Conceptual Bases of Gray Transformations and Discrete Vilenkin-Crestenson Functions Systems

ANATOLY BELETSKY
Department of Electronics,
National Aviation University,
Kyiv-03058, av. Cosmonaut Komarov, 1,
UKRAINE

Abstract: - The article is devoted to the theoretical foundations of the construction of generalized Gray codes. These include the classical "left-side" and the proposed "right-side" Gray's codes. Left-side codes develop from left to right and right-side codes from right to left. Left- and right-handed Gray's codes augmented with reverse permutation operators form a subset of composite Gray's codes. Such codes have found constructive application in solving synthesis and analysis problems of discrete systems of Vilenkin-Crestenson functions (VCF). Particular cases of VCF systems are systems of discrete exponential functions and Walsh functions. The so-called indicator matrices (IM), bijective connected to the VCF systems, form the basis for VCF systems synthesis. The order of IMs is determined by the logarithmic dependence on the order of the VCF system. Unique Walsh-Cooley function systems, the only ones in the set of Walsh function systems that provide linear connectivity of the frequency scales of DFT processors, are developed. Examples of trees of VCF systems give. The orders of IMs are related by logarithmic dependence with the orders of VCF systems. Using IM: (1) estimates of the number of symmetric VCF systems in an unbounded range of changes of their parameters are obtained, (2) structures are determined and (3) rules of the interconnection of the systems established. The fundamental axioms and lemmas corresponding to the VCF systems formulating and directions for further research are outlined.

Key-Words: - Generalized Gray Codes, Vilenkin-Crestenson Function Systems, VCF Indicator Matrixes

Received: March 29, 2022. Revised: January 4, 2023. Accepted: February 7, 2023. Published: March 7, 2023.

1 Introduction

Gray's transformations [1, 2] are considered a generalization of Gray's codes in this paper. The emergence of such codes met the requirements of engineering practice in constructing "angle-code" transducers, providing minimal errors [3, 4]. At the beginning of their appearance, Gray's codes attracted the attention of mathematicians and a wide range of developers of different electronic equipment. A distinctive feature of classical Gray's codes is that in binary space when moving from one number image to the next oldest or lower number, there is a change of digits (1 to 0 or vice versa) only in one digit of the number. Such codes are Hamming unit distance codes [4]. Gray's code is not the only one in this group, as shown in Table 1 on the example of three-digit binary codes.

Table 1. Primary sequences of Gray codes

Primary sequences						Secondary sequences					
1	2	3	4	5	6	7	8	9	10	11	12
000	000	000	000	000	000	000	000	000	000	000	000
001	100	100	001	010	010	100	001	010	010	001	100
011	110	101	101	110	011	101	101	110	011	011	110
010	010	001	100	100	001	111	111	111	111	111	111
110	011	011	110	101	101	110	011	011	110	101	101
111	111	111	111	111	111	010	010	001	100	100	001
101	101	110	011	011	110	011	110	101	101	110	011
100	001	010	010	001	100	001	100	100	001	010	010

Let us pay attention to the properties of the code sequences in the above tables. First, sequences 2-6 in Table 1 produce all possible permutations of the digits of column 1. The number of such permutations is six. Secondly, sequences 7-12 are composed of the corresponding series of 1-6 because of column inversion if the position of the upper zero code remains unchanged. Moreover, thirty, the Hamming distance between the first and last codes of each sequence in the Tables remains equal to one. Let us name the code sequences having the listed properties, closed Gray arrangements.

The application of Gray codes in communication techniques, analog-to-digital conversions, and other areas has proved to be preferable for several reasons, among which we note the following. Firstly, the change of digit values from one code combination to another is twice as rare as in a simple code, and this property allows for higher coding accuracy at the same speed as simple codes [5, 6]. Also secondly, when adding two adjacent code combinations by module 2, a number containing only one unit is formed, regardless of the number of bits of the source code sequence, which allows you to build an effective system of integrity control of the accepted combinations [7, 8].

2 Problem Formulation

In the second half of the previous century, with the rapid development of computer facilities and, on their basis, - digital processing of discrete signals, the problem of the development of new discrete Fourier transforms (DFT) bases. The main requirement to such grounds is to provide in the frequency domain the speed of processing of broadband signals with the much higher rate achieved by processors in the classical basis of discrete exponential functions (DEF). Naturally, we were considering using bases from the Walsh family of systems. Such bases potentially provide a speed of calculating Fourier coefficients an order of magnitude faster than the speed delivered by the basis of DEF. The reason for such a phenomenon is that Walsh's fundamental basis excludes complex multiplication operations accompanying the DEF basis, the implementation of which requires significant machine resources.

However, Walsh bases have at least two significant disadvantages. The first one is that, because of Walsh bases realism, pairs of Fourier coefficients of the same amplitude but different phase signs formed at the output of the DFT processor in the set of spectral components of quadrature signals (and such are the majority of actually processed signals). A noted feature of a spectrum of messages based on Walsh functions leads to ambiguity of the solution when the number of the output channel of the processor with the maximum amplitude response is calculated with the same phase (positive or negative).

Further, none of the known (let us call them classical) Walsh bases, ordered by Hadamard [9], Kachmarz [10], and Paley [11], provides linear connectivity of frequency scales of DFT processors, the essence of which is as follows.

Gray codes and VCF systems, including Walsh functions, are widely represented in numerous monographs, journal articles, conference papers, and the Internet [12-20]. In [21, 22], a rather extensive bibliography of applications of Gray's codes and VCF systems in various fields of science and technology give**.**

Let us name the abscissa of the rectangular coordinate system, on which we will store normalized frequencies of the input complex-exponential signal, the input frequency scale of the DFT. The axis of ordinates, designed to accommodate the channel numbers of the processor, in which responses with the maximum amplitude and selected phase forms, are called the output frequency scale of the processor.

Suppose that for a discrete complex-exponential signal, the integer normalized frequency equals m. Let $\{\omega(k,t)\}$ be the basis of the processor, where k — the number (order) of the basis function coincides with the amount of the basis matrix string and the function argument (normalized discrete-time). If the number k of the output channel of the processor in which the response with the maximum amplitude formed coincides with m, as shown in Fig. 1, it means that the basis delivers $\{\omega(k,t)\}$ linear connectivity to the frequency scales of the DFT processor. For example, a processor with a DEF basis provides such frequency connectivity.

