\sum_{n=1}^{\infty}\left(\frac{e}{e^n}G \odot G\sum_{k=1}^{n}|a \odot \nu_k|_G\right)^{(p_n)_G}$$

$$= G\sum_{n=1}^{\infty} a^{(p_n)_G} \odot \left(\frac{e}{e^n}G \odot G\sum_{k=1}^{n}|\nu_k|_G\right)^{(p_n)_G}$$

$$\leq_G a \odot G\sum_{n=1}^{\infty}\left(\frac{e}{e^n}G \odot G\sum_{k=1}^{n}|\nu_k|_G\right)^{(p_n)_G}$$

$$= a \odot \sigma(\nu).$$

ii) Let $a >_G e$, then we get

$$\sigma(\nu) = G\sum_{n=1}^{\infty}\left(\frac{e}{e^n}G \odot G\sum_{k=1}^{n}|\nu_k|_G\right)^{(p_n)_G}$$

$$= G\sum_{n=1}^{\infty} a^{(p_n)_G} \odot \left(\frac{e}{e^n}G \odot G\sum_{k=1}^{n}\left|\frac{\nu_k}{a}G\right|_G\right)^{(p_n)_G}$$

$$\leq_G a^{M_G} \odot G\sum_{n=1}^{\infty}\left(\frac{e}{e^n}G \odot G\sum_{k=1}^{n}\left|\frac{\nu_k}{a}G\right|_G\right)^{(p_n)_G}$$

$$= a^{M_G} \odot \sigma\left(\frac{\nu}{a}G\right).$$

iii) Let $a \geq_G e$. Since σ is G-convex modular, we can write $\sigma(\nu) \leq_G a \odot \sigma(\nu)$. Also, we find

$$\sigma(a \odot \nu) = G\sum_{n=1}^{\infty}\left(\frac{e}{e^n}G \odot G\sum_{k=1}^{n}|a \odot \nu_k|_G\right)^{(p_n)_G}$$

$$= G\sum_{n=1}^{\infty} |a|_G^{(p_n)_G} \odot \left(\frac{e}{e^n}G \odot G\sum_{k=1}^{n}|\nu_k|_G\right)^{(p_n)_G}$$

$$\geq_G a \odot G\sum_{n=1}^{\infty}\left(\frac{e}{e^n}G \odot G\sum_{k=1}^{n}|\nu_k|_G\right)^{(p_n)_G}$$

$$= a \odot \sigma(\nu).$$

Hence, we obtain the required inequalities.

Theorem 2.10. $Ces^G(p)$ is G-normed space with regard to the Luxemburg G-norm

$$\|\nu\|_G = {}_G\inf\left\{e^{\lambda} >_G 1 : \sigma\left(\frac{\nu}{e^{\lambda}}G\right) \leq_G e\right\}$$

$$= {}_G\inf\left\{e^{\lambda} >_G 1 : G\sum_{n=1}^{\infty}\left(\frac{e}{e^n}G \odot G\sum_{k=1}^{n}\frac{|\nu_k|_G}{e^{\lambda}}G\right)^{(p_n)_G} \leq_G e\right\}.$$

Proof.

GN1) Let $\nu = 1$. Since $G\sum_{n=1}^{\infty}\left(\frac{e}{e^n}G \odot G\sum_{k=1}^{n}\frac{1}{e^{\lambda}}G\right)^{(p_n)_G} = 1$ for every $e^{\lambda} \in \mathbb{R}_G$, then we find $\|\nu\|_G = 1$.

Conversely, let $\|\nu\|_G = 1$. Since $\sigma(\alpha \odot \nu) \leq_G \alpha \odot \sigma(\nu)$ for all $\alpha \in \mathbb{R}_G$ with $1 <_G \alpha <_G e$, we can write

$$G\sum_{n=1}^{\infty}\left(\frac{e}{e^n}G \odot G\sum_{k=1}^{n}\frac{|\nu_k|_G}{e^{\lambda}}G\right)^{(p_n)_G}$$

$$= G\sum_{n=1}^{\infty}\left(\frac{e}{e^n}G \odot G\sum_{k=1}^{n}\frac{e^m \odot \frac{1}{e^m}G \odot |\nu_k|_G}{e^{\lambda}}G\right)^{(p_n)_G}$$

