Application of Operators with Shift in the Modeling of Renewable Systems with Elements in Four States

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Abstract: - In our recent works, we developed models for renewable systems with elements in three states. For the sake of clarity, states were interpreted as being healthy, being sick, and having immunity acquired after recovery. The states differ significantly in their processes of mortality, reproduction, and mutual influence. In this work, along with the three states listed, a fourth state is introduced, which we will interpret as the state of the elements of the system that have undergone vaccination. As a consequence of this, balance relations become more complicated: unknown functions with shifts and iterations from these shifts appear. A need arises to develop the applied mathematical methods. The content of the article is an expansion and development of our works on modeling systems with elements in several states based on the application of new results on solving functional equations with shifts.

Key-Words: - shift, renewable system, degenerate kernel, iterated equations, cyclic model, multiple states.

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1 Introduction

The interest towards investigating equations with shift appeared during the study of singular integral equations and Riemann problems for analytic functions, [1], [2]. For a long time, operators with shifts have been an important point of attention for research. Monographs [2], [3] reflected the results obtained on the theory of solvability of such integral equations and boundary problems. Let us mention some further works on this topic, [4], [5]. In modeling systems with renewable resources [6], in balance relations, functional operators with shift appear. This work is dedicated to the application of operators with a shift in the modeling of renewable systems with elements in four states. In comparison to [6], the balance relations of our model represent more complex equations; they contain not only shift operators but also their iterations. Problem formulations are carried out in Hölder spaces and Hölder spaces with weight — these spaces are convenient for applications, [7], [8]. In solving problems, the theory of continued functions [9], infinite products, and operator theory [10] are used.

We consider a system consisting of elements that can be in four states. For a more concrete perception of the presentation, the elements of the system will also be referred to as individuals, and the states will be interpreted as being healthy, being sick, having temporary immunity acquired after recovery, and being vaccinated. Elements of the system that are in different states have different characteristics. Their mortality coefficients, reproduction terms, and the processes of change of the individual parameters, such as weight gain and loss, are not the same. The system is dynamic, with elements able to transition between states. We will be following the principles of modeling, the main idea of which was to separate the individual parameters (weight of the individuals) and the group parameters (number of individuals of a given weight), as well as to provide a discretization of time. Interpretations of the individual and group parameters are given in parentheses above. These approaches led us to the need for the mathematical apparatus of functional operators with shift.

Section II presents the development of a cyclic model for a renewable system with elements in four states based on the developed models for three states, [6]. Section III includes proof of the lemma on the belonging of some functions to the Hölder space. Here we also prove a theorem about the equivalence of the original equation and the iterated equations, which will be important for what follows. In section IV, theorems from [6] on the solution of the original equation in Hölder spaces and in Hölder spaces with weight are formulated. Then, we return to the obtained balance relations and, using these theorems, we transform them into a system of Fredholm equations of the second kind with degenerate kernels. Applying algorithms for finding solutions for such systems [11], we obtain $n(x)$ - density functions of distribution of the group parameter by the individual parameter for the first state, $k(x)$ - density functions of distribution of the group parameter by the individual parameter for the second state and $m(x)$ - density functions of distribution of the group parameter by the individual parameter for the third state. The equilibrium position of the system is also found. The theory of linear functional operators with shift is the appropriate mathematical instrument for the investigation of renewable systems with multi-state elements.

2 Modeling Systems with Elements in Four States. Evolution of the System during Period T

Following our approach proposed in [6], we will not track how a system changes during the time period (t_0, t_0+T) at every moment but will fix results only at the final timepoint $t = t_0 + T$. For clarity, the individual parameter will be interpreted as the weight of individuals, and the group parameter will be interpreted as the number of individuals.

The initial state of the system S at time t_0 is represented by density functions of distribution of the group parameter by the individual parameter. For the first state (healthy individuals), it is represented by the function $n(x,t_0) = n(x)$, $0 < x < x_{max}$. For the second state (sick individuals), it is represented by the function $k(x,t_0) = k(x)$, $0 < x < x_{max}$. For the third state (individuals with immunity), it is represented by the function $m(x,t_0)=m(x)$, $0 < x < x_{max}$. For the fourth state (vaccinated individuals), it is represented by the function $v(x,t_0)=v(x)$, $0 < x < x_{max}$.

