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Analysis of the Structure of Chaotic Solutions of Differential Equations

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Abstract: - This study deals with the relevant and important area of many fields of mathematics and physics - chaotic systems. Three modified systems of Chua differential equations were considered, and the chaotic structure of their solutions was compared with the structure of solutions of classical Lorentz and Rössler chaotic systems. The following methods were used to achieve the set goal: the Runge-Kutta method, building a phase portrait, determining Lyapunov exponents and noise level, and comparative analysis. A detailed analysis of the structure of chaotic solutions of various differential equations was carried out. It was established that the chaotic solution's structure depends on the differential equation's properties and the initial conditions. According to the obtained results, one of the modifications of the Chua system is significantly superior to classical chaotic systems and can be used as a chaos generator. Prospects for further research involve expanding the scope of the study and the generalization of the obtained results for a wider class of systems of differential equations.

Key-Words: - chaos theory, chaotic solutions, Chua system, differential equations, Lorentz attractor, Rössler attractor.

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1 Introduction

Analysis of the structure of chaotic solutions of various differential equations is one of the key topics of modern Mathematics and Physics. This research is relevant as it is important for understanding various complex systems that are the subject of study in many sciences, such as Physics, Chemistry, Biology, Economics, and others.

Chaos is the non-deterministic behaviour of systems that have a complex structures. These systems usually consist of many interacting elements that can produce unpredictable results. Differential equations are a key instrument for studying complex systems. These equations describe the change of physical quantities over time. Analysis of the structure of chaotic solutions of differential equations enables obtaining new knowledge about the behaviour of systems that usually cannot be described by simple rules.

So, studying the structure of chaotic solutions of various differential equations is a relevant and important area of modern Mathematics and Physics. It can be applied in various fields of science and technology: weather and climate forecasting, control of chaotic systems, cryptography, forecasting the development of pandemics and epidemics, creation of artificial neural networks, etc.

The aim of the research: a study of the structure of chaos in the system of Chua's equations and its modifications, identification of factors affecting the formation of chaotic solutions and their features.

Research objectives:

- Select a set of differential equations of different complexity and perform their simulation;
- Study and compare the structure of chaotic solutions for different parameters and initial conditions;
- Assess the impact of internal and external factors on the structure of chaotic solutions and their behaviour.

2 Literature Review

Henri Poincaré was the first researcher of chaos. In the 1880s, he studied the behaviour of a system with three gravitationally interacting bodies and found that there could be non-periodic orbits that are constantly neither moving away from nor approaching a particular point. Later, many world scientists made a great contribution to the study of chaotic structures, [1]. Many researchers focus on this issue in their studies, given the wide range of applications of chaotic systems. For example, the work, [2], considered the nonlinear Schrödinger equation with the Kerr Law Nonlinearity and demonstrated the existence of various traveling wave reactions. They also described the parametric criteria for all these traveling solutions based on physical factors. In, [3] the researchers also consider the dynamic features of the dispersive extended nonlinear Schrödinger equation (NLSE), where a new method of expanding the F6−model is applied to study solitary waves of the considered model. In, [4] the authors obtained and studied a diverse range of traveling wave structures in the perturbed Fokas–Lenells model (p-FLM) using the extended (G/G2)-expansion approach. In the paper, [5], studied the dynamic characteristics of a meminductor and a memcapacitor through the fractal-factional Caputo–Fabrizio operator. The chaos scheme is modeled for highly nonlinear and non-fractional meminductor and meminductor governing differential equations for studying chaos, hyperchaos, and coexisting attractors. The use of Bernstein and Euler wavelets was considered for solving a nonlinear fractional biological model of two species predator–prey model, [6]. This resulted in new chaotic models of the population of predators and prey. Anees and Iqtadar conducted an analysis based on the chaotic sequences of the Lorentz system and the logistic chaotic map, [7]. They found a serious problem of hacking in symmetric security systems of chaotic communications. In, [8] the authors derived nonlinear governing equations for developing a nonlinear dynamic model for nonlinear frequency and chaotic responses of a doubly curved composite panel reinforced with graphene nanoplates using Hamilton's principle and nonlinear von Kármán theory. In the study, [9], the researchers carry out simulations of several chaotic systems such as multi-spin attractors, self-excited and hidden attractors, period-doubling to chaos, periodic and chaotic explosive oscillations, and various multiple coexisting attractors using a new Atangana–Baleanu time-fractional derivative. In work, [10], proposed a self-hyperchaotic system-based perturbed pseudorandom sequence generator to overcome this problem. This hyperchaotic system is designed to achieve complex dynamic behaviour. In work, [11], also proposed a random number generator using a fractional order Chua chaotic system.

Despite the existing research in this field, the problem remains relevant due to the many unsolved questions and the need for a more detailed understanding of chaoticity. In this work, we try to fill this gap by analyzing the structure of chaotic solutions of differential equations, particularly the system of Chua's equations and their modifications.

3 Methods

The following systems of differential equations are considered in this work: the Lorentz equation, the Rössler equation, and modifications of the Chua equation.