$$\leq_G \frac{1}{e^m}G \odot G\sum_{n=1}^{\infty}\left(\frac{e}{e^n}G \odot G\sum_{k=1}^{n}\frac{e^m \odot |\nu_k|_G}{e^{\lambda}}G\right)^{(p_n)_G}$$

$$= \frac{1}{e^m}G \odot G\sum_{n=1}^{\infty}\left(\frac{e}{e^n}G \odot G\sum_{k=1}^{n}\frac{|\nu_k|_G}{\frac{e^{\lambda}}{e^m}G}G\right)^{(p_n)_G}$$

for all $m \in \mathbb{N}$. Hence, it is obtained that $\sigma(\nu) = G\sum_{n=1}^{\infty}\left(\frac{e}{e^n}G \odot G\sum_{k=1}^{n}\frac{|\nu_k|_G}{e^{\lambda}}G\right)^{(p_n)_G} = 1$ which implies $\nu = 1$.

GN2) Let's take any $\nu \in Ces^G(p)$ and $\alpha \in \mathbb{R}_G$. If $\alpha = 1$, then it is obvious that $\|\alpha \odot \nu\|_G = |\alpha|_G \odot \|\nu\|_G$. Suppose that $\alpha \in \mathbb{R}_G - \{1\}$. We can write

$$\|\alpha \odot \nu\|_G$$

$$= {}_G\inf\left\{e^{\lambda} >_G 1 : G\sum_{n=1}^{\infty}\left(\frac{e}{e^n}G \odot G\sum_{k=1}^{n}\frac{|\alpha \odot \nu_k|_G}{e^{\lambda}}G\right)^{(p_n)_G} \leq_G e\right\}$$

$$= {}_G\inf\left\{e^{\lambda} >_G 1 : G\sum_{n=1}^{\infty}\left(\frac{e}{e^n}G \odot G\sum_{k=1}^{n}\frac{|\nu_k|_G}{\frac{e^{\lambda}}{|\alpha|_G}}G\right)^{(p_n)_G} \leq_G e\right\}.$$

If it is taken as $e^{\lambda} = e^{\lambda'} \odot |\alpha|_G$, then we obtain

$$\|\alpha \odot \nu\|_G = {}_G\inf \left\{ e^{\lambda'} \odot |\alpha|_G >_G 1 : \right.$$

$$\left. G\sum_{n=1}^{\infty}\left(\frac{e}{e^n}G \odot G\sum_{k=1}^{n}\frac{|\nu_k|_G}{e^{\lambda'}}G\right)^{(p_n)_G} \leq_G e \right\}$$

$$= |\alpha|_G \odot {}_G\inf\left\{e^{\lambda'} >_G 1 : \right.$$

$$\left. G\sum_{n=1}^{\infty}\left(\frac{e}{e^n}G \odot G\sum_{k=1}^{n}\frac{|\nu_k|_G}{e^{\lambda'}}G\right)^{(p_n)_G} \leq_G e \right\}$$

$$= |\alpha|_G \odot \|\nu\|_G.$$

GN3) Let any $\nu, y \in Ces^G(p)$ be given. A, B and C be sets of the positive geometric numbers e^{λ_1}, e^{λ_2} and e^{λ_3} hold the following inequalities

$$G\sum_{n=1}^{\infty}\left(\frac{e}{e^n}G \odot G\sum_{k=1}^{n}\frac{|\nu_k|_G}{e^{\lambda_1}}G\right)^{(p_n)_G} \leq_G e,$$

$$G\sum_{n=1}^{\infty}\left(\frac{e}{e^n}G \odot G\sum_{k=1}^{n}\frac{|y_k|_G}{e^{\lambda_2}}G\right)^{(p_n)_G} \leq_G e$$

and

$$G\sum_{n=1}^{\infty}\left(\frac{e}{e^n}G \odot G\sum_{k=1}^{n}\frac{|\nu_k \oplus y_k|_G}{e^{\lambda_3}}G\right)^{(p_n)_G} \leq_G e,$$

respectively. Taken as $D = \left\{e^{\lambda_1} \oplus e^{\lambda_2} : e^{\lambda_1} \in A \wedge e^{\lambda_2} \in B\right\}$. For any $e^{\lambda_1} \oplus e^{\lambda_2}$, we get