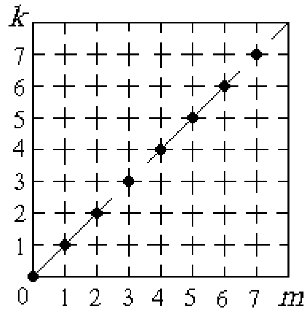

Fig. 1: The ratio of frequency scales of an eight-point DFT processor in the DEF base

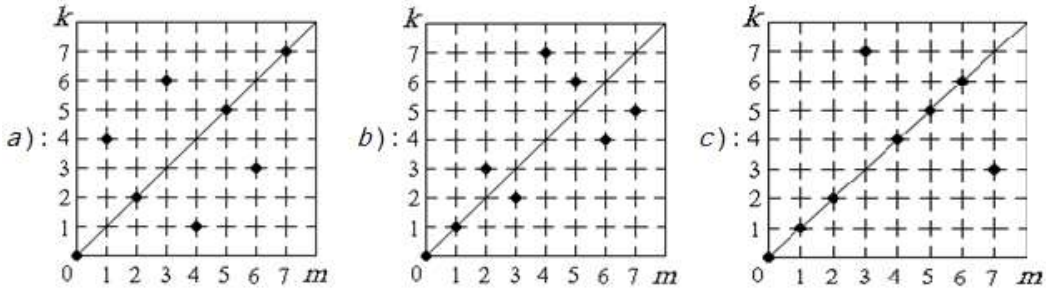

Fig. 2: The ratio of frequency scales of DFT processors in bases:
a) Walsh-Hadamard ($\boldsymbol{H}$); b) Walsh-Kachmazh ($\boldsymbol{W}$); c) Walsh-Paley ($\boldsymbol{P}$)

The graphs illustrate the interrelationship of frequency scales of eight-point DFT processors (far from linear) in the above-mentioned classical Walsh bases for selecting responses with positive phases in Fig. 2. As it follows from this figure, none of the known Walsh bases provide linear connectivity to the frequency scales of FFT processors, which can lead, for example, to a decrease in the efficiency of frequency meters built based on these Walsh bases.

The primary purpose of this study is to develop the theoretical basis of Gray transformations and solve the problem of synthesis of Vilenkin-Crestenson functions in discrete systems.

3 Problem Solution

Let us note one crucial feature of the Gray codes. Both mathematicians and developers of equipment appeared out of sight of the possibility of constructing the codes inverted in the direct formation of classical Gray codes. In the known (conventional) scheme, the creation of forwarding and reverse codes develops from left to right. At the same time, the number transformed upper (left) digit remains unchanged in the case of direct and inverse transformations.

Let's denote the bits of the number $\boldsymbol{x}$ represented in the position code by $x_{n-1}, x_{n-2}, ..., x_1, x_0$ and the Gray code $\boldsymbol{y}$ by $y_{n-1}, y_{n-2}, ..., y_1, y_0$, where n — is the number of bits in the code vectors $\boldsymbol{x}$ and $\boldsymbol{y}$. For the binary numbering system, the rule of direct transformation of the vector $\boldsymbol{x}$ into $\boldsymbol{y}$ a simple one is quite simple and looks like this:

$$y_{n-1-i} = x_{n-i} \overset{2}{\oplus} x_{n-1-i}, \quad i = \overline{0,\ n-1}, \quad x_n = 0, \tag{1}$$

and in the reverse transformation

$$x_{n-1-i} = x_{n-i} \overset{2}{\oplus} y_{n-1-i}, \quad i = \overline{0,\ n-1}, \quad x_n = 0, \tag{2}$$

At the same time, it is possible to construct a transformation scheme in the opposite direction of Gray's classical (*left-side*) transformation. In this class of transformations, called *right-hand*, the values of the lower bits of the numbers to be transformed remain unchanged. The system of equations, providing the right-side conversion of Gray is reparations:

$$y_i = x_i \overset{2}{\oplus} x_{i-1}, \quad i = \overline{0,\ n-1}, \ x_{-1} = 0, \tag{3}$$

– for direct right-side conversion and

$$x_i = y_i \overset{2}{\oplus} x_{i-1}, \quad i = \overline{0,\ n-1}, \quad x_{-1} = 0, \tag{4}$$

– to reverse the right-side transformation of Gray.

It is advisable to lay out the material based on code transformations, relying on structural-logic schemes of code formation. This approach to explaining the essence of algorithms is convenient because it makes the material more understandable for engineers and significantly simplifies the task of the formal mathematical description of the coding procedure. For illustration in Figs. 3 and 4 (the example of four-bit codes), the schemes corresponding to left-side transformations (1) and (2) result, and in Figs. 5 and 6 – for the corresponding (3) and (4) right-side changes.

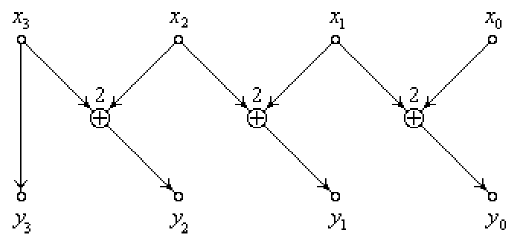

Fig. 3: Block diagram of the straight left-side Gray's transform

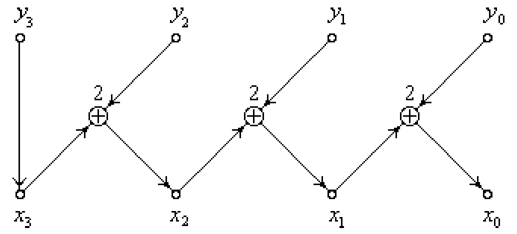

Fig. 4: Block diagram of the inverse left-side Gray's transform

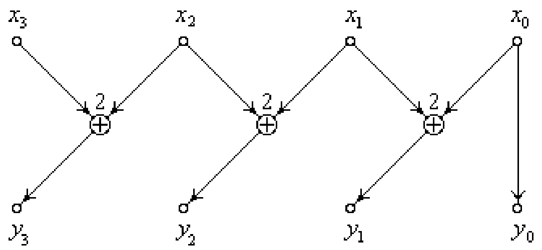

Fig. 5: Block diagram of the straight right-side Gray's transform

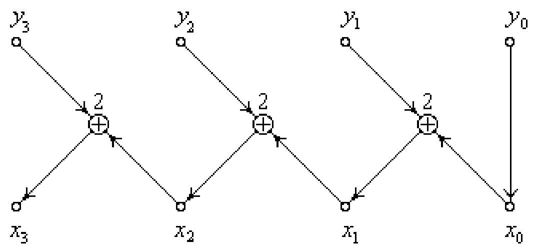

Fig. 6: Block diagram of the reverse right-side Gray's transform

The following is the principal difference between Gray's classical (left-side) transformation and the alternative (right-side) change. Suppose in the left-side Gray transformation the Hamming's distance d between any adjacent pairs (y_i, y_{i+1}), $i=\overline{0, n-1}$; the code combinations are equal to 1 (including the couple formed by the first and the last codes of the closed sequence (i.e., $d(y_0, y_{n-1})=1$), for the right-side transformation. Table 2 shows that this equality is observed only for $i=n/2-1$ and $i=n-1$.

