Formulation, Proposition and Application of Interval Semigraph

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Abstract: - Semigraph represents a versatile extension within graph theory, offering a broader framework for exploration. Among these, intersection graphs hold pivotal roles, serving as essential analytical tools with wide-ranging applications. In this study, we introduce the integration of intersection graphs into the generalized structure of semigraphs, aiming to invigorate researchers' interest and encourage further advancements in this domain. This paper introduces two novel concepts: edge intersection semigraphs and interval semigraphs. Additionally, foundational results pertaining to these concepts are established, elucidating their fundamental properties. Furthermore, we illustrate the practical utility of interval semigraphs through an illustrative example involving their application in optimizing road traffic management systems.

Key-Words: - Semigraph, Intersection Semigraph, Interval Semigraph, Interval Graph, Intersection Graph, Optimization.

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1 Introduction

A graph $G = (V, E)$ is called an intersection graph, [1], for a finite family F of a non-empty set if there is a one-to-one correspondence between F and V such that two sets in F have a non-empty intersection if and only if their corresponding vertices in V are adjacent. We call F an intersection model of G. For an intersection model F, we use $G(F)$ to denote the intersection for F.

An undirected graph $G = (V, E)$ is said to be an interval graph, [1], if the vertex set V can be put into one-to-one correspondence with a set I of intervals on the real line such that two vertices are adjacent in G if and only if their corresponding intervals have non-empty intersection. That is, there is a bijective mapping $f: V \to I$. The set I is called an interval representation of G.

The notion of semigraph, [2], is a generalization of that of a graph. While generalizing a structure, one naturally looks for one in which every concept in the structure has a natural generalization. Semigraph is such a natural generalization of the graph and it resembles a graph when drawn in a plane.

In a graph, one edge contains two vertices i.e. the graph is represented in a situation where these involve only two objects in a relation. But when there is a relation contained among the objects more than two it is not possible to explain in terms of graph. So, to handle this type of relation situation we need a generalized graph and this generalized structure is Semigraph. In semigraph, we see that all properties enjoyed by vertices are also enjoyed by edges.

[2], introduced a new generalization of graphs called semigraphs. In 1994, after being confirmed that the new generalization was natural and worthy of applications, to a wider variety of situations and

decided to build a structural foundation for it. His immediate task was to examine and see how the new idea worked and what impact it had on the existing terminology of graph theory. The immediate impact was overwhelming. Graph theoretic terms and definitions seemed to rally one after another to be accommodated in a semigraph setting either for their extensions or new interpretations. With a generalized graph structure allowing an unusually long list of terms and definitions that were algebraically screened with positive outcomes, it was natural to eye upon its topological aspects, and expectations were fruitfully rewarded. Within a very short period after contributed the new generalization of the graph to the world of Mathematics in 2000, Indian graph theorists took up keen interest in this new area and they began attempting to complete the untouched and unfinished works of its originator. Encouraged by new developments in various directions of semigraph structure new faces are attracted into its fold day by day.

In some situations, where more than two objects/elements are involved in the same relation, e.g. buying or selling goods. A goods we cannot buy directly from a factory or company. To buy goods there is a retailer and distributor between a consumer and a company (factory). At this time, we cannot represent this in graph theory but with the help of semigraph we can easily express the situation.

The quest to characterize interval graphs originated independently in different fields, [3], [4], in combinatorics and, [5], in genetics during 1957-1959. However, the introduction of the semigraph concept in mathematics came later, in the year 2000. Despite this period, the introduction of the interval semigraph remained unaddressed, despite the comprehensive generalization of concept onto semigraphs. This paper addresses this gap by introducing the notions of edge intersection semigraphs, interval semigraphs, and several associated results.

2 Preliminary

Definitions, [2], [6], [7], [8]

Semigraph: A semigraph G is a pair (V, X) where V is a non-empty set whose elements are called vertices of G and X is a set of n-tuples, called edges of G, of distinct vertices, for various $n \geq 2$, satisfying the following conditions.