$$G\sum_{n=1}^{\infty}\left(\frac{e}{e^n}G \odot G\sum_{k=1}^{n}\frac{|\nu_k \oplus y_k|_G}{e^{\lambda_1} \oplus e^{\lambda_2}}G\right)^{(p_n)_G}$$

$$\leq_G G\sum_{n=1}^{\infty}\left(\frac{e^{\lambda_1}}{e^{\lambda_1} \oplus e^{\lambda_2}}G \odot \frac{e}{e^n}G \odot G\sum_{k=1}^{n}\frac{|\nu_k|_G}{e^{\lambda_1}}G\right.$$

$$\left. \oplus \frac{e^{\lambda_2}}{e^{\lambda_1} \oplus e^{\lambda_2}}G \odot \frac{e}{e^n}G \odot G\sum_{k=1}^{n}\frac{|y_k|_G}{e^{\lambda_2}}G\right)^{(p_n)_G}$$

$$\leq_G \frac{e^{\lambda_1}}{e^{\lambda_1} \oplus e^{\lambda_2}} \odot G\sum_{n=1}^{\infty}\left(\frac{e}{e^n}G \odot G\sum_{k=1}^{n}\frac{|\nu_k|_G}{e^{\lambda_1}}\right)^{(p_n)_G}$$

$$\oplus \frac{e^{\lambda_2}}{e^{\lambda_1} \oplus e^{\lambda_2}}G \odot G\sum_{n=1}^{\infty}\left(\frac{e}{e^n}G \odot G\sum_{k=1}^{n}\frac{|y_k|_G}{e^{\lambda_2}}G\right)^{(p_n)_G}$$

$$\leq_G \frac{e^{\lambda_1}}{e^{\lambda_1} \oplus e^{\lambda_2}}G \oplus \frac{e^{\lambda_2}}{e^{\lambda_1} \oplus e^{\lambda_2}}G = e.$$

Therefore, we have $D \subset C$. It is obtained that ${}_G\inf C \leq_G {}_G\inf D = {}_G\inf A \oplus {}_G\inf B$, because of $D \subset C$. So, we get $\|\nu \oplus y\|_G \leq_G \|\nu\|_G \oplus \|y\|_G$.

Now, we discuss some relations between the G-modular and Luxemburg G-norm on $Ces^G(p)$.

Proposition 2.11. For any $\nu \in Ces^G(p)$, we have
i) If $\|\nu\|_G <_G e$, then $\sigma(\nu) \leq_G \|\nu\|_G$
ii) If $\|\nu\|_G >_G e$, then $\sigma(\nu) \geq_G \|\nu\|_G$
iii) If $\|\nu\|_G = e \Leftrightarrow \sigma(\nu) = e$
iv) If $\|\nu\|_G <_G e \Leftrightarrow \sigma(\nu) <_G e$
v) If $\|\nu\|_G >_G e \Leftrightarrow \sigma(\nu) >_G e$.

Proof.

i) Let $\varepsilon >_G 1$ be such that $1 <_G \varepsilon <_G e \ominus \|\nu\|_G$. Hence we can write $\|\nu\|_G \oplus \varepsilon <_G e$. By the definiton of $\|.\|_G$, there exists $e^{\lambda} >_G 1$ such that $\|\nu\|_G \oplus \varepsilon >_G e^{\lambda}$ and $\sigma\left(\frac{\nu}{e^{\lambda}}G\right) \leq_G e$. From Proposition 2.9 (i) and (iii), we find

$$\sigma(\nu) \leq_G \sigma\left(\frac{\|\nu\|_G \oplus \varepsilon}{e^{\lambda}}G \odot \nu\right)$$

$$\leq_G (\|\nu\|_G \oplus \varepsilon) \odot \sigma\left(\frac{\nu}{e^{\lambda}}G\right)$$

$$\leq_G e \odot (\|\nu\|_G \oplus \varepsilon)$$

$$= \|\nu\|_G \oplus \varepsilon.$$

Thus $\sigma(\nu) \leq_G \|\nu\|_G \oplus \varepsilon$, for all $\varepsilon \in (1, e \ominus \|\nu\|_G)$. Taken as $A = \{\|\nu\|_G \oplus \varepsilon : 1 <_G \varepsilon <_G e \ominus \|\nu\|_G\}$, then we see that $\|\nu\|_G = {}_G\inf A$. Since $\sigma(\nu)$ is a lower bound of A, we have $\sigma(\nu) \leq_G \|\nu\|_G$.
ii) Let $\varepsilon >_G 1$ be such that $1 <_G \varepsilon <_G \frac{\|\nu\|_G \ominus e}{\|\nu\|_G}G$, hence we can write $\|\nu\|_G >_G \|\nu\|_G \odot (e \ominus \varepsilon) >_G e$. By the definiton of $\|.\|_G$ and Proposition 2.9 (i), we find