Table 2. Distances of adjacent right-side Gray Codes Hemmings

i	$\boldsymbol{x}$	$\boldsymbol{y}$	$d(y_i, y_{i+1})$	i	$\boldsymbol{x}$	$\boldsymbol{y}$	$d(y_i, y_{i+1})$
0	000	000	2	4	100	100	2
1	001	011	2	5	101	111	2
2	010	110	2	6	110	010	2
3	011	101	1	7	111	001	1

The combination of left- and right-side Gray transformations (both forward and reverse) together with the inverse permutation operator (the "exchange" matrix [12]) led to the possibility of constructing combined or compound Gray codes (CGC). The use of CGC was very successful in determining the structure and interrelation of Walsh's symmetric function systems [13], Vilenkin-Crestenson discrete functions (WCF) [14-17] in cryptography [18, 19], coding [20], and other applications [21].

Walsh's function systems and the DEF systems are a case of discrete Vilenkin-Crestenson systems functions (VCF), as shown in Fig. 7 [13, 19, 20].

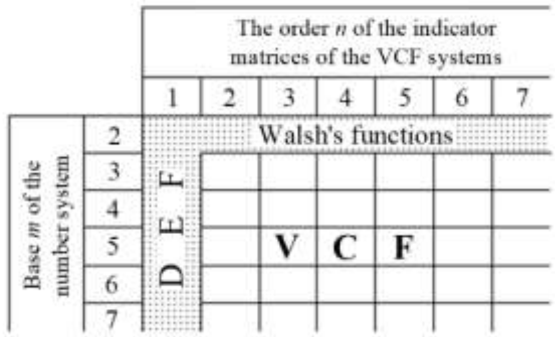

Fig. 7: Areas of identification of VCF systems and their particular case

VCF systems are square complex-valued matrices (except for Walsh systems, which are real matrices). The function determines the order of VCF systems

$$N = m^n,$$

where m — the base of the number system, and n — is the order of the indicator matrices of the VCF systems.

Enter a designation $\boldsymbol{V}_{m,n}$ for a particular system of VCF with parameters m, n and let it be $L_{m,n}$ — the number of symmetrical designs $\boldsymbol{V}_{m,n}$. The definition of an indicator matrix (IM) for the VCF system will introduce below.

The number of symmetric systems $L_{m,1}$ of DEF coincides with the number mutually simple with m:

$$L_{m,1} = \varphi(m),$$

where $\varphi(m)$ is the Euler function.

The exact estimate of the number of symmetric systems of Walsh $L_{2,n}$ functions was first obtained in [21] and looked like it:

$$L_{2,n} = \prod_{i=1}^{n} (2^n - n(\mathrm{mod}\, 2)). \quad (5)$$

Assessment $L_{2,n}$, calculated using formula (5) for several values n, is shown in Table 3.

Table 3 − Number of Walsh symmetrical function systems

n	1	2	3	4	5	6	7	8
N	2	4	8	1	32	64	128	256
$L_{2,n}$	1	4	2	4	13'	888'	112'88	28'897'7

The paper [15] notes that the classical systems of Walsh functions are interconnected by the codes (transformations) of Gray, as shown in Fig. 8, in which it marked: GC (RGC) — the direct (reverse) code of Gray, BIP — binary-inverse permutation.

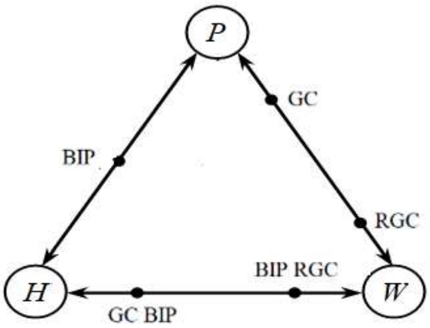

Fig. 8: The interrelation of numbers of essential functions in Walsh's various classic systems

The complex significance of the VCF matrices can avoid by using an isomorphic representation of the phase multipliers $W = \exp\{-j2\pi / N\}$. An isomorphism establishes by a simple correspondence $\{W^k\} \leftrightarrow \{k\}$. For example, relation (6) illustrates the image of an eighth-order Walsh-Paley matrix.

$$\boldsymbol{P}_8 = \{p(k,t)\} = \begin{array}{c c} & \begin{array}{cccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \quad t \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ k \end{array} & \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} \end{array} . \quad (6)$$

The transition from the complex elements of VCF systems to their images entails replacing the bitwise multiplication operations with bitwise addition operations modulo m.

Following the structural-logic schemes presented in Fig. 1, the composition of the system linear modular equations four-point direct (7) and inverse (8) left-side Gray transformations

Finally, the number of IMs is the same as that of VCF systems, meaning each VCF system

$$\begin{aligned} y_3 &= x_3; \\ y_2 &= x_3 + x_2; \\ y_1 &= \quad\; x_2 + x_1; \\ y_0 &= \qquad\quad x_1 + x_0; \end{aligned} \qquad (7)$$

$$\begin{aligned} x_{3,} &= y_3; \\ x_2 &= y_3 + y_2; \\ x_1 &= y_3 + y_2 + y_1; \\ x_0 &= y_3 + y_2 + y_1 + y_0, \end{aligned} \qquad (8)$$

whose matrix shapes look like

$$\boldsymbol{Y} = \boldsymbol{X} \cdot 2 = (x_3, x_2, x_1, x_0) \cdot \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}; \qquad (9)$$

$$\boldsymbol{X} = \boldsymbol{Y} \cdot 3 = (y_3, y_2, y_1, y_0) \cdot \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad (10)$$

where 2 and 3 are the symbols of forward and reverse Gray left-side transformation operators. Indicator matrices correspond to systems of Walsh functions of order 16.

Note that in matrix products (9) and (10), the data vector line multiplies by the matrices on the left. Multiplying the matrix by the vector column on the right is accepted. That explains why this multiplication method of matrix operations 2 and 3 can be formed by the direct transformation of single matrix lines $\boldsymbol{E}$, the Walsh-Paley system IM ($\boldsymbol{P}$), according to the scheme shown in Fig. 3.

The VCF indicator matrices have some remarkable properties. Firstly, the IMs are right-side symmetric (i.e., symmetric to the auxiliary diagonal matrices). Call square form, symmetrical to the main diagonal, left-side symmetric matrices. Second, the IMs are unborn matrices. Let us clarify the concept of nondegenerate VCF modular IM systems. Let us denote $\boldsymbol{J}$, the indicator matrix of some system of the VCF $\boldsymbol{V}_{m,n}$ with arbitrary parameters m, n. The non-renewal of the IM $\boldsymbol{J}$ system means that the determinant $\boldsymbol{J}$ by module m is mutually simple, with m and independent of n.

corresponds to only one IM. The reverse is true: IM unequivocally explains its only relevant VCF

Let us introduce some constructive principles for the theory of discrete systems of the VCF. The proof can either be omitted because of the evidence or through illustrative examples, confirming the rules' validity. For engineering applications, such a simplification, which does not distort the results, is quite acceptable because it equips the application developer with the necessary working tools, relieving him from the need to "wander in the mathematical labyrinths".