1. Lorentz equation is a system of inhomogeneous differential equations that describe the dynamics of a system capable of transiting from one state to another, where each state corresponds to some parameters in this system. The Lorentz equations were developed by Edward Lorentz in 1963 to describe turbulent fluid flow. These equations were introduced to understand the behaviour of atmospheric processes.

The Lorentz equation consists of three coupled first-order differential equations:

$$\begin{cases} \dot{X} = \sigma(y - x) \\ \dot{Y} = x(\alpha - z) - y \\ \dot{Z} = xy - \beta z \end{cases} \quad (1)$$

where *x(t)* — convective movement intensity; *y(t)* — the temperature difference of the ascending and descending liquid flows; *z(t)* — deviation of the vertical temperature distribution from the linear regime; σ — the Prandtl number, a parameter that affects the stretching of the system in the x direction, where y is greater than x; α — a parameter that affects the clustering or stretching of the system in the z direction when the value of z is less than or greater than a certain threshold (2); β — a parameter that affects the interaction between x and z by decreasing the value of z when x and y are greater than a certain threshold (3).

$$\alpha = \frac{gaH^3 \Delta T / (\nu k)}{\pi^4 (1 + a^2)^3 / a^2} \quad (2)$$

$$\beta = \frac{4}{1 + a^2} \quad (3)$$

where g — gravity acceleration; a — coefficient of thermal expansion; H — the height of the liquid layer; ΔT — the temperature difference between the upper and lower levels; v - kinematic viscosity of the liquid; k — thermal conductivity of the liquid.

It is known that under condition (4) unstable limit cycles assemble into stationary points, and stationary points lose their stability, forming a Lorentz attractor, which will be considered in this study.

$$\alpha > \frac{\sigma(\sigma+\beta+3)}{\sigma-\beta-1} \quad (4)$$

2. Rössler attractor is an attractor found in the Rössler system of differential equations (5) discovered in 1976, which describes the dynamics of chemical reactions occurring in some stirred mixture.

$$\begin{cases} \frac{dx}{dt} = -y - z \\ \frac{dy}{dt} = x + ay \\ \frac{dz}{dt} = b + z(x+c) \end{cases} \quad (5)$$

where *a*, *b*, and *c* – are parameters that determine the system behaviour. The classic Rössler attractor occurs with the following values of the parameters: *a=0.2; b=0.2; c=5.7.*

3. Chua circuit proposed in 1983 is the simplest electrical circuit that demonstrates modes of chaotic oscillations. The circuit (Figure 1) consists of two capacitors, one inductor, a linear resistor, and a nonlinear resistor with negative resistance — a Chua diode.

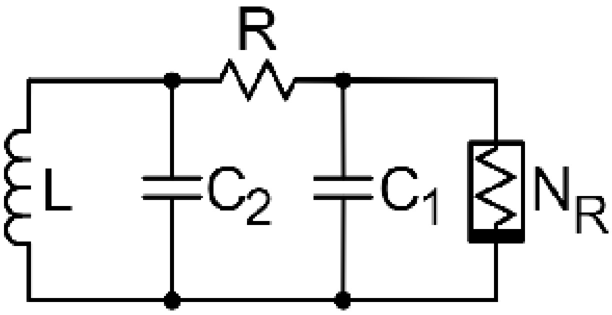

Fig. 1: Chua circuit

The system of differential equations describing the processes of this circuit looks as follows:

$$\begin{cases} \frac{dx}{dt} = \alpha(y - x - h(x)) \\ \frac{dy}{dt} = x - y + z \\ \frac{dz}{dt} = -\beta y \end{cases} \quad (6)$$

where

$$h(x) = m_1 x + \frac{1}{2}(m_0 - m_1)(|x+1| - |x-1|); \quad (7)$$

$$m_0 = \frac{G_a}{G}; \quad (8)$$

$$m_1 = \frac{G_b}{G}; \quad (9)$$

$$\alpha = \frac{C_2}{C_1}; \quad (10)$$

$$\beta = \frac{C_2}{LG^2}; \quad (11)$$

$$\tau = \frac{tG}{C_2}; \quad (12)$$

$$x = \frac{\upsilon c_1}{E}; \quad (13)$$

$$y = \frac{\upsilon c_2}{E}; \quad (14)$$

$$x = \frac{i_L}{EG}; \quad (15)$$

where *h(x)* — a nonlinear function involving feedback through a resistor, a capacitor, and a Chua diode; G_a i G_b – resistors; *G* — an active element implemented through an operational amplifier, transistor, or other element; C_1 and C_2 – containers; *L* – inductance; i_L – the current flowing through the inductor; *v*, c_1, c_2 – the initial values of the voltage on the capacitors C_1, C_2 and voltages on the Chua diode; *t* – time; *E* – electromotive force.