(a) Any two edges have at most one vertex in common.

(b) Two edges $(u_1,u_2,\ldots,u_n)$ and $(v_1,v_2,\ldots,v_m)$ are considered to be equal if and only if
  (i) $m = n$ and
  (ii) either $u_i = v_i$ for $1 \leq i \leq n$, or $u_i = v_{n-i+1}$ for $1 \leq i \leq n$.

An edge e is represented by a simple open Jordan curve which is drawn as a straight line whose end points are called the end vertices of the edge e and the m-vertices of the edge e each of which is not an m-vertex of any other edge of the semigraph G are denoted by small circles placed on the curve in between the end vertices, in the order specified by e. The end vertices of edges which are not m-vertices are specially represented by thick dots. If an m-vertex of an edge e is an end vertex of an edge e' i.e. an (m, e) vertex, we draw a small tangent to the circle at the end of the edge e' .

Example 2.1: Let $G = (V, X)$ be a semigraph. Then the edges of the semigraph in Figure 1 are $(v_0, v_1, v_2), (v_2, v_6, v_7, v_8), (v_1, v_3, v_4), (v_4, v_5),$ (v_5, v_6)

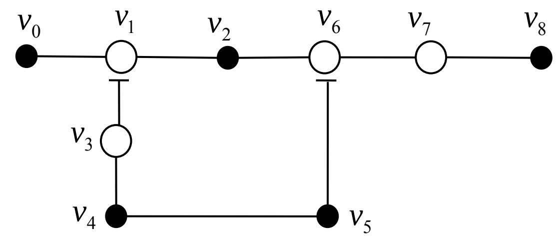

Fig. 1: Semigraph Example

Subedges: A subedge of an edge $E = (v_1, v_2, \ldots, v_n)$ is a k-tuple $E' = (v_{i_1}, v_{i_2}, \ldots, v_{i_k})$ where $1 \leq i_1 < i_2 < \cdots < i_k \leq n$ or $1 \leq i_k < i_{k-1} < \cdots < i_1 \leq n$. We say that the subedge E' is induced by the set of vertices $\{v_{i_1}, v_{i_2}, \ldots, v_{i_k}\}$.

Example 2.2: Let $G = (V, X)$ be a semigraph. Then the sub edges of the semigraph from the Figure 1 are $(v_0, v_2), (v_2, v_7, v_8), (v_3, v_4), (v_4, v_5), (v_5, v_6)$.

Partial edge: A partial edge of E is a $(j - i + 1)$-tuple $E(v_i, v_j) = (v_i, v_{i+1}, \ldots, v_j)$, where $1 \leq i \leq n$. Thus a subedge E' of an edge E is a partial edge if and only if , any two consecutive vertices in E' are also consecutive vertices of E .

Example 2.3: Let $G = (V, X)$ be a semigraph. Then the partial edges of the semigraph from the Figure 1 are $(v_6, v_7, v_8), (v_0, v_1), (v_1, v_2)$.

Subsemigraph: A semigraph $G' = (V', E')$ is a subsemigraph of a semigraph $G = (V, E)$ if $V' \subseteq V$ and the edges in G' are subedges of G.

Induced subsemigraph: A semigraph $G' = (V', E')$ is an induced subsemigraph of a semigraph $G = (V, E)$ if the edges in G' are subedges of G induced by the vertices V'.

Consecutive ones property: Consecutive ones property, [9], is a property of two-dimensional matrices whose entries are only 0 and 1. These matrices are called binary matrices and the binary matrix has the consecutive ones property (C1P) for columns when its rows can be permuted so that in each of its columns the 1's appear consecutively.

3 Intersection and Interval Semigraph

Definition 3.1: Let $G = (V, X)$ be a semigraph and F be a set of collection of partial edges of the semigraph G, i.e. $F=\{S_i = p_i |\ p_i$ is a partial edges of some edge in $G\}$. Then $\tau(F)$ is said to edge intersection semigraph in G where the edge set of $\tau(F)$ consists of the edge of the types:

(a) $E^* = (p_{i_1}, p_{i_2}, ..., p_{i_r}), r \geq 2$, where $p_{i_j}, p_{i_{j+1}}, 1 \leq j \leq r-1$ are the consecutive partial edges of the same s-edge $E^* \in X$.