Based on the above properties of the IM, we formulate (as a theorem) the definition of these matrices.

system.

Theorem: Indicator matrices of discrete VCF systems $\boldsymbol{V}_{m,n}$ of the order $N = m^n$, the elements of which are non-negative integers $v_{i,j}$, belonging to the set Z_m, are square right-side symmetric matrices $\boldsymbol{J}_v$ of the order n (necessary conditions), not born in the ring of deductions on the module (sufficient conditions).

The theorem conditions make it possible to merely calculate the number $L_{m,n}$ of symmetric systems $\boldsymbol{V}_{m,n}$ on computers. The calculation results show in Table 4.

Table 4. Estimates of the number of symmetric systems of the VCF

m	Parameter value n				
	1	2	3	4	5
2	1	4	28	448	13.888
3	2	18	468	37.908	9.173.736
4	2	32	1.792	458.752	455.081.984
5	4	100	12.400	7.750.000	24.211.000.000
6	2	72	13.104	16.982.784	127.404.845.568
7	6	294	100.548	241.415.748	4.057.233.060.888
8	4	256	114.688	469.762.048	14.912.126.451.71
9	6	486	341.172	2.238.429.492	131.633.084.706.5
10	4	400	347.200	3.472.000.000	336.242.368.000.0
11	10	1.210	1.609.30	23.560.152.000	–
12	4	576	838.656	17.390.370.816	–
13	12	2.028	4.453.48	127.196.070.76	–
14	6	1.176	2.815.37	108.154.255.10	–
15	8	1.800	5.803.20	293.787.000.00	–

The estimates derived from the simulation are located in columns 1-3 of Table 4 and columns 4 and 5 above the double lines. Based on the data of Table 4, analytical estimates of number $L_{m,n}$ were empirically obtained, namely [1, 2]:

- If m — is a simple number, then:

$$L_{m,n} = \varphi(m)\prod_{i=1}^{n}(2^n - n(\mathrm{mod}\,2)),$$

where $\varphi(m)$ — Euler function.

- If m — is the $k-$degree of a simple number m_p, (i.e., $m = m_p^k$), then:

$$L_{m,n} = L_{m_p^k,n} m_p^{n(n+1)/2}.$$

- If m — is the composite number is such that $m = \prod_{i=1}^{l} m_i^{k_i}$, in which m_i — a simple number, then:

$$L_{m,n} = \varphi(m) \cdot \prod_{i=1}^{n} L_{m_p^{k_i},n}.$$

The aggregate of 28 third-order IMs, corresponding to the set of Walsh's eighth-order function systems, is shown in Table 5. Fig. 9 shows the graph of the interconnection of Walsh systems eighth-order containing all 28 indicator matrices [1, 2, 19, 20].

