A Fractional Reduced Differential Transform Method for Solving Multi-Fractional Telegraph Equations

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Abstract: - This paper presents a novel modification of the Fractional Reduced Differential Transform Method (FRDTM) to solve space-time multi-fractional telegraph equations. The telegraph equation is crucial in modeling voltage and current distribution in electrical transmission lines, and its solutions have applications in physics, economics, and applied mathematics. The proposed method effectively simplifies the fractional differential equations by omitting one fractional derivative term, allowing for the transformation of the remaining terms using the FRDTM. The solutions demonstrate the method's accuracy and efficiency in fractional partial differential equations. This study advances the analytical solutions of fractional telegraph equations by providing a straightforward yet powerful approach to fractional differential problems.

Key-Words: - Fractional Reduced Differential Transform Method; FRDTM; Fractional telegraph Equation.

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1 Introduction

In the new technology phase, the fractional derivative has been strongly applied in applied science to solve the fractional controlled price equations, discretize the time-independent space-fractional models, nonlinear mechanism by dynamical complexity, and fractional controlled diffusion processes. Some controlling problems have been analyzed and performed in nonlinear differential problems, especially the Markov process. The fractional derivative has also been construed as the state-of-the-art performance for the problems in physics science, economics, or other aspects, [1], [2], [3], [4], [5], [6]. Large-scale papers about fractional derivative branches were published, which played a significant role in research and applied to different aspects of life. Besides that, the fractional derivative also showed a broad and efficient use in investigating the behavior of real groups, [7]. Moreover, fractional calculus could become an effective tool to express the real world by representing arguments and rate diffusion changes. Fractional calculus is also applied in the utilization of computer science, and probability distribution, especially in financial risk management. The solutions of fractional partial differential equations focus on fat-tailed, and stable distribution. The financial business is essentially worldwide for the government to set up algorithms for adjusting and controlling inflation related to market price, total income, or debasement. Mathematicians have supported sustained efforts to produce more and more mathematical tools for establishing the fundamental foundations. The purpose is to support managers, official agents, businessmen, and the government in solving the financial distribution models and time-continuous systems, [8].

Various methods are created to find the best solutions for nonlinear problems, especially by combining other algorithms to seek the solutions. For example, the homotopy perturbation method combining the expansion method is applied in solving many aspects of economic problems, [9], [10]. The method is based on the inductive algorithm to find the approximate solutions. Another performance showing a variation iteration method is to find the approximate solutions relying on integrating the equations approaching exact solutions, [11]. This method also has the best performance when the errors approach zero. Moreover, a combination of a Laplace transform method has depicted effective results in applied physics and biology science, [12]. Laplace transforms have been demonstrated successfully when solving the economic-financial equations, and space-time fractional telegraph equations, [13], [14]. In addition, the Adomian decomposition method is applied to find the zeroes of Volterra equations and depict the approximate solutions on Mapple. The results showed the important summary of solving some complicated Lighthill singular integral equations, [15].

Based on the effective methods mentioned above, the reduce transform method was introduced and performed with the effective application, [16], [17], [18], [19], [20], [21], [22]. The technique is applied to solve many kinds of partial equations composed of heat and wave equations using linear or nonlinear terms in normal or fractional derivative, [23], [24], [25], [26], [27], [28], [29]. The technique has also illustrated the approximate solution and approach to the exact solution when nth terms come to infinity where fractional integration problems are considered, [30]. Many researchers are supporting the facilities to find the best illustration of the solutions in many kinds of fractional differential equations, [31]. Two terms of space and time fractional derivative are considered for applying the integration of fractional derivative of the fractional term and the left one keeping on for FRDTM, [32], [33], [34], [35], [36], [37], [38], [39], [40], [41], [42]. This paper will integrate FRDTM and propositions of fractional derivatives to find solutions to the one-dimensional fractional differential equations, [43], [44], [45], [46], [47], [48], [49], [50], [51], [52]. The new integration will be expressed using the new modification for the fractional reduced derivative transformation method and the analytic solutions of the fractional telegraph equations written (1.1) as the following, [31], [53], [54], [55], [56], [57], [58], [59], [60], [61], [62]:

$$ {}_0^C\mathscr{D}_x^{\alpha} f(x,t) = {}_0^C\mathscr{D}_t^{\beta} f(x,t) + N[f(x,t)] + L[f(x,t)], \quad (1.1) $$