(b) $E^* = (p_i, p_j)$, where p_i and p_j are not the partial edges of the same s-edge and they have a vertex in common and also $\tau(F) \cong G$.

Example 3.1: Consider $G = (V, X)$ be a semigraph and clearly from the Figure 2, the edges of the semigraph are (1, 2, 3, 4, 5), (3, 6, 7), (4, 8, 9, 10).
Now, if F is the collection of partial edges of G then $F= \{S_1, S_2,\ldots, S_n\}$, where
$S_1 = \{(1, 2, 3), (3, 4), (4, 5)\}$
$S_2 = \{(4, 8), (8, 9, 10)\}$
$S_3 = \{(3, 6), (6, 7)\}$
Clearly, we can say that $\tau(F) \cong G$. Thus G is an edge intersection semigraph.

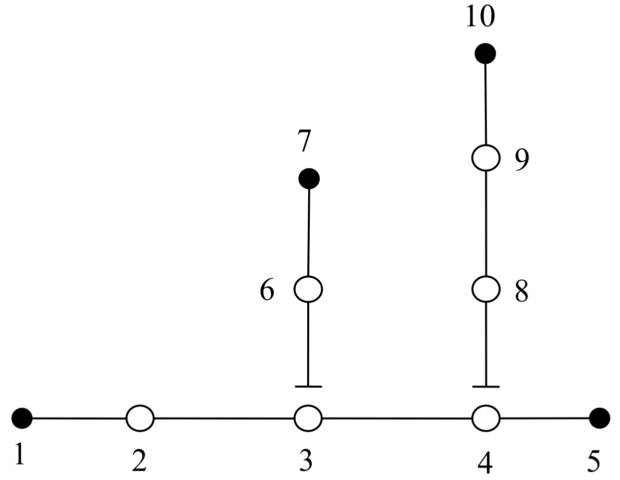

Fig. 2: Intersection Semigraph Example

Definition 3.2: Let $G = (V, X)$ be a semigraph and F be a set of collections of partial edges of some edge E of the semigraph G, i.e. $F =\{S_i = p_i |\ p_i$ is a partial edge of some edge E of $G\}$. Then p_i corresponds to an interval $[a, b]$ in $\Re$, $\forall i$ provided that if p_i and p_j are adjacent then our corresponding interval is $[a, b] \cap [c, d] \neq \emptyset$.

Example 3.2: Consider $G = (V, X)$ be a semigraph then from Figure 2, the partial edges of G correspond to an interval on a real line where two partial edges are adjacent. Thus, G is an interval semigraph.

Definition 3.3: An undirected semigraph $G = (V, X)$ is called a **triangulated semigraph** if every cycle of length strictly greater than 3 possesses a chord that is an edge joining two non-consecutive vertices of the cycle.

Theorem 3.1: Every semigraph is an edge intersection semigraph.

Proof: Let $G = (V, X)$ be a semigraph and S be any non-empty set and $F = \{S_i = p_i |\ p_i$ is a partial edge of some edge E of $G\}$, where $S_i \subseteq S, \forall i$
Then each edge of G is adjacent to the partial edges p_i . Thus, every semigraph is an edge intersection semigraph.

Theorem 3.2: Every edge intersection semigraph is an interval semigraph and conversely.

Proof: Let $G = (V, X)$ be an edge intersection semigraph, where $V=V(G)$ and $X=X(G)$ and let S be a non-empty set of closed intervals on the real line with each vertex corresponding to a closed interval. Then the set of collections of partial edges of some edge in G is isomorphic to G. By the definition of interval semigraph, the set of intervals is also isomorphic to G. Therefore, every edge intersection semigraph is an interval semigraph.