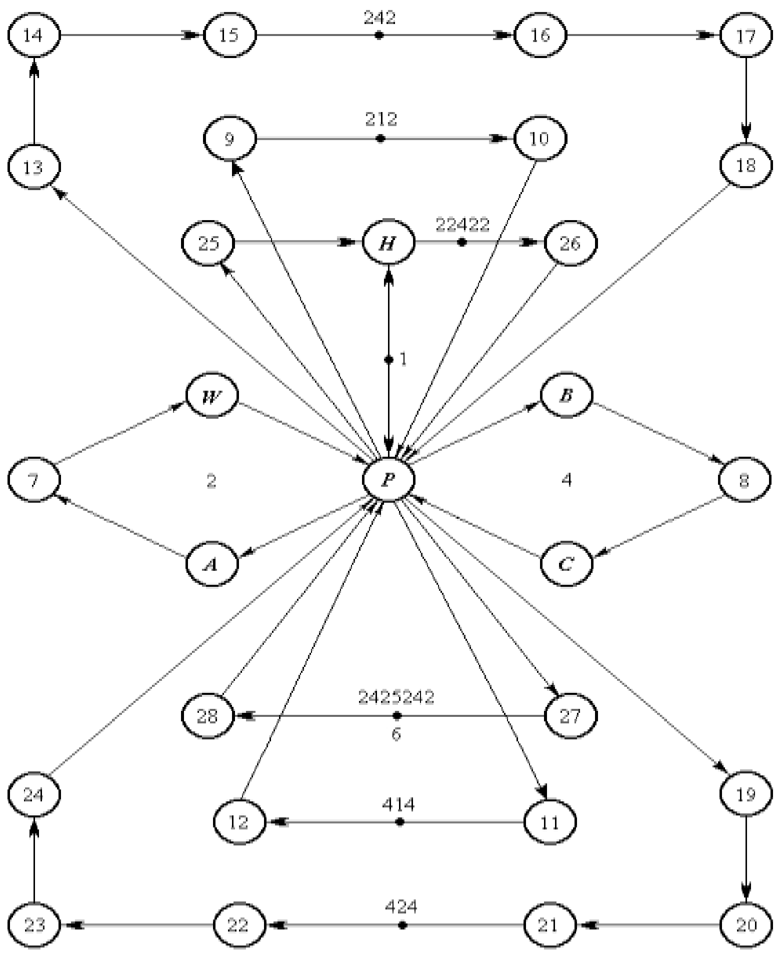

Fig. 9: The total count of Paley's eighth-order related function system Walsh

Table 5. Indicator matrixes of Walsh eighth-order function systems

$\boldsymbol{P}=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$	$\boldsymbol{A}=\begin{bmatrix}1&1&0\\0&1&1\\0&0&1\end{bmatrix}$	$\boldsymbol{B}=\begin{bmatrix}1&0&0\\1&1&0\\0&1&1\end{bmatrix}$	$\boldsymbol{H}=\begin{bmatrix}0&0&1\\0&1&0\\1&0&0\end{bmatrix}$	$\boldsymbol{W}=\begin{bmatrix}1&1&1\\0&1&1\\0&0&1\end{bmatrix}$	$\boldsymbol{C}=\begin{bmatrix}1&0&0\\1&1&0\\1&1&1\end{bmatrix}$
$7=\begin{bmatrix}1&0&1\\0&1&0\\0&0&1\end{bmatrix}$	$8=\begin{bmatrix}1&0&0\\0&1&0\\1&0&1\end{bmatrix}$	$9=\begin{bmatrix}0&1&0\\1&0&1\\1&1&0\end{bmatrix}$	$10=\begin{bmatrix}1&0&1\\1&0&0\\1&1&1\end{bmatrix}$	$11=\begin{bmatrix}0&1&1\\1&0&1\\0&1&0\end{bmatrix}$	$12=\begin{bmatrix}1&1&1\\0&0&1\\1&0&1\end{bmatrix}$
$13=\begin{bmatrix}0&1&1\\1&1&1\\0&1&0\end{bmatrix}$	$14=\begin{bmatrix}1&0&1\\1&1&0\\1&1&1\end{bmatrix}$	$15=\begin{bmatrix}0&0&1\\1&0&0\\1&1&0\end{bmatrix}$	$16=\begin{bmatrix}0&1&0\\0&1&1\\1&0&0\end{bmatrix}$	$17=\begin{bmatrix}1&1&1\\1&0&1\\0&1&1\end{bmatrix}$	$18=\begin{bmatrix}1&1&0\\0&0&1\\1&0&1\end{bmatrix}$
$19=\begin{bmatrix}0&1&0\\1&1&1\\1&1&0\end{bmatrix}$	$20=\begin{bmatrix}1&1&1\\0&1&1\\1&0&1\end{bmatrix}$	$21=\begin{bmatrix}0&1&1\\0&0&1\\1&0&0\end{bmatrix}$	$22=\begin{bmatrix}0&0&1\\1&1&0\\0&1&0\end{bmatrix}$	$23=\begin{bmatrix}1&1&0\\1&0&1\\1&1&1\end{bmatrix}$	$24=\begin{bmatrix}1&0&1\\1&0&0\\0&1&1\end{bmatrix}$
$25=\begin{bmatrix}1&1&0\\1&1&1\\0&1&1\end{bmatrix}$	$26=\begin{bmatrix}0&1&1\\1&1&1\\1&1&0\end{bmatrix}$	$27=\begin{bmatrix}0&1&0\\0&0&1\\1&0&0\end{bmatrix}$	$28=\begin{bmatrix}0&0&1\\1&0&0\\0&1&0\end{bmatrix}$		

Table 6. Gray's elementary operators

Designation operator	Operation performed
0	Saving the original combination
1	Inverse transposition
2	Straight left-side transformation of Gray
3	Inverse left-side transformation of Gray
4	Straight right-side transformation of Gray
5	Inverse right-side transformation of Gray
6	Cyclic shift one digit to the right
7	Cyclic shift one digit to the left

Table 7. A subset of simple Walsh's eighth-order Gray system codes

$0=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$	$2=\begin{bmatrix}1&1&0\\0&1&1\\0&0&1\end{bmatrix}$	$4=\begin{bmatrix}1&0&0\\1&1&0\\0&1&1\end{bmatrix}$	$6=\begin{bmatrix}0&1&0\\0&0&1\\1&0&0\end{bmatrix}$
$1=\begin{bmatrix}0&0&1\\0&1&0\\1&0&0\end{bmatrix}$	$3=\begin{bmatrix}1&1&1\\0&1&1\\0&0&1\end{bmatrix}$	$5=\begin{bmatrix}1&0&0\\1&1&0\\1&1&1\end{bmatrix}$	$7=\begin{bmatrix}0&0&1\\1&0&0\\0&1&0\end{bmatrix}$

Large Latin letters in the nodes of the graph highlight the invariant fundamental Walsh systems, so named because the algorithm of their synthesis does not depend on the Walsh systems. Numbers in the nodes of the contours indicate the numbers of systems that do not belong to a subset of Walsh's invariant fundamental systems. The tree nodes are connected by directed edges, each assigned a weight equal to a compound or simple Gray's code. A full description of Gray's elementary operators is given in Table 6, and Table 7 presents their IMs. Fig. 10 shows, as an example, the system $\boldsymbol{V}_{3,2}$. The IMs elements P_1 and P_2 connect by the operator $\times 2$ (multiplication by 2 modulo 3).

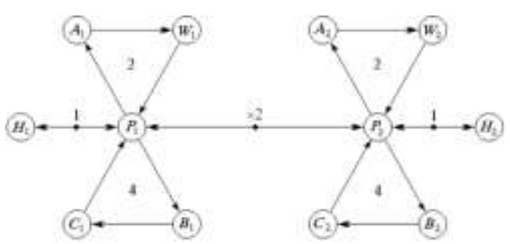

Fig. 10: VCF of the system $\boldsymbol{V}_{3,2}$

The generating systems in Fig. 10 are the VCF-matrices $\boldsymbol{H}_1$ and $\boldsymbol{H}_2$ the second Cronecker degree of DEF matrices $\boldsymbol{E}_1$ and $\boldsymbol{E}_2$:

$$\boldsymbol{E}_1=\begin{bmatrix}0&0&0\\0&1&2\\0&2&1\end{bmatrix},\quad \boldsymbol{E}_2=\begin{bmatrix}0&0&0\\0&2&1\\0&1&2\end{bmatrix}.$$

Statement 1. Indicator matrices $\boldsymbol{J}_v$ of Paley-linked systems of VCF $\boldsymbol{V}_{m,n}$ coincide with the Gray code (in general - composite), using which the transition from the VCF-Paley $\boldsymbol{P}_{m,n}^{(r)}$ to the system $\boldsymbol{V}_{m,n}$ carries out.

The upper index (if any) in $\boldsymbol{P}_{m,n}^{(r)}$ indicates the $r-$th matrix of VCF-Paley systems. For example, Fig. 11 shows the interconnection graph of systems $\boldsymbol{V}_{5,2}$. The IMs elements P_1 - P_4 connect by the operator $\times 4$ (multiplication by 4 modulo 5).

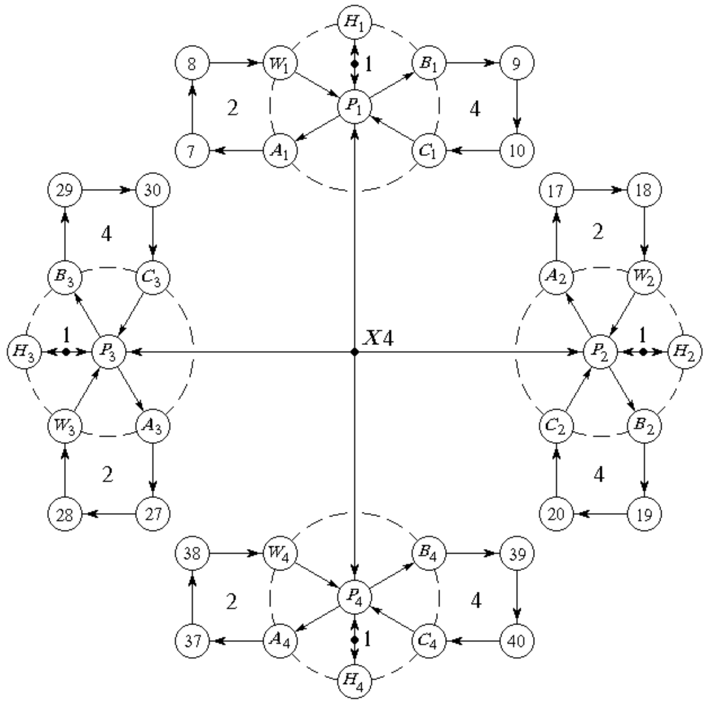

Fig. 11: The graph of the interconnection of the fundamental systems of the VCF $\boldsymbol{V}_{5,2}$