with the initial condition as the following $f(0,t) = g(t), f_t(0,t) = h(t)$, where ${}_0^C\mathscr{D}_x^{\alpha} = \frac{\partial^\alpha}{\partial x^\alpha}$, and ${}_0^C\mathscr{D}_t^{\beta} = \frac{\partial^\beta}{\partial t^\beta}$, $0 < \alpha \leq 1$, $1 < \beta \leq 2$ are Caputo's derivatives. L, N denote the linear, and nonlinear operator existing partial derivatives, $f(x,t)$ is a given function, [14], [30], [63], [64], [65], [66], [67].

2 Fundamental Functions

We will summarize some related propositions in fractional derivative theory, [27], [28]. First of all, we consider some definitions as follows

Definition 1 (Gamma function). *Given values $z \in \mathbb{C}$, and $Re(z) > 0$, the integration is defined*

$$ \Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt. $$

We have some specific identities related to the gamma function

$$ \Gamma(z+1) = z\Gamma(z), \quad \Gamma(n+1) = n!. $$

Definition 2. *[45] Mittag-Leffler functions are usefully deployed in the form of solutions given below*

$$ E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha+\beta)}, \quad \alpha > 0, \beta > 0. $$

Based on the definition, we have some useful identities as follows

$$ E_{1,1}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k+1)} = e^z; E_{1,2}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k+2)} = \frac{e^z - 1}{z}; $$

$$ E_{1,3}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k+3)} = \frac{e^z - 1 - z}{z^2}; $$

$$ E_{2,1}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(2k+1)} = \cosh(z); $$

$$ E_{2,2}(z^2) = \sum_{k=0}^{\infty} \frac{z^{2k}}{\Gamma(2k+2)} = \frac{\sinh(z)}{z}. $$

Definition 3. *[28] Considering a continuous function $y = f(x)$ with an arbitrary constant $n-1 < \alpha \leq n$, the Caputo's fractional derivative of the order α is given by*

$$ {}_0^C\mathscr{D}_x^{\alpha} f(x) = \frac{1}{\Gamma(n-\alpha)} \int_a^x (x-s)^{n-\alpha-1} f^{(n)}(s) ds. \quad (2.1) $$

Based on Definition 3, we have derivative of function $f(x) = C$, (C is a constant)

$$ {}_0^C\mathscr{D}_x^{\alpha} f(x) = 0. $$

The inverse operator of ${}_0^C\mathscr{D}_x^\alpha$, called J_x^α is fractional integral operator of order α is given as the following

$$J^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-s)^{\alpha-1} f(s)ds. \qquad (2.2)$$

We have some of Caputo's fractional derivative properties

$$J^\alpha \left({}_0^C\mathscr{D}_x^\alpha\right) f(x) = f(x) - \sum_{k=0}^{n-1} f^{(k)}(0)\frac{x^k}{k!}, x>0. \quad (2.3)$$

For particular values, we have some useful propositions

Definition 4. *For $m-1<\alpha\le m$, we have the following propositions, [29], [59]:*

Caputo's fractional derivatives of $f(x)=x^\beta$:

$${}_0^C\mathscr{D}_x^\alpha f(x) = \frac{\Gamma(1+\beta)}{\Gamma(1+\beta-\alpha)} x^{\beta-\alpha}, \beta>-1, \beta\in\mathscr{R}.$$

Caputo's fractional derivatives of $f(x)=e^{\lambda x}$:

$${}_0^C\mathscr{D}_x^\alpha f(x) = \lambda^m x^{m-\alpha} E_{1,m-\alpha+1}(\lambda x). \qquad (2.4)$$

Caputo's fractional derivatives of $f(x)=\sin\lambda t$:

$${}_0^C\mathscr{D}_x^\alpha f(x) = \frac{1}{2i}(i\lambda)^m x^{m-\alpha}(E_{1,m-\alpha+1}(i\lambda x) - (-1)^m x^{m-\alpha} E_{1,m-\alpha+1}(-i\lambda x)). \qquad (2.5)$$

Caputo's fractional derivatives of $f(x)=\cos\lambda t$:

$${}_0^C\mathscr{D}_x^\alpha f(x) = \frac{1}{2}(i\lambda)^m x^{m-\alpha}(E_{1,m-\alpha+1}(i\lambda x) + (-1)^m x^{m-\alpha} E_{1,m-\alpha+1}(-i\lambda x)). \qquad (2.6)$$