Conversely, let us assume that $G = (V, X)$ is an interval semigraph where $V=V(G)$ and $X=X(G)$.

By the definition of interval semigraph, there exists intervals on a real line and the set of intervals is isomorphic to G. Since, the set of intervals is isomorphic to G so the set of collection of partial edges of some edge in G is also isomorphic to G.
Therefore, an interval semigraph is also an edge intersection semigraph.

Theorem 3.3: An induced subsemigraph of an interval semigraph is an interval semigraph.

Proof: Let $G' = (V', E')$ be an induced subsemigraph of an interval semigraph $G = (V, X)$, then by the definition of interval semigraph there exist an interval representation $\{I_e\}_{e \in E\{G\}}$ for the semigraph $G = (V, E)$. Also then $\{I_e\}_{e \in E'(G')}$ is an interval representation for the induced subsemigraph $G' = (V', E')$. Therefore, an induced subsemigraph of an interval semigraph is an interval semigraph.

Theorem 3.4: An interval semigraph satisfies the triangulated semigraph property.

Proof: Let the interval semigraph $G = (V, X)$ contains a chordless cycle $(v_0, v_1, v_3, \dots, v_{n-1}, v_0)$ with $n > 3$. Suppose I_u denotes the interval corresponding to the partial edge (v_k, v_{k+1}) . Now for $j = 1,2,3, \dots, n-1$, consider an interval for the interval semigraph $I_i \in I_{j-1} \cap I_j$. Since, $I_{j-1}$and I_{j+1} do not intersect so the interval I_i constitutes an increasing or decreasing sequence. Therefore, it is not possible for I_0 and I_{n-1} to intersect, contradicting the fact that $v_0 v_{n-1}$ is a partial edge of the interval semigraph.

Theorem 3.5: A connected semigraph is an interval semigraph if and only if, for every three partial edges, one of them lies between the other two.

Proof: Let $G = (V, X)$ be a connected semigraph and there is a set F of collection of partial edges of the semigraph. Then there is a partial edge p_i for which each S_i belongs to X corresponds to the partial edge p_i of F . Then, a partial edge p_j lies between any two partial edge p_i and p_k .

Conversely, suppose that $G = (V, X)$ satisfies the condition that for every three partial edges, one of them lies between the other two partial edges. We have to show that $G = (V, X)$ is a connected interval semigraph. Consider a semigraph (V, X') with $X \subseteq X'$ such that the semigraph satisfies the condition in the theorem and X' is maximal.

Let p_i and p_j are two distinct partial edges in the semigraph $G = (V, X)$. If p_i and p_j do not belongs to S then $p_i, p_j \notin X$ and the semigraph does not satisfy the condition in the statement. Let the semigraph contain distinct p_x, p_y and p_z partial edges with the length of shortest path of the partial edges

$(p_x, E_1, \dots, E_u, p_y)$ with $p_z \notin (E_1 \cup \dots \cup E_u)$

$(p_y, E_1', \dots, E_v', p_z)$ with $p_x \notin (E_1' \cup \dots \cup E_v')$

$(p_x, E_1'', \dots, E_w'', p_z)$ with $p_y \notin (E_1'' \cup \dots \cup E_w'')$

Where the set of partial edge $S = \{p_i, p_j\} \in \{E_1, \dots, E_u, E_1', \dots, E_v'', E_1'', \dots, E_w''\}$