This study covers three modifications of Chua equation systems:

3.1. Chua system of equations (16) with an unsteady motion function *h(x)* (17) and additional parameters:

$$\begin{cases} \frac{dx}{dt} = a(y - x - h(x)) \\ \frac{dy}{dt} = x - y + z \\ \frac{dz}{dt} = -by/c \end{cases} \quad (16)$$

$$h(x) = \begin{cases} d(x+1) - e, & if \;\; x < 1; \\ d(x-1) + e, & if \;\; x > 1; \\ -dx, & if \;\; x = 1; \end{cases} \quad (17)$$

3.2. Chua system of equations (18) with a non-constant function *h(x)* (19) and additional parameters:

$$\begin{cases} \frac{dx}{dt} = a(y - x - h(x)) \\ \frac{dy}{dt} = x - y + z \\ \frac{dz}{dt} = -by + c_1 x + c_2 z \end{cases} \qquad (18)$$

$$h(x) = \begin{vmatrix} m_1 x + m_1 r, & if \quad x < -r; \\ m_1 x - m_1 r, & if \quad x > r; \\ m_0 x, \quad if & -r < x < r; \end{vmatrix} \qquad (19)$$

3.3. Chua system of equations (20) containing an additional equation that describes the current control element

$$\begin{cases} \frac{dx}{dt} = a\left(y \frac{2x^3}{7}\right) \\ \frac{dy}{dt} = x - y + z \\ \frac{dz}{dt} = -by \\ \frac{dA}{dt} = c(x - A) \end{cases} \qquad (20)$$

where A —is the current-controlling element and c — is the system parameter.

A comparative analysis of the chaoticity of modified Chua systems with classical systems of Lorentz and Rössler differential equations (1, 5) was carried out by creating a phase portrait and calculating Lyapunov exponents (indicators) (21). Various methods of analyzing chaotic systems, such as fractal geometry and spectral analysis, were proposed in the reviewed literature sources. In this work, the analysis of phase portraits and the Lyapunov exponent were chosen, since these methods allow for a more detailed study of the structure of chaotic solutions and to determine their characteristics, such as the degree of chaos and sensitivity to initial conditions. This approach makes it possible to make a comparative analysis with other chaotic systems and to determine the peculiarities of the system of Chua's equations. For this purpose, they were solved using the Runge-Kutta fourth-order method.

$$\lambda = \lim \frac{1}{t} \ln \| d_x F^t \|; \qquad (21)$$

where $|d_x F^t|$ shows the distance between these two trajectories in the state space of the system.

The values of the Lyapunov exponents indicate the rate of increase or decrease of the distance between two close trajectories over time, which is a measure of the stability or chaos of the system. When the system has all negative Lyapunov exponents, all trajectories converge to a fixed point and are stable. When at least one of the exponents is positive, the system is unstable and has chaotic behaviour.

A phase portrait is a geometric image of the solutions of a dynamic system in the phase space, which consists of the system coordinates, where each coordinate corresponds to the state of the system, and the geometric image reflects the nature of changes in the system states over time.

The phase portrait in stable systems is usually a set of nodes, foci, and centers. A node represents a steady fixed state of a system, and a focus represents periodic solutions. The center displays periodic solutions that do not coincide but coincide in time on average. As a rule, phase trajectories in stable systems converge to fixed points or cycles, and there are characteristic areas of convergence around these points. This means the system has certain regions of convergence, where the phase trajectories deviate from these stable solutions and reflect chaotic dynamics. In our study, we analyze these convergence regions in detail and study the structure of chaotic solutions of the system of Chua equations.

The phase portrait in chaotic systems has a more complex structure, reflecting system states' high-frequency change. The difference between phase portraits in chaotic systems is the curves that look like randomly mixed bands, so-called Poincaré bands. These bands show that phase trajectories can be very sensitive to initial conditions. In other words, small changes in initial conditions can result in very different trajectories. This is called the "butterfly effect", where small changes in the initial conditions can significantly affect the system's behaviour in the future.

The effectiveness of the studied systems as generators of chaos was compared through the noise level assessment. The coefficient of variation, the ratio of the root-mean-square deviation to the absolute mean value of the sample were used as an indicator for determining the noise level:

$$K_v = \frac{\sigma}{|\bar{x}|} 100\% \qquad (22)$$

Modelling of equations and analysis of their chaotic structure was carried out in the PyCharm environment by building models using Python.

Various approaches to the analysis of phase portraits and the determination of Lyapunov exponents are considered. Using the methods of numerical modeling and differential geometry, a detailed analysis of the phase portraits of the system of Chua's equations was carried out. The calculation of Lyapunov exponents for different sets of initial conditions was performed, which made it possible to estimate the degree of chaos of the system.

4 Results

1. The system of Lorentz equations was analyzed according to its classical parameters: $\sigma = 10, \alpha = 28, \beta = 8/3$. With two sets of initial conditions:

1) $x_0=1, y_0=1, z_0=1$
2) $x_0=-1, y_0=-1, z_0=-1$

Consider the phase portraits of this system with different sets of initial conditions (Figure 2).