$$e <_G \sigma\left(\frac{\nu}{(e \ominus \varepsilon) \odot \|\nu\|_G}G\right)$$

$$\leq_G \frac{e}{(e \ominus \varepsilon) \odot \|\nu\|_G}G \odot \sigma(\nu).$$

So, we can write $(e \ominus \varepsilon) \odot \|\nu\|_G <_G \sigma(\nu)$ for all $\varepsilon \in \left(1, \frac{\|\nu\|_G \ominus e}{\|\nu\|_G}\right)$. Taken as

$$A = \left\{(e \ominus \varepsilon) \odot \|\nu\|_G : 1 <_G \varepsilon <_G \frac{\|\nu\|_G \ominus e}{\|\nu\|_G}\right\},$$

then we see that $\|\nu\|_G = {}_G\sup A$. Since $\sigma(\nu)$ is an upper bound of A, we have $\|\nu\|_G \leq_G \sigma(\nu)$.

iii) Let $\varepsilon >_G 1$. Suppose that $\|\nu\|_G = e$. From the definiton of $\|.\|_G$, there exists $e^\lambda >_G 1$ such that $(e \oplus \varepsilon) >_G e^\lambda >_G \|\nu\|_G$ and $\sigma\left(\frac{\nu}{e^\lambda} G\right) \leq_G e$. Since $e^\lambda >_G e$,

$$\sigma(\nu) \leq_G \left(e^\lambda\right)^{M_G} \odot \sigma\left(\frac{\nu}{e^\lambda} G\right)$$

$$\leq_G \left(e^\lambda\right)^{M_G} <_G (e \oplus \varepsilon)^{M_G}$$

by Proposition 2.9 (ii). Hence $\left(\sigma(\nu)\right)^{\left(\frac{1}{M}\right)_G} <_G e \oplus \varepsilon$ for all $\varepsilon >_G 1$ which implies $\sigma(\nu) \leq_G e$. Assume that $\sigma(\nu) <_G e$, we can choose $a \in (1, e)$ such that $\sigma(\nu) <_G a^{M_G} <_G e$. Hence we have

$$a^{M_G} \odot \sigma\left(\frac{\nu}{a} G\right) \leq_G \sigma(\nu)$$

$$\sigma\left(\frac{\nu}{a} G\right) \leq_G \frac{e}{a^{M_G}} G \odot \sigma(\nu) <_G e$$

by Proposition 2.9 (i). Therefore, we get $\|\nu\|_G \leq_G a <_G e$ which contradicts to our assumption that $\|\nu\|_G = e$. Hence $\sigma(\nu) = e$.

Conversely, suppose that $\sigma(\nu) = e$. The definition of Luxemburg G-norm $\|.\|_G$, we conclude that $\|\nu\|_G \leq_G e$. If $\|\nu\|_G <_G e$, then we have by (i) that $\sigma(\nu) \leq_G \|\nu\|_G <_G e$ which contradicts to our assumption that $\sigma(\nu) = e$. Therefore, $\|\nu\|_G = e$.

iv) If $\|\nu\|_G <_G e$, then we have by (i) that $\sigma(\nu) \leq_G \|\nu\|_G <_G e$.

Conversely, assume that $\sigma(\nu) <_G e$. It follows from (i) and (ii) $\|\nu\|_G <_G e$.

v) It follows from (iii) and (iv).

Proposition 2.12. Let $\nu \in Ces^G(p)$ and $M = \sup_{n\in\mathbb{N}} p_n$.
i) If $1 <_G a <_G e$ and $\|\nu\|_G >_G a$, then $\sigma(\nu) >_G a^{M_G}$.
ii) If $a \geq_G e$ and $\|\nu\|_G <_G a$, then $\sigma(\nu) <_G a^{M_G}$.