Caputo's fractional derivatives of the function given by $f(x)=(x-a)^{\beta-1}E_{\alpha,\beta}^{p,q}(\lambda(x-a)^\alpha)$ is

$${}_0^C\mathscr{D}_{x,a^+}^\mu f(x) = (x-a)^{\beta-\mu-1} E_{\alpha,\beta-\mu}^{p,q}(\lambda(x-a)^\alpha), (2.7)$$

where

$$E_{\alpha,\beta}^{p,q}(x) = \sum_{k=0}^\infty \frac{p_{qk}x^k}{\Gamma(k\alpha+\beta)k!}, p_{qk} = \frac{\Gamma(p+qk)}{\Gamma(p)},$$
$$\mu,\alpha,\beta,p,q\in\mathbb{C}, Re(\mu,\beta)>0, Re(p)>0, q\in\mathbb{N}.$$

3 Methods

In this section, we will illustrate the results of the FRDTM, [55]. Given T_f be the transformation of this method, and $g(x,t), h(x,t)$ are fundamentally analytic functions corresponding to $G_k = T_f(g), H_k = T_f(h)$ the output of the transform. Using the FRDTM, some basic functions are created shown in Table 1, [42]. FRDTM is to find the approximate solution for fractional differential equations, [34]. Regarding two-variable functions expressed as $f(x,t)$, we set up the following steps, [32]

Step 1: Integrating the Eq. (1.1), and applying the Caputo's fractional derivatives properties Eq. (2.3) to transform the space fractional differential terms.

Step 2: Express the terms in the form

$$F_k(x) = (\sum_{i=0}^\infty u(i)x^i)(\sum_{j=0}^\infty v(j)t^j) = \sum_{k=0}^\infty H_k(i,j)t^k,$$

where $H_k(i,j) = u(i)v(j)$ is the compression of $F(x,t)$.

The term of fractional reduced differential transform method of $F(x,t)$ is formed by

$$H_k(x) = \frac{1}{\Gamma(k\alpha+1)}[\frac{\partial^{k\alpha}F_k(x,t)}{\partial t^{k\alpha}}]_{t=t_0}, \qquad (3.1)$$

where α denotes the order of time fractional derivative.

Step 3: Then the inverse transformation of H_k is defined by

$$F_k(x,t) = \sum_{k=0}^\infty H_k(t)(t-t_0)^{k\alpha}. \qquad (3.2)$$

Combine Eq. (3.1) and Eq. (3.2), we have

$$F_k(x,t) = \sum_{k=0}^\infty \frac{1}{\Gamma(k\alpha+1)}[\frac{\partial^{k\alpha}F_k(x,t)}{\partial t^{x\alpha}}]_{x=x_0}(t-t_0)^{k\alpha}. (3.3)$$

We can choose $t_0=0$, from Eq. (3.3), we have

$$F_k(x,t) = \sum_{k=0}^\infty \frac{1}{\Gamma(k\alpha+1)}[\frac{\partial^{k\alpha}F_k(x,t)}{\partial t^{k\alpha}}]t^{k\alpha}. \qquad (3.4)$$

Applying the inductive method for equation (3.4), we have

$$f(x,t) = \lim_{k\to\infty} F_k(x,t).$$

In a special case, the fractional derivative is the component of the integer and fractional derivative, we suppose

$$f(x,t) = \sum_{k=0}^\infty F_k(x)(t-t_0)^{\frac{k}{\alpha}},$$

where α is the order of the fractional derivative and F_k is the transformation of $f(x)$.

With $k=0,1,...,(\alpha q-1)$, we apply the transformation formula

$$F_k(x,t) = \begin{cases} 0 & \text{if } \frac{k}{\alpha}\notin\mathbb{Z}^+ \\ \frac{1}{(\frac{k}{\alpha})!}(\frac{d^{\frac{k}{\alpha}}}{dt^{\frac{k}{\alpha}}}f(x,t))_{t=t_0} & \text{if } \frac{k}{\alpha}\in\mathbb{Z}^+. \end{cases}$$