Then, the state of the system is described as follows:

$$n(x,t_0+T) = d_n(x)\alpha'(x)n(\alpha(x)) + N^R + N^K + \zeta_n(x),$$

$$k(x,t_0+T) = d_k(x)\beta'(x)k(\beta(x)) + K^R + K^N + K^M + \zeta_k(x),$$

$$m(x,t_0+T) = d_m(x)\gamma'(x)m(\gamma(x)) + M^R + M^K + \zeta_m(x),$$

$$v(x,t_0+T) = d_m(x)\delta'(x)m(\delta(x)).$$

Here, the change in the distribution of the group parameter by the individual parameter (distribution of the number of individuals by their weights) over time is represented by the displacements $\alpha(x)$, $\beta(x)$, $\gamma(x)$, which must depend on the state in which the individuals are. We account for the natural mortality by the coefficients $d_n(x)$, $d_k(x)$, $d_m(x)$, $d_v(x)$, for the elements which are in different states. Of course, mortality rates for healthy (first state) and sick (second state) individuals are different. The process of reproduction will be represented by the terms:

$$N^R(x) = N^r(x)C_n^r, \quad C_n^r = \int_0^1 \rho_n^r(\tau)n(\tau)d\tau,$$

$$K^R(x) = K^r(x)C_k^r, \quad C_k^r = \int_0^1 \rho_k^r(\tau)k(\tau)d\tau,$$

$$M^R(x) = M^r(x)C_m^r, \quad C_m^r = \int_0^1 \rho_m^r(\tau)m(\tau)d\tau,$$

and the reciprocal influence of the states of the elements will be represented by the terms:

$$N^K(x) = N^k(x)C_n^k, \quad C_n^k = \int_0^1 \rho_n^k(\tau)k(\tau)d\tau,$$

$$K^N(x) = K^n(x)C_k^n, \quad C_k^n = \int_0^1 \rho_k^n(\tau)n(\tau)d\tau,$$

$$K^M(x) = K^m(x)C_k^m, \quad C_k^m = \int_0^1 \rho_k^m(\tau)m(\tau)d\tau.$$

Self-influence will also be described by integrals with degenerate kernels:

$$K^K(x) = K^k(x)C_k^k, \quad C_k^k = \int_0^1 \rho_k^k(\tau)k(\tau)d\tau.$$

We did not strive to describe the general form of the degeneracy of kernels, but instead focused on keeping the essence of the approach and the clarity of presentation. In describing the evolution of the system S over the time interval $j = (t_0, t_0+T)$, we used works on modeling systems with renewable resources. Here, terms $\zeta_n(x), \zeta_k(x), \zeta_m(x)$ that do not depend on $n(x)$, $k(x)$, $m(x)$, but have an impact on them, have been introduced. These terms can reflect various changes occurring in the system S; for example, migration processes, removal of individuals from the system (fish catching), and introduction of individuals into the system (release of fry and spawners). We note that the presence of these terms does not affect the method and the technique applied in this work.

Then, the state of the system is described as follows:

$n(x,t_0+T) = d_n(x)\alpha'(x)n(\alpha(x)) + N^R(x) + N^K(x) + \zeta_n(x)$

,

$k(x,t_0+T) = d_k(x)\beta'(x)k(\beta(x)) + K^R(x) + K^N(x) + K^M(x) + \zeta_k(x),$

$m(x,t_0+T) = d_m(x)\gamma'(x)m(\gamma(x)) + M^R(x) + M^K(x) + \zeta_m(x)$,

$v(x,t_0+T) = v(d_v(x)\gamma'(x)v(\delta(x)).$

3 Transitions of States, Balance Relationships and the Cyclic Model

Let us introduce a coefficient $[1-\Theta(x)]$, which separates the proportion of individuals who receive different medical care: $[1-\Theta(x)]d_k(x)\,\beta'(x)\,k[\beta(x)]$. We can finally write down the balance ratios at the very end of the period:

$k(x,t_0+T) = [1-\Theta(x)]d_k(x)\beta'(x)k(\beta(x)) + K^R(x) + K^N(x) + K^K(x) + \zeta_k(x)$