Proof.

i) Assume that $1 <_G a <_G e$ and $\|\nu\|_G >_G a$. Hence we can write $\left\|\frac{\nu}{a} G\right\|_G >_G e$. We have $\sigma\left(\frac{\nu}{a} G\right) \geq_G \left\|\frac{\nu}{a} G\right\|_G >_G e$ by using Proposition 2.11 (ii). Since $1 <_G a <_G e$, we obtain $\sigma(\nu) \geq_G a^{M_G} \odot \sigma\left(\frac{\nu}{a} G\right) >_G a^{M_G} \odot e = a^{M_G}$ from Proposition 2.9 (i).

ii) Assume that $a \geq_G e$ and $\|\nu\|_G <_G a$. Hence we can write $\left\|\frac{\nu}{a} G\right\|_G <_G e$. We find $\sigma\left(\frac{\nu}{a} G\right) \leq_G \left\|\frac{\nu}{a} G\right\|_G <_G e$ from Proposition 2.11 (i). If $a = e$, then we get $\sigma(\nu) <_G e = a^{M_G}$. If $a >_G e$, we obtain $\sigma(\nu) \leq_G a^{M_G} \odot \sigma\left(\frac{\nu}{a} G\right) <_G a^{M_G} \odot e = a^{M_G}$ from Proposition 2.9 (ii).

Proposition 2.13. Let (ν_k) be a sequence in $Ces^G(p)$.
i) If ${}_G\lim_{k\to\infty} \|\nu_k\|_G = e$, then ${}_G\lim_{k\to\infty} \sigma(\nu_k) = e$.
ii) If ${}_G\lim_{k\to\infty} \sigma(\nu_k) = 1$, then ${}_G\lim_{k\to\infty} \|\nu_k\|_G = 1$.

Proof.
i) Assume that ${}_G\lim_{k\to\infty} \|\nu_k\|_G = e$. Let $\varepsilon \in (1, e)$. Then there exists $k_0 \in \mathbb{N}$ such that $(e \ominus \varepsilon) <_G \|\nu_k\|_G <_G (\varepsilon \oplus e)$ for all $k > k_0$. By Proposition 2.12 (i) and (ii), we find $(e \ominus \varepsilon)^{M_G} <_G \sigma(\nu_k) <_G (e \oplus \varepsilon)^{M_G}$ for all $k > k_0$ which implies that $\sigma(x_k) \overset{G}{\to} e$ as $k \to \infty$.
ii) Assume that ${}_G\lim_{k\to\infty} \|\nu_k\|_G \neq 1$. Then there exists $\varepsilon \in (1, e)$ and a subsequence (ν_{k_n}) of (ν_k) such that $\left\|\nu_{k_n}\right\|_G >_G \varepsilon$ for all $n \in \mathbb{N}$. By Proposition 2.12 (i), we have $\sigma\left(\nu_{k_n}\right) >_G \varepsilon^{M_G}$ for all $n \in \mathbb{N}$. This implies $\sigma(\nu_k) \overset{G}{\nrightarrow} 1$ as $k \to \infty$.

Proposition 2.14. Let $\nu = (\nu_s)$, $\nu_k = \left(\nu_k(s)\right)_{s=1}^\infty \in Ces^G(p)$ for all $s \in \mathbb{N}$. If $\sigma(\nu_k) \overset{G}{\to} \sigma(\nu)$ as $k \to \infty$ and $\nu_k(s) \overset{G}{\to} \nu(s)$ as $k \to \infty$ for all $s \in \mathbb{N}$, then $\nu_k \to \nu$ as $k \to \infty$.

Proof.
Let $\varepsilon >_G 1$ be given. Since $\sigma(\nu) = G\sum_{n=1}^\infty \left(\frac{e}{e^n} G \odot G\sum_{s=1}^n |\nu(s)|_G\right)^{(p_n)_G} <_G \infty$, there exists $n_0 \in \mathbb{N}$ such that

$$G\sum_{n=n_0+1}^{\infty}\left(\frac{e}{e^n}G\odot G\sum_{s=1}^{n}|\nu(s)|_G\right)^{(p_n)_G}$$
$$<_G \frac{\varepsilon}{e^3}G\odot\frac{e}{e^{2M+1}}G. \tag{7}$$