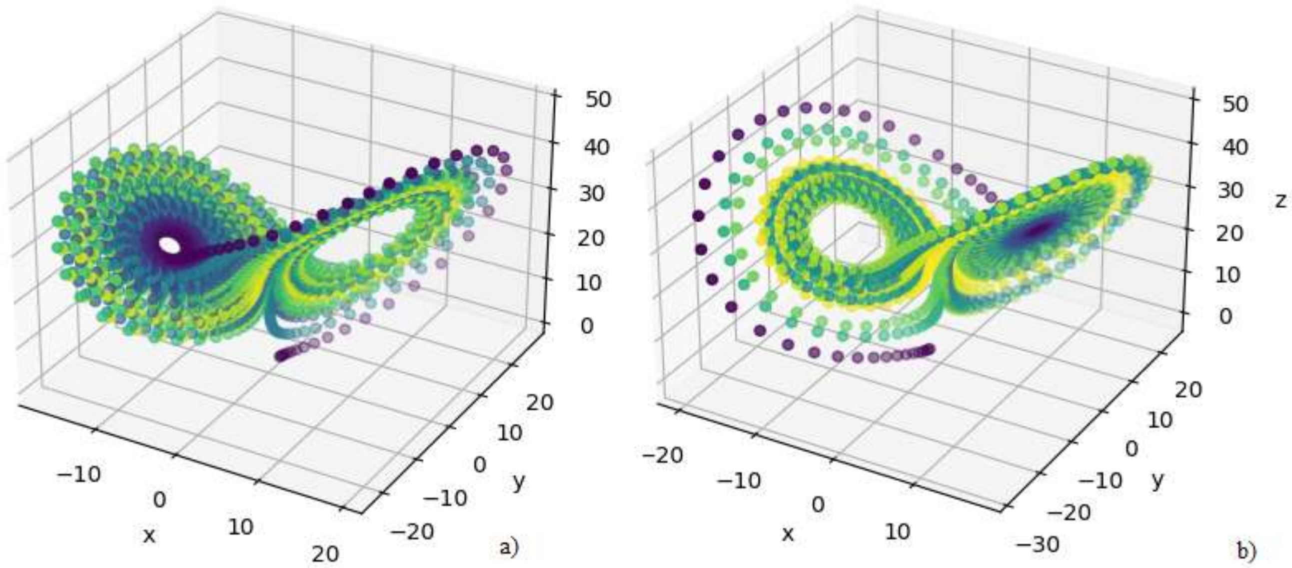

Fig. 2: Lorentz attractor: a) under initial conditions $x_0=1, y_0=1, z_0=1$; б) under initial conditions $x_0=-1, y_0=-1, z_0=-1$

The Lorentz attractor, which occurs in the case of the parameters: $\sigma = 10, \alpha = 28, \beta = 8/3$, is characterized by chaotic behaviour, which manifests itself as complex, non-periodic oscillations. Phase portraits of the Lorentz attractor (Figure 2) have a characteristic shape resembling two twisted scales that are unevenly stretched into space. Moreover, as Figure 2a and Figure 2b demonstrate, a “butterfly effect” — sensitivity to initial conditions — is observed for this equation system.

The respective Lyapunov exponents for two sets of parameters are:

1) $\lambda_1 = 0.9293; \lambda_2 = -14.572; \lambda_3 = -8.022$;
2) $\lambda_1 = -0.9293; \lambda_2 = -14.572; \lambda_3 = -8.022$.

It is worth noting that the negative sign of λ1 for the second initial conditions indicates that these conditions are an attractor particle that is symmetric with respect to the origin of coordinates.

2. The system of Rössler’s equations was analyzed according to two sets of parameters:

1) $a = 0.2; b = 0.2; c = 5.7$;
2) $a = 0.1; b = 0.1; c = 14$,

and initial conditions: $x_0 = 1, y_0 = 0.2, z_0 = 3$.

The phase portraits of these two experiments are shown in Figure 3.

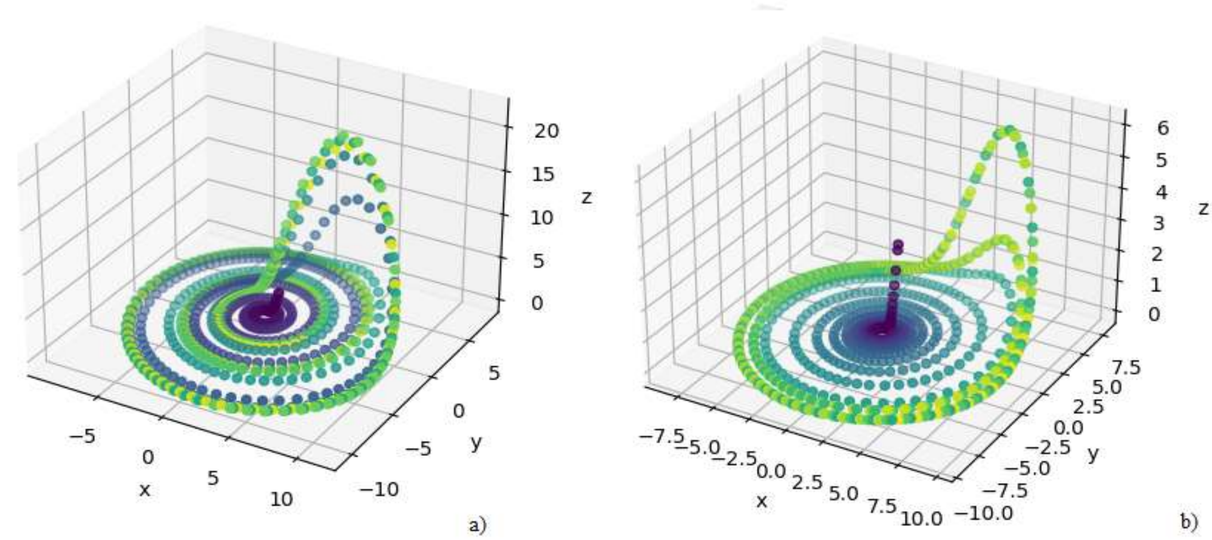

Fig. 3: Rössler attractor: a) for the parameters $a = 0.2; b = 0.2; c = 5.7$; б) for the parameters $a = 0.1; b = 0.1; c = 14$