Let us make some assumptions. First, we denote by $\Lambda(x)$ the incidence rate; by $[\Lambda(x)]d_n(x)\alpha'(x)n[\alpha(x)]$ the proportion of organisms that were healthy at the beginning $t=t_0$, but became infected during the time period $[t_0,t_0+T]$ and by $[1-\Lambda(x)]d_n(x)\alpha'(x)n[\alpha(x)][1-\Theta(x)]d_k(x)\,\beta'(x)\,k[\beta(x)]$ the proportion of organisms that did not get sick and remained healthy. Second, the duration of the illness is shorter than the length of the period and, moreover, if a certain organism was initially sick or infected, then by the end of the period it either recovers or leaves the system. This is taken into account by the mortality rate and, therefore, the term indicates the number of recovered patients and the number of healthy individuals. Patients do not pass to the next period in that state. These processes are reflected in changes in balance equations; we obtain:

$$n(x,t_0+T) = [1-\Lambda(x)]d_n(x)\alpha'^{(x)}n[\alpha(x)] + d_m(x)\gamma'(x)m(\gamma(x)) + N^R(x) + N^K(x) + K^R(x) + K^N(x) + M^R(x) + \zeta(x), \tag{1}$$

where $\zeta(x) = \zeta_n(x) + \zeta_m(x)$,

$$k(x,t_0+T) = [\Lambda(x)]d_n(x)\alpha'^{(x)}n[\alpha(x)], \tag{2}$$

$$m(x,t_0+T) = [1-\Theta(x)]d_k(x)\beta'(x)k(\beta(x)) + K^K(x) + \zeta_k, \tag{3}$$

$$v(x,t_0+T) = v(x). \tag{4}$$

Equations (1), (2), (3), (4) are called balance relations.

Let our goal be to find the equilibrium state of the system S, that is, to find such an initial distribution of group parameters by the individual parameters $n(x),\ k(x), m(x)$, that after all transformations during the time interval (t_0,t_0+T), it would coincide with the final distribution:
$n(x,t_0) = n(x,t_0+T),\ k(x,t_0) = k(x,t_0+T),$
$0 \le x \le 1.$
From here and the balance equations (1), (2), (3), (4), it follows that

$$n(x) = [1-\Lambda(x)]d_n(x)\alpha'(x)n[\alpha(x)] + d_m(x)\gamma'(x)m(\gamma(x)) + N^R(x) + N^K(x) + \zeta(x), \tag{5}$$

$$k(x) = [\Lambda(x)]d_n(x)\alpha'(x)n[\alpha(x)], \tag{6}$$

$$m(x) = [1-\Theta(x)]d_k(x)\beta'(x)k(\beta(x)) + K^K(x) + \zeta_k. \tag{7}$$

This model is a cyclic model. Relations (5), (6), (7) are equilibrium proportions or balance equations of the cyclic model. The system of balance equations of the cyclic model is represented through lineal functional equations with the shift. The discretization of time and the special attention which we have paid to the study of density distributions of the group parameters by the individual parameters lead us to functional operators with shift. These relations correspond to a complex system of integral equations with functional operators with shifts. Consider the case when the conditions

$$\beta(x) = x,\ \gamma(x) = \alpha(x), \delta(x) = \alpha(x), \tag{8}$$

are fulfilled. The first ratio means that infected organisms (in the second state) do not change their weight and do not reproduce. The second ratio means that healthy organisms without immunity (in the first state) develop and gain weight in the same way as organisms that have just recovered from illness with acquired immunity (in the third state). If requirements (8) are met, then relations (7) take the form:

$$m(x) = [1-\Theta(x)]d_k(x)\cdot k(x) + K^K(x) + \zeta_k(x).$$

For further analysis of this model, we define the space in which we consider the balance equations and we formulate some statements about the operators located there. Let us introduce a coefficient $[1-\Theta(x)]$ which separates the proportion of patients who are in a hospital and receive full medical care:

$$[1-\Theta(x)]d_k(x)\,\beta'(x)\,k[\beta(x)].$$

Let us simplify our model. We assume that the affected and the recovered individuals do not reproduce and that their states do not affect each other:

$K^R(x) = K^N(x) = N^K(x) = M^K(x) = M^R(x) = 0.$

Without loss of generality, we assume below that x_{max}=1.

These assumptions, as well as the simplified choice of degenerate kernels, do not affect the methods used, but allow us to avoid unnecessary encumberment in the content of the article. We note that different states of elements in this work are similar to different resources in our previous works. An essential difference is that representatives of the same resource cannot transform into representatives of another resource and thus do not have different states. If the distribution $n(x)$ can be found, it will not be difficult to calculate the distributions $k(x)$ and $m(x)$, so we will focus on analyzing equation (5). Let us take a look at the density distribution *n(x)*. Substituting m from (6) into relation (5), we obtain:

$n(x) = [1 - \Lambda]d_n\alpha'(x)n[\alpha(x)] + d_m\alpha'(x) \times$
$\{[1 - \Theta(\alpha)]d_k(\alpha)\cdot k(\alpha(x)) + K^K(\alpha(x)) + \zeta_k(\alpha)\} +$
$M^R(x) + N^R(x) + N^K(x) + \zeta(x).$

From (2), we have:

$$n(x) = b(x)n(\alpha(x)) + c(x)n\left(\alpha(\alpha(x))\right) + d_m(x)\alpha'(x)K^K(\alpha(x)) + d_m(x)\alpha'(x)\zeta(\alpha(x)) + M^R(x) + N^R(x) + N^K(x) + \zeta(x), \tag{9}$$

where $b(x)$and $c(x)$ can be written out in terms of known functions from (5), (6), (7).