Since

$$\sigma(\nu_k)\ominus G\sum_{n=1}^{n_0}\left(\frac{e}{e^n}G\odot G\sum_{s=1}^{n}|\nu_k(s)|_G\right)^{(p_n)_G}$$
$$\xrightarrow{G}\sigma(\nu)\ominus G\sum_{n=1}^{n_0}\left(\frac{e}{e^n}G\odot G\sum_{s=1}^{n}|\nu(s)|_G\right)^{(p_n)_G}$$

as $k\to\infty$ and $\nu_k(s)\xrightarrow{G}\nu(s)$ as $k\to\infty$, there exists $k_0\in\mathbb{N}$ such that

$$\sigma(\nu_k)\ominus G\sum_{n=1}^{n_0}\left(\frac{e}{e^n}G\odot G\sum_{s=1}^{n}|\nu_k(s)|_G\right)^{(p_n)_G}$$
$$<_G \sigma(\nu)\ominus G\sum_{n=1}^{n_0}\left(\frac{e}{e^n}G\odot G\sum_{s=1}^{n}|\nu(s)|_G\right)^{(p_n)_G}$$
$$\oplus\frac{\varepsilon}{e^3}G\odot\frac{e}{e^{2M+1}}G \tag{8}$$

for all $k>k_0$, and

$$G\sum_{n=1}^{n_0}\left(\frac{e}{e^n}G\odot G\sum_{s=1}^{n}\left|\nu_k(s)\ominus\nu(s)\right|_G\right)^{(p_n)_G} <_G \frac{\varepsilon}{e^3} \tag{9}$$

for all $k>k_0$. It follows from (7), (8) and (9) that for all $k>k_0$,

$$\sigma(\nu_k\ominus\nu)$$
$$=G\sum_{n=1}^{\infty}\left(\frac{e}{e^n}G\odot G\sum_{s=1}^{n}|\nu_k(s)\ominus\nu(s)|_G\right)^{(p_n)_G}$$
$$=G\sum_{n=1}^{n_0}\left(\frac{e}{e^n}G\odot G\sum_{s=1}^{n}|\nu_k(s)\ominus\nu(s)|_G\right)^{(p_n)_G}$$
$$\oplus G\sum_{n=n_0+1}^{\infty}\left(\frac{e}{e^n}G\odot G\sum_{s=1}^{n}|\nu_k(s)\ominus\nu(s)|_G\right)^{(p_n)_G}$$
$$<_G\frac{\varepsilon}{e^3}\oplus e^{2M}\odot$$
$$\left[G\sum_{n=n_0+1}^{\infty}\left(\frac{e}{e^n}G\odot G\sum_{s=1}^{n}|\nu_k(s)|_G\right)^{(p_n)_G}\right.$$
$$\left.\oplus G\sum_{n=n_0+1}^{\infty}\left(\frac{e}{e^n}G\odot G\sum_{s=1}^{n}|\nu(s)|_G\right)^{(p_n)_G}\right]$$
$$=\frac{\varepsilon}{e^3}\oplus e^{2M}\odot$$
$$\left(\sigma(\nu_k)\ominus G\sum_{n=1}^{n_0}\left(\frac{e}{e^n}G\odot G\sum_{s=1}^{n}|\nu_k(s)|_G\right)^{(p_n)_G}\right.$$
$$\left.\oplus G\sum_{n=n_0+1}^{\infty}\left(\frac{e}{e^n}G\odot G\sum_{s=1}^{n}|\nu(s)|_G\right)^{(p_n)_G}\right)$$
$$<_G\frac{\varepsilon}{e^3}\oplus e^{2M}\odot$$
$$\left(\sigma(\nu)\ominus G\sum_{n=1}^{n_0}\left(\frac{e}{e^n}G\odot G\sum_{s=1}^{n}|\nu(s)|_G\right)^{(p_n)_G}\right.$$
$$\oplus\frac{\varepsilon}{e^3}G\odot\frac{e}{e^{2M+1}}G$$
$$\left.\oplus G\sum_{n=n_0+1}^{\infty}\left(\frac{e}{e^n}G\odot G\sum_{s=1}^{n}|\nu(s)|_G\right)^{(p_n)_G}\right)$$
$$<_G\frac{\varepsilon}{e^3}\oplus e^{2M}\odot$$
$$\left(G\sum_{n=n_0+1}^{\infty}\left(\frac{e}{e^n}G\odot G\sum_{s=1}^{n}|\nu(s)|_G\right)^{(p_n)_G}\right.$$
$$\oplus\frac{\varepsilon}{e^3}G\odot\frac{e}{e^{2M+1}}G$$
$$\left.\oplus G\sum_{n=n_0+1}^{\infty}\left(\frac{e}{e^n}G\odot G\sum_{s=1}^{n}|\nu(s)|_G\right)^{(p_n)_G}\right)$$
$$=\frac{\varepsilon}{e^3}\oplus e^{2M}\odot$$
$$\left(e^2\odot G\sum_{n=n_0+1}^{\infty}\left(\frac{e}{e^n}G\odot G\sum_{s=1}^{n}|\nu(s)|_G\right)^{(p_n)_G}\right.$$
$$\left.\oplus\frac{\varepsilon}{e^3}G\odot\frac{e}{e^{2M}}G\right)$$
$$<_G\frac{\varepsilon}{e^3}G\oplus\frac{\varepsilon}{e^3}G\oplus\frac{\varepsilon}{e^3}G=\varepsilon.$$