Studying the phase portraits of the Rössler dynamic system, we also observe the occurrence of chaotic attractors. For the considered parameters, the Lyapunov exponents are as follows:

1) $\lambda_1 = -1.09868411; \lambda_2 = 0.4384828245735064; \lambda_3 = -0.4384828245735064$;

2) $\lambda_1 = 0.11727913; \lambda_2 = -1.57868094; \lambda_3 = -9.29501389$.

The equation $\lambda 2$ is positive for the first set of parameters and the second $\lambda 1$, confirming that the considered system is chaotic.

The initial conditions were randomly generated in the range [-1;1] to study three modified Chua systems. A total of 1,000 experiments were conducted for each option of the system of equations.

3. Chua system of equations (16) was studied with the following parameters: a = 1.1428, b = 0.7142, c = 0.4285, d = -1, e = -0.7. Figure 4 shows phase portraits for two sets of initial conditions from the variety of results obtained. One positive Lyapunov exponent was obtained for one of them, in the second case all are negative (more details in Table 1).

Sets of initial conditions for solving the system of Chua equations (16):

1) $x_0 = 0.21; y_0 = -0.54; z_0 = 0.85$;

2) $x_0 = 0.74; y_0 = -0.95; z_0 = 0.73$.

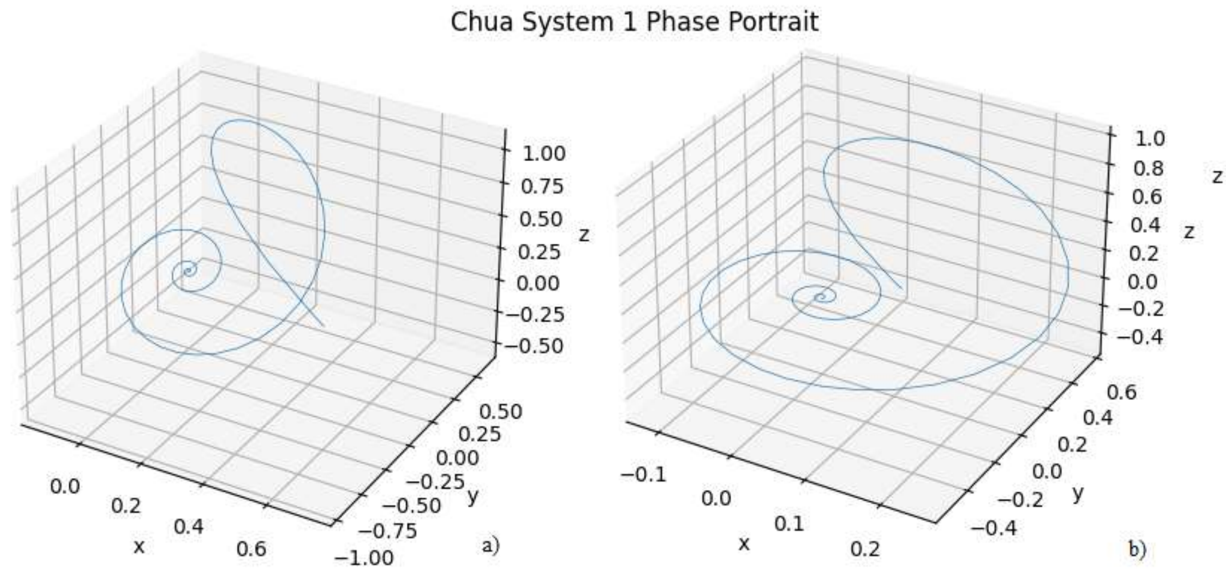

Fig. 4: Phase portraits of the Chua 1 system: a) under initial conditions: $x_0 = 0.21; y_0 = -0.54; z_0 = 0.85$; b) under initial conditions: an $x_0 = 0.74; y_0 = -0.95; z_0 = 0.73$

Both phase portraits shown in Figure 4 have a complex and non-periodic structure. We observe a high sensitivity to the initial conditions: the "butterfly effect", which indicates that the hypothesis of the chaotic nature of the system under study is not rejected at this research stage. However, the values of the Lyapunov exponents listed in Table 1 should be considered.

Table 1. The value of the Lyapunov exponent for the Chua 1 system

Initial conditions			Lyapunov exponents		
x_0	y_0	z_0	λ_1	λ_2	λ_3
0.21	-0.54	0.85	**0.0039**	-2.05	-0.42
0.74	-0.95	0.73	-1.47	-0.37	-0.14
0.40	-0.33	0.35	-0.23	-1.11	-0.24
0.76	-0.16	0.12	-0.38	-0.86	-1.08
0.99	-0.94	0.99	-0.95	-0.39	-0.17

According to Table 2, only one of the considered sets of initial conditions has a positive Lyapunov exponent. Therefore, it can be argued that the system shows high sensitivity to initial conditions and manifests chaotic properties.