Statement 2. The numbers k_v of the functions system $\boldsymbol{V}_{m,n}$ are related to the numbers k_p of the VCF-Paley system $\boldsymbol{P}_{m,n}$ by

$$k_v = k_p \cdot \boldsymbol{J}_v . \qquad (11)$$

Using the ratio (11), let us show how we calculate, for example, the Walsh system of functions indicated in Fig. 9 with the symbol $\boldsymbol{C}$. Let us substitute the indicator matrix $\boldsymbol{C}$ from Table 5 to (11) and replace it with k_v to k_c,

$$k_c = \underbrace{(x_1, x_2, x_3)}_{k_p} \otimes \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}}_{\boldsymbol{J}_c}, \quad x_i \in \{0, 1\},$$

By rearranging the basic functions (matrix rows) of the Walsh-Paley system (8) according to the rule given in Table 8, we obtain the matrix named in [22, 23] by the Walsh-Cooley system. The basis for this name is that was first obtained this system by rotating the unit complex circle containing the phase multipliers of the Cooley-Tukey algorithm [24] counterclockwise. The Walsh-Cooley matrix of the eighth order is shown below as an example.

$$\boldsymbol{C}_8 = \begin{array}{ccc} & & \begin{array}{cccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \; t \\ \begin{array}{c} 0 \\ 3 \\ 6 \\ 5 \\ 4 \\ 7 \\ 2 \\ 1 \end{array} & \begin{array}{c} \to \\ \to \\ \to \\ \to \\ \to \\ \to \\ \to \\ \to \end{array} \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \end{array} & \begin{bmatrix} 0&0&0&0&0&0&0&0 \\ 0&0&1&1&1&1&0&0 \\ 0&1&1&0&0&1&1&0 \\ 0&1&0&1&1&0&1&0 \\ 0&1&0&1&0&1&0&1 \\ 0&1&1&0&1&0&0&1 \\ 0&0&1&1&0&0&1&1 \\ 0&0&0&0&1&1&1&1 \end{bmatrix} \\ k_p & k_c & \end{array} . \qquad (12)$$

Walsh-Cooley function systems have a unique property, which can formulate as follows. Among all the classical symmetric Walsh function systems used as the DFT basis, the only system that provides linear connectivity of the frequency scales of the DFT processor is the Walsh-Cooley function system. In addition, we will outline the essence of Walsh's forward and backward tasks. The convenience of using such bases, for example, in Spectro analyzers, is that one can make an unambiguous decision for the normalized integer frequency of the input signal using the number of the DFT processor output channel with the maximum response and negative phase.

In addition, we will outline the essence of Walsh's so-called forward and backward tasks.

The direct of Walsh's task is to calculate a matrix $\boldsymbol{W}$ of order $N = 2^n$ using a given indicator matrix $\boldsymbol{J}_w$.

The reverse of Walsh's task is to calculate the indicator matrix $\boldsymbol{J}_w$ for a given Walsh matrix $\boldsymbol{W}$.

The first is more straightforward — the reverse Walsh problem, in which the sequence of calculations illustrates Walsh-Cooley's system of functions of the eighth order (12).

1) Let us present the binary system of Walsh functions in a reduced form $\boldsymbol{W}_N^{(*)}$, including n basic functions $w(k,t)$ of the order $k = 2^i$, $i = \overline{0, n-1}$. For the system (12) we have

$$C_8^{(*)} = \begin{array}{c|c} & \begin{matrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & t \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 4 \\ k \end{matrix} & \begin{bmatrix} 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix} \end{array} . \qquad (13)$$

Moving from a shortened form of Walsh systems to a common structure is simple enough. The function of the zero-order restores trivially. The functions of order $k \neq 2^i$ unambiguously defines by a set of functions, the order of which is a power of two. I.e., the missing functions were composed of functions already available in the reduced matrices of Walsh systems.

2) The number of columns of the reduced matrix can reduce to the number of its rows if you select those and only those matrix columns whose numbers t, as well as the row numbers k, are the degree of two (i.e., for $t = 2^i$, $i = \overline{0, n-1}$). Because of reducing the columns of the reduced matrix (13), we come to the square matrix, which is called $\boldsymbol{J}^*$ the first native matrix of the Walsh functions system

$$\boldsymbol{J}^* = \begin{array}{c|c} & \begin{matrix} 1 & 2 & 4 & t \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 4 \\ k \end{matrix} & \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \end{array} . \qquad (14)$$

3) The matrix (14) is left-side symmetric, while the IMs are right-side symmetric. By inversion of matrix rows, which achieve by multiplying it on the left by the matrix (operator), we bring the original matrix to the right-side symmetric inverse permutation

$$\boldsymbol{J}^+ = 1 \cdot \boldsymbol{J}^* = \begin{array}{c|c} & \begin{matrix} 1 & 2 & 4 & t \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 4 \\ k \end{matrix} & \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \end{array} . \qquad (15)$$

4) The subsequent request of the matrix (15) will receive the Walsh-Cooley function system IMs $\boldsymbol{J}_c$ (12).

The solution to Walsh's straightforward task implies performing calculations in the inverse sequence of estimates for Walsh's inverse task. That means the fourth step performs first, then the third, and so on.

Let us formulate some fundamental statements that form the basis for constructing Gray's theory of transformations.

Axiom 1. Every simple g and arbitrary composite $\boldsymbol{G}$ binary Gray's code, given by indicator matrices of order n, forms elements (generators) of a multiplicative group. Among the composite codes $\boldsymbol{G}$, there is at least one symmetric CGC that generates a group of maximal order L_n defined by the relation $L_n = 2^n - 1$.

Axiom 2. The right-side transposition operator reverses quadratic matrices concerning the auxiliary diagonal.

Axiom 3. An arbitrary right-side symmetric matrix, multiplied from left or right by the operator 1, becomes left-side symmetric, and vice versa.

Axiom 4. An inverse permutation matrix 1 rearranges all the rows of an arbitrary square matrix $\boldsymbol{A}$ in reverse order if multiplied from the left by that matrix and rearranges the columns of matrix $\boldsymbol{A}$ in reverse order if multiplied from the right by matrix $\boldsymbol{A}$.

Axiom 5: Inverse permutation of columns and rows of a square matrix $\boldsymbol{A}$ is equivalent to a joint left- and right-side transposition of the matrix. From axioms 4 and 5 follow

Corollary: Since $g^* = g^T$ and, in addition, matrices g also are invariant to the right-side transposition operation $\perp$, then

$g^* = g^{T\perp} = 1 \cdot g \cdot 1$, as well as, $g = 1 \cdot g^* \cdot 1$. (16)

Inequalities (16) mean, in particular, that the matrices corresponding to the set of the original g and the conjugate g^* Gray operators are similar.