In [3], invertibility conditions were found for the two-term functional operator with shift $A = aI + bB_\alpha$, $B_\alpha\varphi(x) = \varphi[\alpha(x)]$, acting in Lebesgue space $L_p(J)$ and the inverse operator was constructed for it. Applying the research scheme for the operator A from [3], but acting in the Hölder space with weight, $H^{\,0}_{\mu}(J,\rho)$, we obtained similar results on the conditions of invertibility and the construction of the inverse operator. These mathematical tools were sufficient to study systems with renewable resources. However, in this article, when modeling a system S with elements that can be in four states, we came to the need to study three-term operators and tree-term equations with shift. Definitions and description of spaces $H_\mu(J)$ and $H^{\,0}_{\mu}(J,\rho)$ can be found in [2], [7].

4 On the Belonging of Some Functions to the Hölder class. Theorem on the Equivalence of the Initial and Iterated Equations

To simplify the proof of the statements in this section we will add an additional requirement to those imposed on the shift: $\alpha(x)$ has a second derivative and $\alpha''(1) \neq 0$, $\alpha''(1) \neq 0$. In what follows, we will assume that $\alpha(x)$ has this property. Thus, we write the weight function from space $H^0_\mu(J,\rho)$:

$$\rho(x) = (x-0)^{\mu_0}(1-x)^{\mu_1},$$

where $\mu < \mu_0 < 1 + \mu$, $\mu < \mu_0 < 1 + \mu$.

Lemma 1. Function $F(x) = \frac{x}{\alpha(x)}$ belongs to $H_1(J)$, $\frac{\rho(x)}{B_\alpha\rho(x)}$ belongs to $H_\mu(J)$ and $\frac{\rho(x)}{B_\alpha{}^2\rho(x)}$ belongs to $H_\mu(J)$.

Proof. First, we prove that $F(x)$ belongs to the Lipschitz class $H_1(J)$: $|F(x_1) - F(x_2)| < C|x_1 - x_2|$. We use Lagrange's Theorem: if some function Φ is continuous on closed segment $[x_1,x_2]$ and differentiable on an open interval (x_1,x_2) then the following relation holds:

$\Phi(x_1) - \Phi(x_2) = \Phi'(c)(x_1 - x_2),\quad x_1 < c < x_2.$

For $\Phi(x) = F(x)$, we have:

$$\frac{x_1}{\alpha(x_1)} - \frac{x_2}{\alpha(x_2)} = [\frac{x}{\alpha(x)}][\frac{x}{\alpha(x)}]'_{x=c}(x_1 - x_2). \tag{10}$$

The function $[\frac{x}{\alpha(x)}]'_{x=c}$ is bounded by some constant M. We calculate this constant.

The derivative of the function $[\frac{x}{\alpha(x)}]$ is equal to $[\frac{\alpha(x) - x\alpha'(x)}{(\alpha(x))^2}]$ and, therefore, $\left(\frac{x}{\alpha(x)}\right)' = [\frac{x}{\alpha(x)}]^2\,[\frac{\alpha(x) - x\alpha'(x)}{(x)^2}]$.

The derivative has no singularities at $x \neq 0$. In order to assert the continuity of the derivative, we consider the limit when x tends to zero

$$\lim_{x\to 0}\left(\left[\frac{x}{\alpha(x)}\right]^2\right)\left(\frac{\alpha(x) - x\alpha'(x)}{(x)^2}\right) = [\frac{1}{\alpha'(0)}]^2,$$
$$\lim_{x\to 0}\frac{\alpha'(x) - \alpha'(x) - x\alpha''(x)}{2x} = \frac{1}{(\alpha'(0))^2}\frac{-\alpha''(0)}{2}.$$

The continuous function $\left[\frac{x}{\alpha(x)}\right]'$ is defined on the segment $[0,1]$ and therefore reaches its extreme values on this segment, which we will denote by F'_{min} and F'_{max}. We came to an estimate:

$$|\left(\frac{x}{\alpha(x)}\right)'| \le M = \{\max|F'_{min}|,|F'_{max}|\}.$$

Coming back to (10), we conclude that the function $F(x)$ is a Lipschitz function.