Table 1. The value of the Lyapunov exponent for the Chua 2 system

Initial conditions			Lyapunov exponents		
x_0	y_0	z_0	λ_1	λ_2	λ_3
-0.27	0.18	-0.05	**0.0511**	0	-10.506
0.48	-0.35	0.47	**0.2723**	-1.2711	-17.201
-0.9	0.45	0.95	**0.5434**	-1.2303	-25.912
-0.19	0.97	-0.65	**1.5629**	-2.0178	-29.977
0.65	-0.81	-0.8	-0.4492	-0.4305	-27.765

The analysis of the phase portraits and the Lyapunov exponents of the system of Chua equations (16) gives grounds to summarize: as the phase portrait shows chaotic properties, it can be stated that the system is chaotic, regardless of the sign of the Lyapunov exponent. It can also be noted that the studied system shows complex and unpredictable dynamics, which are inherent in chaotic systems.

4. The studied modification of the Chua system of equations (18) considered the parameters: a = 15.6, b = 28, c_1 = -1, c_2 = -0.7, r = 1.2, m_0 = -8/7, m_1 = -5/7. For example, Figure 5 shows phase portraits for two sets of initial conditions:

1) $x_0 = -0.272; y_0 = 0.177; z_0 = -0.05$;
2) $x_0 = -0.189; y_0 = 0.969; z_0 = -0.653$.

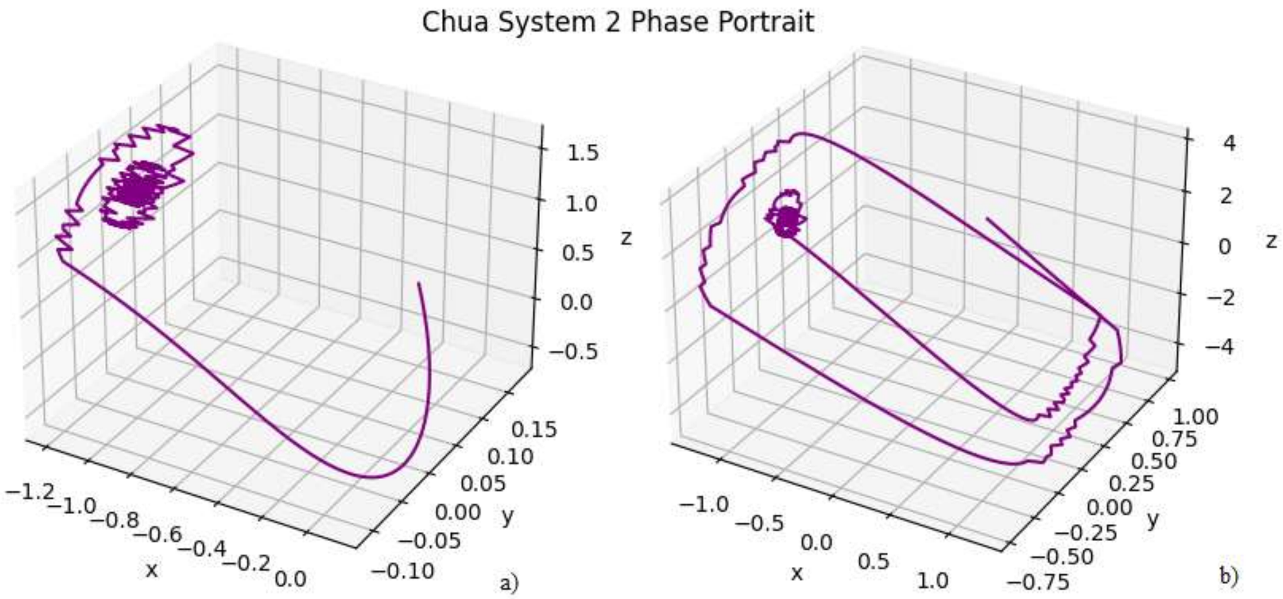

Fig. 5: Phase portraits of the Chua 2 system: a) under initial conditions: $x_0 = -0.272; y_0 = 0.177; z_0 = -0.05$; b) under initial conditions: $x_0 = -0.189; y_0 = 0.969; z_0 = -0.653$

As the phase portraits in Figure 5 show, trajectories of the system are non-repetitive and show unpredictable dynamics, being also significantly dependent on the initial conditions. The value of the Lyapunov exponent for some randomly selected sets of initial conditions listed in Table 2 was considered to confirm the hypothesis about the chaotic nature of the system.

As Table 2 shows, the array of Lyapunov exponent for each experiment has one positive value. Therefore, the obtained results indicate the chaotic nature of the Chua system (18).

3.3 The studied third modification of the Chua system of equations (18) with the current control equation is considered for the parameter values: $\alpha = 15.6, \beta = 28, \gamma = -1, \sigma = 0.1$. For example, Figure 6 and Figure 7 demonstrate phase portraits and dot plots for two sets of initial conditions:

1) $x_0 = 0.048; y_0 = 0.0319; z_0 = 0.6017; A_0 = 0.83374$

2) $x_0 = 0.1; y_0 = 0.08; z_0 = 1; A_0 = 1.5$.