Lemma 1. The right-side CGC transposition is equivalent to the inversion of this code, which reduces to a rearrangement of the order of Gray's simple codes.

Proof. That is followed by the fact that any simple Gray's code is symmetric about the auxiliary diagonal and invariant to right-side transposition and thus $\left(g_1 \cdot g_2\right)^{\perp} = g_2{}^{\perp} \cdot g_1{}^{\perp} = g_2 \cdot g_1$.

Lemma 2. A symmetric CGC corresponds to a right-side symmetric transformation matrix.

Proof. By definition, a symmetric composite code is a code $\boldsymbol{G}_s = \boldsymbol{G} \cdot \omega \cdot \boldsymbol{G}^{\perp}$ in which $\boldsymbol{G}$ is an arbitrary CGC and ω is a kernel, a simple or symmetric Gray's mixed code. We have

$$\boldsymbol{G}_s^{\perp} = \left(\boldsymbol{G} \cdot \omega \cdot \boldsymbol{G}^{\perp}\right)^{\perp} = (\boldsymbol{G}^{\perp})^{\perp} \cdot \omega^{\perp} \cdot \boldsymbol{G}^{\perp} = $$
$$= \boldsymbol{G} \cdot \omega \cdot \boldsymbol{G}^{\perp} = \boldsymbol{G}_s .$$

Lemma 3. A composite Gray's code $\boldsymbol{G} \cdot \boldsymbol{G}^T$, where $\boldsymbol{G}$ is a right-symmetric matrix of arbitrary CGC, corresponds to a left-symmetric matrix.

Proof. From the fact that $\left(\boldsymbol{A} \cdot \boldsymbol{B}\right)^T = \boldsymbol{B}^T \cdot \boldsymbol{A}^T$, it follows that $\left(\boldsymbol{G} \cdot \boldsymbol{G}^T\right)^T = \left(\boldsymbol{G}^T\right)^T \cdot \boldsymbol{G}^T = \boldsymbol{G} \cdot \boldsymbol{G}^T$, which is true if and only if $\boldsymbol{G} \cdot \boldsymbol{G}^T$ is a left-side symmetric matrix.

Lemma 4. The right-symmetric matrix corresponds to a composite Gray's code $\boldsymbol{G} \cdot \boldsymbol{G}^{\perp}$, where $\boldsymbol{G}$ is a left-symmetric matrix of arbitrary CGC.

The proof of Lemma 4 is similar to the defense of Lemma 3. Namely, $\left(\boldsymbol{G} \cdot \boldsymbol{G}^{\perp}\right)^{\perp} = \left(\boldsymbol{G}^{\perp}\right)^{\perp} \cdot \boldsymbol{G}^{\perp} = \boldsymbol{G} \cdot \boldsymbol{G}^{\perp}$, which is true if and only if $\boldsymbol{G} \cdot \boldsymbol{G}^{\perp}$ is a right-side symmetric matrix.

Let us support the proof of Lemma 4 with a numerical example. Let $\boldsymbol{G} = 24$ be a left-symmetric CGC. We have $\boldsymbol{G}^{\perp} = (24)^{\perp} = 4^{\perp}2^{\perp} = 42$. Hence, composite Gray's code $\boldsymbol{G} \cdot \boldsymbol{G}^{\perp} = 2442$ is a symmetric code which, according to Lemma 2, corresponds to the right-side symmetric matrix.

Lemma 5. The generators of cyclic groups that form a tree of Paley-linked systems of Walsh functions are symmetric CGCs of exclusively odd length.

Proof. As follows from Fig. 9, all graph contours are formed either by simple Gray's codes of length 1 or by composite codes of length = 3, 5, and 7. Suppose a symmetric CGC $\boldsymbol{G} = gg^* \cdot g^* g$ of length $K = 4$, for example, code $\boldsymbol{G} = 2442$, is chosen as the generator of the group. According to the above indicator matrices of VCF systems introduced. The order n of IMs coincides with the exponent at the base of the number system m, which determines the order N of the VCF system, i.e., $N = m^n$. There is a Corollary, the two codes g^* can represent CGC $1g1$ and thus code $\boldsymbol{G}$ is reduced to a second-degree CGC of odd length since $\boldsymbol{G} = g1g\underset{0}{1 \cdot 1}g1g = (g1g)^2$.

Let us next consider a CGS with parameter $K = 6$. By direct computation, it is easy to see that, for example, $\boldsymbol{G} = g^* g^* \cdot gg \cdot g^* g^* = (g)^2$, whereas the code $253352 = 535$. Lemma 5 also confirmed for the trees of Paley-linked systems of Walsh functions with fourth-order IM and, most likely, for orders greater than four, so there is no objective reason why this should not be the case.

4 Discussion

We can point out at least one direction in which it is reasonable to develop further research. Its essence is to create Walsh-like sequential systems. We refer to such (0,1)-sequent functions as Walsh-like ones where the number of zeros and ones in each half of the determination interval is not necessarily equal, as is the case with representations of classical Walsh functions. The algorithm's simplicity for synthesizing systems of sequential functions and the high speed of spectral processing of discrete signals provided by such bases open up broad prospects for using Walsh-like methods in various fields of science and technology.

5 Conclusions

The problem of developing mathematical and algorithmic support for synthesizing discrete systems of Vilenkin-Crestenson functions, including systems of Walsh functions, is solved.

The scientific novelty of obtained results is as follows. First, the GGC proposed, in addition to the classical left-side codes, contains the so-called right-side codes. If in classical Gray's codes, the process of their formation develops in the direction from left to right, then in right-side codes — from right to left. Using GGC, the problem of construction of VCF systems is solved enough. Second, an isomorphic transformation proposes, employing not only the transition from complex-valued to integers of matrices of VCFs was possible but also the technology of synthesis of VCF systems substantially facilitated. Third, the so-called one-to-one correspondence between matrices of VCF systems and their IMs, which allows for replacement operations on VCF systems of large orders by processes on IMs of small orders. And fourth, the straight and inverse Walsh problems are formulated, through which the relationship between Walsh systems and their indicator matrices are established.

The practical significance of obtaining results is as follows. Original Walsh-Cooley function systems with unique properties have been developed. Bases based on Walsh-Cooley function systems are the only Walsh bases that provide linear coupling of frequency scales of DFT processors. The convenience of such bases, for example, in a Spectro analyzer, is that it can use the output channel number of the maximal response and negative phase DFT processor to make a one-valued decision for a normalized integer frequency of the input signal.

Prospects for further research lie in the development of Walsh-like functions in which the number of zeros and ones in each half of the definition interval need not be the same as with classical Walsh systems.