Note that $1 - \alpha(x) = \alpha(x)(1 - x)$ and that the product of a Hölder function from $H_\mu(J)$ and a Lipschitz function from $H_1(J)$ gives us a Hölder function from $H_\mu(J)$. The following equalities hold:

$$\frac{\rho(x)}{B_\alpha\rho(x)} = \frac{(\alpha(x)-0)^{\mu_0}(1-\alpha(x))^{\mu_1}}{(\alpha(x)-0)^{\mu_0}(1-\alpha(x))^{\mu_1}} = \left(\frac{x}{\alpha(x)}\right)^{\mu_0}\left(\frac{1-x}{1-\alpha(x)}\right)^{\mu_1}$$

belongs to $H_\mu(J)$. Now, we move on to:

$$\frac{\rho(x)}{B_\alpha{}^2\rho(x)} = \frac{\rho(x)}{B_\alpha\rho(x)}\frac{B_\alpha\rho(x)}{B_\alpha{}^2\rho(x)} = \frac{\rho(x)}{B_\alpha\rho(x)}B_\alpha\left(\frac{\rho(x)}{B_\alpha\rho(x)}\right).$$

From the requirements imposed on the shift follows the belonging to Hölder class with exponent μ. This implies that B_α(F) ϵ $H_\mu(J)$.

Let us make a remark: in [6], the statements of this lemma were used, but the proof was not given. Here we present a complete proof of the statements in Lemma 1.

We consider an initial equation $\varphi(x) - KB_\alpha\varphi(x) = g(x)$ and will now proceed to describe the process of constructing iterated equations. We write the initial equation in the recurrent form:

$$\varphi(x) = KB_\alpha\varphi(x) + g(x) \qquad (11)$$

Substituting the expression for the unknown function into the right side of the same equation, we get an equation after the first iteration:
$\varphi(x) = (KB_\alpha)[KB_\alpha\varphi(x) + g] + g,$
$\varphi = (KB_\alpha)(KB_\alpha)\varphi(x) + (KB_\alpha)g + g.$

We denote the obtained equation as the first iterated equation. Using the same algorithm, we construct the second iterated equation and move operator B_α to the end of it:
$\varphi = (b)\cdot(B_\alpha b)\cdot(B_\alpha^2 b)\cdot B_\alpha^2\varphi + b\cdot(B_\alpha b)B_\alpha^2 g + bB_\alpha g + g$

Here, we have indicated the results of the first and the second iteration. Continuing the iterative process, at step n, we obtain n-th iterated equation:

$\varphi(x) = (KB_\alpha)^{n+1}\varphi(x) + G_n(x),$
where $G_n = (KB_\alpha)^n g + \cdots + (KB_\alpha)g + g.$

We represent the operator $(KB_\alpha)^{n+1}$, obtained after n iterations, in another way:
$(KB_\alpha)^{n+1}K(B_\alpha KB_\alpha{}^{-1})(B_\alpha{}^2\,KB_\alpha{}^{-2})\ldots(B_\alpha{}^n\,KB_\alpha{}^{-n})B_\alpha{}^{n+1}$

We denote by Ω_{n-1} the operator
$K(B_\alpha KB_\alpha{}^{-1})(B_\alpha{}^2\,KB_\alpha{}^{-2})\ldots(B_\alpha{}^n\,KB_\alpha{}^{-n}).$

The n-th iterated equation will be written as:
$\varphi(x) = \Omega_{n-1}B_\alpha^n\varphi(x) + G_n(x).$

Let us formulate and prove an important theorem.
Theorem 1. The original equation and the iterated equations are equivalent to each other.