This shows that $\sigma(\nu_k\ominus\nu)\xrightarrow{G}1$ as $k\to\infty$. Therefore, by Proposition 2.13 (ii) $\|\nu_k-\nu\|_G\xrightarrow{G}1$ as $k\to\infty$.

Theorem 2.15. $Ces^G(p)$ is a G-Banach space under the Luxemburg G-norm.

Proof. Let $\left(\nu^{(n)}\right)=\left(\nu_k^{(n)}\right)$ be G-Cauchy sequence in $Ces^G(p)$. Given any $\varepsilon\in(1,e)$. Hence there exists $n_0\in\mathbb{N}$ such that

$$\left\|\nu^{(n)}\ominus\nu^{(s)}\right\|_G<_G\varepsilon^{M_G}$$

for all $n,s>n_o$. By Proposition 2.11 (i), we have

$$\sigma\left(\nu^{(n)}\ominus\nu^{(s)}\right)<_G\left\|\nu^{(n)}\ominus\nu^{(s)}\right\|_G<_G\varepsilon^{M_G} \tag{10}$$

for all $n,s>n_o$. Hence, we can write

$$G\sum_{N=1}^{\infty}\left(\frac{e}{e^N}G\odot G\sum_{k=1}^{N}\left|\nu_k^{(n)}\ominus\nu_k^{(s)}\right|_G\right)^{(p_k)_G} <_G \varepsilon^{M_G}$$

Because of

$\left(\frac{e}{e^N}G\odot G\sum_{k=1}^{N}\left|\nu_k^{(n)}\ominus\nu_k^{(s)}\right|_G\right)^{(p_k)_G} <_G \varepsilon^{M_G}$ for all $N\in\mathbb{N}$, we find

$$G\sum_{k=1}^{N}\left|\nu_k^{(n)}\ominus\nu_k^{(s)}\right|_G <_G \varepsilon\odot e^N=\varepsilon^N.$$

This implies

$$\left|\nu_k^{(n)}\ominus\nu_k^{(s)}\right|_G <_G \varepsilon^k$$

for all $k\in\mathbb{N}$ and for all $n,s>n_o$. Since $\left(\nu_k^{(n)}\right)$ be a G-Cauchy sequence in $\mathbb{R}_G$, there exists $\nu_k\in\mathbb{R}_G$ such that $\nu_k^{(n)}\overset{G}{\to}\nu_k$ for all $k\in\mathbb{N}$. Let $\nu=(\nu_k)$, we shall show that $\nu\in Ces^G(p)$. Taken as $\nu|_r=(\nu_1,\nu_2,\ldots,\nu_r,1,1,1,\ldots)$. For all $r\in\mathbb{N}$ and $n,s>n_o$, we find

$$\sigma\left(\left(\nu^{(n)}\ominus\nu^{(s)}\right)\Big|_r\right) <_G \varepsilon^{M_G}$$

by using (10). Since $\nu_k^{(s)}\overset{G}{\to}\nu_k$ for all $k=1,2,3,\ldots,r,$ we find

$$\sigma\left(\left(\nu^{(n)}\ominus\nu^{(s)}\right)\Big|_r\right)\overset{G}{\to}\sigma\left(\left(\nu^{(n)}\ominus\nu\right)\Big|_r\right)$$