The phase portrait illustrated in Figure 6a shows more complex dynamics with many starting points, the boundaries on the phase portrait are more blurred. However, there are several shutters, which may indicate invariant areas or symmetries in the system.These images (Figure 6 and Figure 7) may indicate complex chaotic dynamics in the system. It should also be noted that there is a striking difference between the phase portraits in Figures 6 and Figure 7 with a slight change in the initial conditions, which confirms the hypothesis of high chaotic system. This may indicate an invariant plane or other symmetry in the system. The image has sharp boundaries, indicating certain areas in the phase space with different dynamics. Table 3 shows the values of Lyapunov exponents for the studied system of equations with several sets of initial conditions.

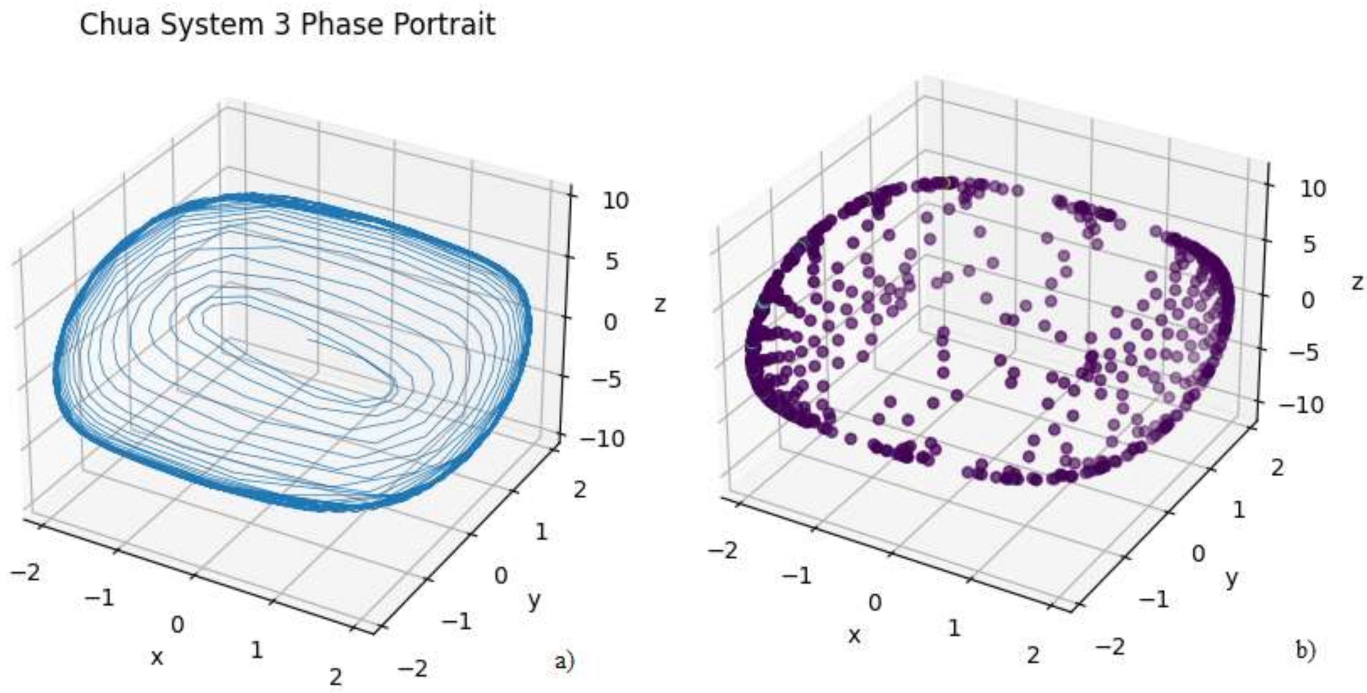

Fig. 6: a) Phase portrait of the Chua 3 system under initial conditions: $x_0 = 0.1; y_0 = 0.08; z_0 = 0.1; A_0 = 1.5$; b) Phase portrait in the form of a scatter plot.