Proof. In [3], it has been proved that the solutions of the original equations are solutions of the iterated equations. Now, we will prove that the solutions of the first iteration equation are solutions of the original equation. So, let $f(x)$ be the solution of the first iteration equation: $f(x) = (LB_\alpha)^2 f(x) + G_1(x)$, $G_1(x) = (LB_\alpha)g + g$, then the function f will be the solution of the second iteration equation: $f = (LB_\alpha)[(LB_\alpha)^2\varphi(x) + G_1] + G_2$, $G_2 = LB_\alpha G_1 + g$. Now, we add and subtract the term $LB_\alpha f(x) + g(x)$ and perform some transformations,
$f = LB_\alpha f + g + LB_\alpha[(LB_\alpha)^2 f + G_1] + G_2 - LB_\alpha f - g$
,

$f = LB_\alpha f + g + LB_\alpha[(LB_\alpha)^2 f + G_1 - f - G_1] + LB_\alpha G_1 + G_2 - g,$

$$f = LB_\alpha f + g - LB_\alpha G_1 + G_2 - g.$$

Here, the fact that f is a solution of the first iterated equation $f(x) = [(LB_\alpha)^2 f(x) + G_1]$ was used. Counting $-LB_\alpha G_1 + G_2 - g = 0$ we come to initial equation $f(x) = LB_\alpha f(x) + g(x)$. The theorem has been proved.

5 Solving the Balance System of Equations

Now, we formulate Theorem 2 from [2] on the solution of the original equation in Hölder spaces $H_\mu(J)$ and in Hölder spaces with weight $H_\mu^0(J,\rho)$.

Let the sequence of operators $\{\Omega_n\}$ converge in operator norm to the operator Ω acting in $H_\mu(J)$ and the functional sequence $\{G_n(x)\}$ converge to some function $G(x)$ from space $H_\mu(J)$ and $1 - b(1) - c(1) \neq 0$. Then, the equation in space $H_\mu(J)$,

$$\varphi(x) - b(x)B_\alpha\varphi(x) - c(x)B_\alpha{}^2\varphi(x) = g(x),$$

where the coefficients and the free term belong to $H_\mu(J)$ and the unknown function is searched from $H_\mu(J)$, has a unique solution that is determined by the formula

$$\varphi(x) = \left(\Omega\frac{g(1)}{1-b(1)-c(1)}\right)(x) + G(x).$$

To formulate the second part of the theorem, we introduce these operators:

$$\Omega_n^S = S(B_\alpha SB_\alpha{}^{-1})(B_\alpha{}^2\,SS^{-2})\ldots(B_\alpha{}^n\,SB_\alpha{}^{-n}),$$

where $S = vI + wB_\alpha$ $v(x) = \frac{\rho(x)\,b(x)}{B_\alpha\rho(x)}$, $w(x) = \frac{\rho(x)c(x)}{B_\alpha{}^2\rho(x)}$
and

$$\Gamma_n^S = I + (SB_\alpha) + (SB_\alpha)^2 + \cdots + (SB_\alpha)^n.$$

Note that Lemma 1 functions $v(x)$ and $w(x)$ belong to $H_\mu(J)$.

Let the sequence of operators $\{\Omega_n^S\}$ converge in operator norm to the operator Ω^S acting in $H_\mu(J)$ and the functional series $\{\Gamma_n^S\,\rho(x)h(x)\}$ converge in norm of space $H_\mu(J)$ to the some function $\Gamma^S\rho(x)h(x)$ from the space $H_\mu(J)$ and $1-b(1)-c(1)\neq 0$. Then, the equation $\psi(x)-b(x)B_\alpha\psi(x)-c(x)B_\alpha{}^2\psi(x)=h(x)$, where coefficients belong to $H_\mu(J)$ and free term and unknown function $\psi(x)$ belongs to $H_\mu^0(J,\rho)$, has a unique solution $\psi(x)=\rho^{-1}(x)\Gamma^S[\rho(x)h(x)]$, where $\rho(x)=x^{\mu_0}(1-x)^{\mu_1}$, $\mu<\mu_i<1+\mu, i=0,1$.

Since the density function of vaccinated individuals is only included in the free terms, it does not affect the construction of the inverse operators and further analysis of the balance equations is analogous to the content of the last section from the work, [6]. As a result, we reduce the balance equations to Fredholm integral equations of the second kind with degenerate kernels. Applying the method for solving such systems of equations with degenerate kernels, we find $n(x), k(x), m(x)$. This cumbersome procedure in a short form and without details was presented in work, [6].

6 Conclusion

In the present article, while developing a model for renewable systems with elements in different states, in section 5, functional equations with shift and an iteration of this shift were considered. In developing more general models, functional equations with shift and several iterations of this shift appear in balance relations. In reducing operators with several iterations to a composition of operators with shift and without iterations, we arrive at the need for the study of a certain class of nonlinear equations. Authors plan to publish results on the solutions of such nonlinear equations in the future. This will further expand the possibilities of modeling and extend the toolbox of means that can be applied to the analysis of balance equations of the considered systems.

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