as $s\to\infty$. Hence, we get $\sigma\left(\left(\nu^{(n)}\ominus\nu\right)\Big|_r\right) <_G \varepsilon^{M_G}$ for all $r\in\mathbb{N}$ and $n>n_o$. This implies

$$\sigma\left(\nu^{(n)}\ominus\nu\right) <_G \varepsilon^{M_G}$$

for all $n>n_o$. By Proposition 2.12 (i), we obtain

$$\left\|\nu^{(n)}\ominus\nu\right\|_G <_G \varepsilon$$

for all $n>n_o$. This means that $\nu^{(n)}\overset{G}{\to}\nu$ where $n\to\infty$. Also, we see that $\nu=\nu^{(n)}\ominus\left(\nu^{(n)}\ominus\nu\right)\in Ces^G(p)$, because of $\nu^{(n)}\ominus\nu\in Ces^G(p)$. Therefore, $Ces^G(p)$ is G-Banach space.

Definition 2.16. Let V be a G-Banach space. A point $\nu\in S(V)=\{\upsilon\in V:\|\upsilon\|_G=e\}$ is referred to as G-nonsquare point if for every $y\in S(V)$ the condition $\min\left\{\left\|\frac{\nu\oplus y}{e^2}G\right\|_G,\left\|\frac{\nu\ominus y}{e^2}G\right\|_G\right\} <_G e$ holds true.

Definition 2.17. A G-Banach space V is called G-nonsquare, if all element ν in $S(V)$ is G-nonsquare point.

Proposition 2.18. Let $p=(p_n)$ be a bounded sequence of positive real numbers with $p_n>1$ for all $n\in\mathbb{N}$. Then $\nu\in S\left(Ces^G(p)\right)$ is a G-nonsquare point $\Leftrightarrow \sigma(\nu)=e$.

Proof. From Proposition 2.11 (iii), it seen that if $\nu\in Ces^G(p)$, then $\sigma(\nu)=e$. Now let $\sigma(\nu)=e$ and suppose that ν is not G-nonsquare point. Then there exits $y\in S\left(Ces^G(p)\right)$ such that $\|\nu\oplus y\|_G=\|\nu\ominus y\|_G=e$. Hence, we can write $\sigma(\nu\oplus y)=\sigma(\nu\ominus y)=e$ by using Proposition 2.11 (iii). Since $p_n>1$ for all $n\in\mathbb{N}$

$$e=\sigma(\nu)=\sigma\left(\frac{\nu\oplus y}{e^2}G\oplus\frac{\nu\ominus y}{e^2}G\right)$$

$$<_G \frac{e}{e^2}G\odot\sigma(\nu\oplus y)\oplus\frac{e}{e^2}G\odot\sigma(\nu\ominus y)=e$$

is obtained due to strict G-convexity. This constitutes a contradiction.

Theorem 2.19. Let $p=(p_n)$ be a bounded sequence of positive real numbers with $p_n>1$ for all $n\in\mathbb{N}$. Then $Ces^G(p)$ is G-nonsquare.

Proof. It follows from Proposition 2.11 (iii) and Proposition 2.18.

3 Conclusion

The concepts of modular, modular spaces, and Luxemburg norm are given from a new perspective using geometric arithmetic. We define the generalized geometric Cesàro sequence space and construct a G-modular on this space. Luxemburg G-norm, produced by the G-modular, is built into the generalized geometric Cesàro sequence space. The relationships between G-modular and Luxemburg G-norm are investigated. Also, we provide evidence that the generalized geometric Cesàro sequence space is, in fact, a G-Banach space under the Luxemburg G-norm. Moreover, one gets that the generalized geometric Cesàro sequence space is G-nonsquare when $p_n>1$ for all $n\in\mathbb{N}$. This sets the way for our future work, which will look into the dual spaces of the geometric Cesàro sequence space and establish the relevant matrix transformations. Since the theory of sequence space and geometric calculus is quite active and has extensive applications, we believe many researchers will use our newly acquired results for future works and applications in related fields.

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Contribution of Individual Authors to the Creation of a Scientific Article (Ghostwriting Policy)

The authors equally contributed in the present research, at all stages from the formulation of the problem to the final findings and solution.

Sources of Funding for Research Presented in a Scientific Article or Scientific Article Itself
No funding was received for conducting this study.

Conflict of Interest
The author declares that they have no conflict interests.