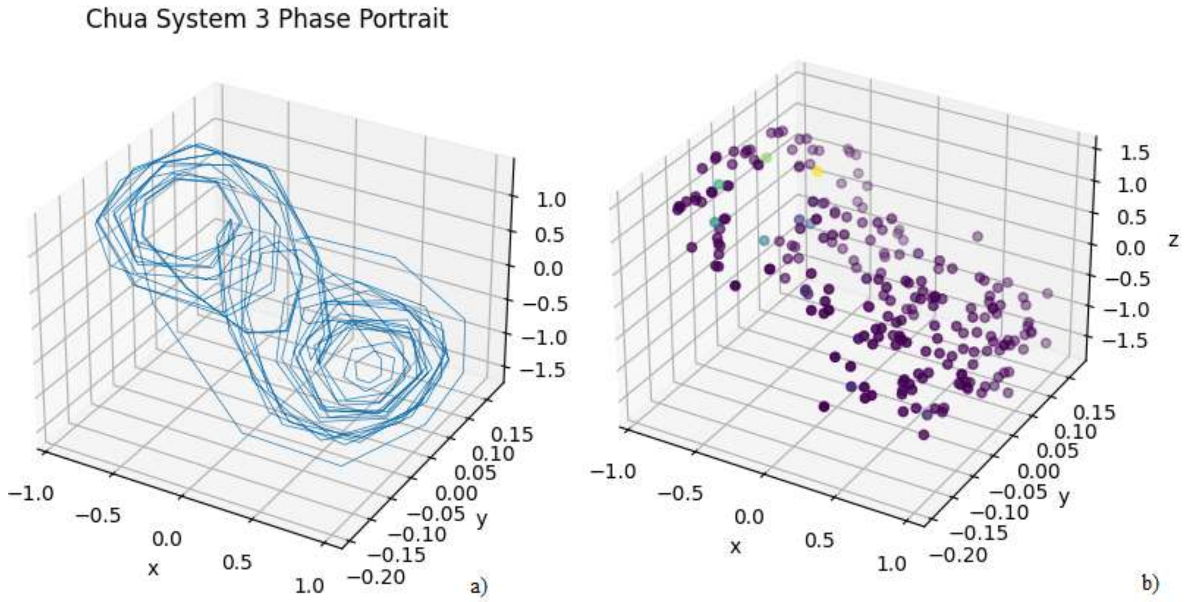

Fig. 7: a) a shutter centered at the point (0,0) dividing the phase portrait into two symmetrical parts; b) a chaotic placement of points, without any patterns.

Table 3. Values of Lyapunov exponents for the Chua 3 system

Initial conditions				Lyapunov exponents			
x_0	y_0	z_0	A_0	λ_1	λ_2	λ_3	λ_4
0.048	0.0319	0.6017	0.8337	**0.0537**	-0.0675	-2.4923	-0.1005
0.4	0.02	0.59	0.82	**0.0492**	-0.0621	-2.4929	-0.1092
0.03	0.01	0.58	0.81	**0.0455**	-0.0569	-2.4935	-0.1167
0.02	0.001	0.4	0.7	**0.0035**	-0.0044	-0.3063	-0.0049
-0.01	-0.02	0.2	0.5	-0.0239	**0.0301**	-2.4944	-0.0671
0.1	0.08	1	1.5	**0.0645**	-0.0826	-2.4915	-0.1367

There is one positive exponent in each set, therefore, it can be concluded, based on the analysis of the phase portraits above that the system is chaotic.

5 Discussion

Table 4 compares the noise level (%) of the studied systems based on the Chua equation system with the classical Lorentz and Rössler chaotic systems.

Table 2. The value of the coefficient of variation, % for the solutions of the considered systems

System of equations	x	y	z	A
Lorentz	424,22	450,67	31,36	
Rössler	2424,35	1638,74	578,50	
Chua 1	264,50	1816,90	315,99	
Chua 2	22,38	602,55	45,23	
Chua 3	19437,73	15329,24	46345,01	690,42

So, Chua systems of equations 1 and 2 have a noise level similar to the system of Lorenz equations and are inferior to the system of Rössler equations. Instead, the modified Chua 3 system of equations far surpasses the results of other systems and can be used as a chaos generator.

Unfortunately, a recent study that considered a chaos generator based on the classical Chua equation, does not explain how they estimated the noise level, so a comparison cannot be made, [11].

Many authors studied chaos generators. For example, the paper, [12], studied a new system of chaotic generation: a circulating chaotic system. The researchers studied the dynamics of the system and showed that it has some important characteristics of chaotic systems, such as sensitivity to the initial conditions and randomness, and also showed that the system has many stable attractors and can have a wide variety of behaviours depending on the parameters of the system. In, [13] the authors proposed an electro-optical source of chaos based on the theory of converting phase modulation into intensity modulation and an analog-digital hybrid time-delay feedback loop.

However, the results presented in this work were obtained in a simpler way and can be practically implemented.

6 Conclusions

The growing importance of chaotic systems in various fields of science and technology tasks scientists to analyze the structure of chaotic solutions. The research carried out in this work is focused on the analysis of the structure of chaotic solutions of several differential equations.

The results of the study give grounds to draw the following conclusions. All the studied differential equations have chaotic solutions with a highly complex structure. It was found that the structure of chaos in these solutions depends on the initial conditions and parameters of the equation. It was shown that analysing the structure of chaotic solutions is an important step in understanding the behaviour of chaotic systems.

The significance of the research is that the analysis of chaotic solutions of differential equations allows a deeper understanding of the structure of complex systems and their behaviour in various conditions. This can find practical applications in various sciences and fields, from theoretical physics and mathematics to biology and engineering.

Our main contribution is developing a modified Chua equation system that includes new parameters and current control equations. This modification revealed high chaoticity and complex dynamics, distinguishing it from the classical system of Chua and others. The modified system of Chua's equations can be applied to solve various applied problems, such as cryptography, pseudorandom number generation, steganography, and communication systems. Due to its chaos and complex dynamics properties, this modified system opens up new possibilities for creating reliable and efficient algorithms in these areas.

An important direction of future research is extending our analysis to other classes of chaotic systems and determining their specific features. It is recommended to discuss the advantages and limitations of using our analysis, particularly in the context of various applied problems.

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