References:

[1] Beletsky A. Ya. *Gray code combinatory*, Kyiv: Pub. house «KVIC», 2003, 506 p.
[2] Beletsky A. Ya. *Gray transformations:* monograph in two vol., Vol 1: *Basic theory*, Kyiv: Pub. house «NAU», 2007. — 512 p; Vol 2: *Applied aspects*, Kyiv: Pub. house «NAU», 2007, 644 p.
[3] Gray F. *Pulse code communication*, Pat USA, № 2632058, 1953.
[4] Grechishnikov V. M., Butco A. D. *Diagnostics of optical and electrical channels of the optoelectronic angle converter*, Samara: Pub. House ANC RAN, v. 20, # 4, 2018, p.p. 138–143.
[5] Gytis E. I. *Information converters for elementary digital computing devices*, Moscow: Pub. House «Energy», 1970, 399 p.
[6] Hamming R. *Coding and Information Theory*, Prentice-Hall, 1980, ISBN 0-13-139139-9
[7] Korotayev B. B., Prokofiev A. V., Timofeev A. N., *Optoelectronic measuring transducers of linear and angular motion*, St. Petersburg: Research Institute of ITMO, 2012, 116 p.
[8] Domrachev V. A. *Digital angle converters: construction principles, accuracy theory, control methods*, Moscow: Energoizdat, 1984, 328 p.
[9] Hadamard M. J., *Buii. Sci. Math*, 1923, A17, p.p. 240-246.
[10] Walsh I. L., *Amer. J. Math*., 1923, 45, p.p 5-24.
[11] Paley B. E., *Proc. London Math. Soc*. (2), 1932, 34, p.p. 241-279.
[12] Savage C. A. *Survey of Combinatorial Gray Codes,* Wikipedia [online], Available at: https: //people.engr.ncsu.edu/savage/available for mailing/survey.pdf
[13] Pham D. M., Premkumar A. B., Madhukumar A. *Error Detection and Correction in Communication Channels Using Inverse Gray RSNS Codes*, IEEE Transactions on Communications, # 59(4), 2011, p.p. 975 – 986.
DOI: 10.1109/TCOMM.2011.022811.100092
[14] Pallam S. W., Audu G. A., Musa S. Y., Visa I. M., *Performance Analysis of Convolutional and Gray Coding Techniques in Wireless Communications*, Int. J. of Scientific & Engineering Research, Vol. 5, Issue 3, March-2014 843 ISSN 2229-5518
[15] Petchmaneelumka W., Julsereewong A., *A Gray-code algorithmic analog-to-digital converter based on operational conveyors*, Int. Conference on Control, Automation and Systems, 27-30 Oct. 2010 Gyeonggi-do, Korea (South)
DOI: 10.1109/ICCAS.2010.5669636
[16] Chaikla A., Arayawat S. and Riewruja V., *OTA-based Gray-code Algorithmic ADC*, Proc. SICE-ICASE Int. Joint Conference, 2006, p.p. 5787-5791.
[17] Arayawat S., Thubthong U., Julsereewong P., Riewruja V., and Julsereewong A., *A Voltage-mode Gray-code Algorithmic ADC*, Proc. SICE Annual Conference, 2008, p.p. 605-608.
[18] B. Jackson, B. Stevens, G. Hurlbert, *Research problems on Gray codes and universal cycles*, Discrete Math. 309 (2009), 5341–5348, Problem 483.
[19] Diaconis P., Holmes S., *Gray codes for randomization procedures*, Wikipedia [online], Available at: https://statweb.stanford.edu/~cgates/PERSI/papers/graycodes.pdf
[20] Alvarez G., Arroyo D., Nunez J., *Application of Gray code to the cryptanalysis of chaotic cryptosystems*, Wikipedia [online], Available at: https://www. researchgate.net/ publication/215732242
[21] Mütze T. *Combinatorial Gray Codes — an Updated Survey*, Wikipedia [online], Available at: https://arxiv. org/pdf/2202.01280.pdf
[22] Berger U., Schwichtenberg K., Tsuiki H., Miyamoto H., *Logic for Gray-code Computation*, Wikipedia [online], Available

at: https://www.mathematik.uni-muenchen.de/~ schwicht/papers/ proof13/gray1509 28.pdf

[23] Richard E. Blahut, *Theory and Practice of Error Control Codes*, Addison-Wesley Publishing Company, 1983, 500 p. ISBN-13: 978-0201101027

[24] Trahtman A. M., Trahtman V. A., *Fundamentals of the theory of discrete signals on final machines*, Moscow: Sov. Radio, 1975, — 208 p.

[25] Artemiev M. Yu., Gaev G. P., Krenkel N. E., Skotnikov A. P., *Algorithm of formation of symmetric systems of Walsh functions*, Radio engineering and electronics, № 7, 1978, p.p. 1432-1440.

[26] Yuen C. *Walsh Functions and Gray Code*, IEEE Trans., 1971, EMC-13 (3), p.p. 68 -73.

[27] Wilenkin N. Ya. *About one class of complete orthogonal systems*, News of the USSR Academy of Sciences. Ser. mat., Vol. 11, No. 4, 1947, p.p. 363-400.

[28] Crestenson H. E. *A class of generalized Walsh functions*, Pacific J. Math., vol. 5, 1955, p.p. 17-31.

[29] Beletsky A. Ya. *Discrete orthogonal bases of Wilenkin-Crestenson functions*, Palmarium Academic Publishing, Germany, 2015, 232 p. ISBN 978-3-659-60300-6

[30] Beletsky A. Ya. *Generalized Gray Transforms and Synthesis Symmetric Systems of Walsh functions*, Telecommunications and Radio Engineering, Vol. 78, # 4, 2019, p.p. 305-326. DOI: 10.1615/TelecomRadEng.v78.i4.30

[31] Beletsky A. I. *Quasi-equidistant codes*, Kyiv: Pub. House NAU-Druk, 2008, 460 p.

[32] Loginov V.P. *Functions of Walsh and their applications (review)*, Foreign Radioelectronics, # 4, 1973, p.p. 73-101.

[33] Beletsky A. Ya. *Synthesis and analysis of the system of Walsh-Cooley basis functions*, NIKON-2000: XIII International Conference, Wroclaw, 2000.

[34] Cooley J.W. and Tukey J.W. *An algorithm for the complex Fourier series machine calculation*, Math. Comp., 19, 1965, p.p. 297-306.

Contribution of Individual Authors to the Creation of a Scientific Article (Ghostwriting Policy)

The authors equally contributed in the present research, at all stages from the formulation of the problem to the final findings and solution.

Sources of Funding for Research Presented in a Scientific Article or Scientific Article Itself

No funding was received for conducting this study.

Conflict of Interest

The authors have no conflicts of interest to declare that are relevant to the content of this article.