But if each occurrence of $\{p_i, p_j\}$ among E_u's, E_v''s and E_w'''s is replaced by $S \in X'$, then by condition in theorem one of p_x, p_y and p_z will be between other two. Without loss of generality, let the partial edge p_y is between the partial edge p_x and p_z. That means that $\{p_i, p_j\} = E_t''$ for some $1 \le t \le w$ and so $p_i \ne p_y \ne p_j$ and $p_y \in S$. Without loss of generality, using minimality of lenth of partial edge $p_x, E_1'', \dots, E_w'', p_z$, suppose that $p_i \in E_{t-1}''$.By assumed maximality of X', we can assume $P = E_1'' \cup \dots \cup E_{t-1}'' \in X'$. Thus there exists $P \in X'$ such that $p_i, p_x \in P$ and $p_j, p_y \notin P$. Similarly, there exists $Q \in X'$ such that $p_j, p_z \in Q$ and $p_i, p_y \notin Q$. Again, using maximality of X', we assume $S \setminus P, S \setminus Q \in X'$ and so p_i and p_j are connected by partial edge $p_i, S \setminus Q, p_y, S \setminus P, p_j$ in (V, X'). Thus, every partial edges $S \in X'$ are linked by a partial edge in (V, X') whose partial edges are subsets of S and that every partial edge $S \in X^*$ has cardinality two otherwise $S \setminus Q \in X^*$ would contradicts S's assumed minimality. Therefore (V, X^*) is a semigraph and the assumed maximality of X' implies $V \in X'$ so (V, X^*) is connected and the condition implies that (V, X^*) is a path and every partial edge $S \in X'$ can be seen as a connected subset of (V, X') by the argument on cardinality of S .

4 Application

To minimize traffic with minimum time interval:

Let V be the set of places such as villages, towns, cities, etc. in a state interconnected by a road network. Consider a State Transport Corporation running its buses to various destinations with a minimum time interval in the state. For the purposes of economy, to reduce running time and to avoid the stretch of any road being traversed by the buses plying on two different routes can minimize the traffic with a minimum time.

The corporation can plan its routes in such a way that any two routes may be planned as an interval semigraph on the available road network where the vertices represent the village, towns, cities, etc. and the partial edges are determined with the corresponding time intervals. From the Figure 3, we can define the end vertices v_1, v_4, v_6, v_7, v_9 as the terminal of the road network and the middle vertices v_2, v_3, v_5, v_8 as through destinations. The partial edges $(v_1, v_2, v_5), (v_5, v_6), (v_5, v_7)$, etc. are defined as the corresponding time interval of the interval semigraph. Now by the definition of interval semigraph, if any two partial edges are adjacent then their corresponding interval has a non-empty intersection. Thus, for any two partially connected routes their running time interval must be non-empty from one destination to another destination. If the corporation has planned their routes in this way then they can minimize their traffic problem and also they can reduce the running time from one terminal to another.

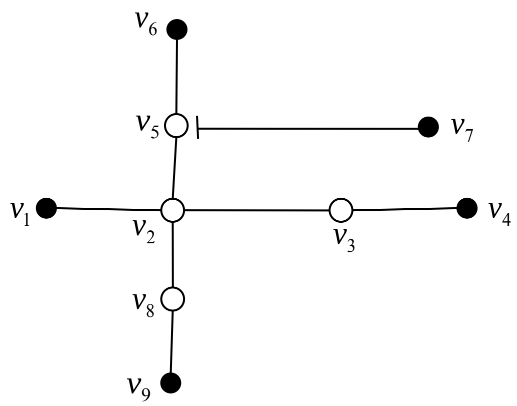

Fig. 3: Semigraphical Model

5 Conclusion

In this research, we delve into the intricate relationship between intersection semigraphs and their specialized subset, the interval semigraphs. The interval semigraph, a distinctive category within intersection semigraph theory, serves as a focal point for our investigation. Our exploration highlights the uncharted terrain within interval graph theory, indicating ample opportunities to unravel the underlying structures of interval semigraphs. Through this study, we anticipate the emergence of substantial findings that will significantly contribute to the evolving landscape of interval semigraph theory.

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Contribution of Individual Authors to the Creation of a Scientific Article (Ghostwriting Policy)

- Abdur Rohman and Surajit Kr. Nath conceived of the presented idea and developed the theory.
- Surajit Kr. Nath and Sheetal Sonawane verified the analytical methods and contributed to the design and implementation of the research to write the manuscript.

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Conflict of Interest

The authors have no conflicts of interest to declare.