Table 1. Basic results using FRDTM method

Original Functions	FRDTM for some fundamental functions
$f(x,t)=ag(x,t)\pm bh(x,t)$	$F_k(x)=aG_k(x)\pm bH_k(x)$
$f(x,t)=g(x,t)h(x,t)$	$F_k(x)=\sum_{r=0}^{k}G_r(x)H_{k-r}(x)$
$f(x,t)=(x-x_0)^p$	$F_k(x)=\delta(k-\alpha p)=\begin{cases}1 & \text{if} \quad k=\alpha p\\ 0 & \text{if} \quad k\neq\alpha p\end{cases}$
$f(x,t)={}_0^C\mathscr{D}_x^{\alpha}g(x,t)$	$F_k(x)=\frac{\Gamma(\alpha(k+1)+1)}{\Gamma(1+k\alpha)}G_{k+1}(x)$
$f(x,t)=\sin t$	$H_k(x)=\frac{\sin(\frac{k\pi}{2})}{k!}$
$f(x,t)=ax^mt^n$	$H_k(x)=ax^m\delta(k-m)$
$f(x,t)=e^t$	$H_k(x)=\frac{1}{k!}$

4 Applications

In this section, we illustrate the solutions of one-dimensional fractional differential equations by using the extension of FRDTM via the examples as the following, [26], [46], [47], [48], [52]:

Example 1. *Consider the equation, [49], in the form*

$$ {}_0^C\mathscr{D}_t^{\alpha}f(x,t)-{}_0^C\mathscr{D}_x^{\beta}f(x,t)-f(x,t)=0, \quad (4.1)$$

satisfy the terminal conditions

$$f(x,0)=1+\sin x, f(0,t)=e^t, f_x(0,t)=1,$$

where $0<\alpha\leq 1, 1<\beta\leq 2$.

Transform the J_x^{β} on both sides Eq. (4.1) using Eq. (2.3), we have

$$J_x^{\beta}\left({}_0^C\mathscr{D}_t^{\alpha}\right)f(x,t)=f(x,t)-(x+e^t)+J_x^{\beta}f(x,t). (4.2)$$

Using FRDTM on both sides for Eq. (4.2), we have

$$J_x^{\beta}H_{k+1}(x)=\frac{\Gamma(k\alpha+1)}{\Gamma(\alpha(k+1)+1)}[H_k(x)-(\delta(k)x+\frac{1}{k!}) + J_x^{\beta}H_k(x)].$$

(Derivative of $\sin x$ and x by Caputo's fractional derivatives of the order β)
Transform the initial condition $H_0(x)=1+\sin x$, using iterative method, Eq.(2), Eq. (2.5), we have

$$H_1(x)=\frac{1}{\Gamma(\alpha+1)}[{}_0^C\mathscr{D}_x^{\beta}(\sin x-x)+1-\sin x];$$

$$H_2(x)=\frac{\Gamma(1+\alpha)}{\Gamma(2\alpha+1)}[{}_0^C\mathscr{D}_x^{\beta}(H_1-1)+H_1];$$

$$H_3(x)=\frac{\Gamma(2\alpha+1)}{\Gamma(3\alpha+1)}[{}_0^C\mathscr{D}_x^{\beta}(H_2-\frac{1}{2})+H_2];$$

$$H_4(x)=\frac{\Gamma(3\alpha+1)}{\Gamma(4\alpha+1)}[{}_0^C\mathscr{D}_x^{\beta}(H_3-\frac{1}{3})+H_3];\cdots.$$

The solution gained is

$$f(x,t)=\sum_{k=0}^{\infty}H_k(t)x^{k\alpha}$$
$$=\sin x+1+H_1(x)t^{\alpha}+H_2(x)t^{\alpha}+H_3(x)t^{\alpha}+\cdots.$$

This solution will be convergent to the exact solution when $\beta=2$

$$f(x,t)=\sin x+\sum_{k=0}^{\infty}\frac{t^{k\alpha}}{\Gamma(k\alpha+1)}=\sin x+E_{1,1}(t^{\alpha}).$$

When $\alpha=1,\beta=2$, the exact solution attained is $f(x,t)=\sin x+e^t$. The solutions are performed in Figure 1, Figure 2, and Table 2 in Appendix. Table 2 (Appendix) shows the solution by comparing the values $\alpha_1=0.5,\alpha_2=0.7,\alpha=1$. error_1 shows the difference between α_1,α, and error_2 shows the difference between α_2,α.

Example 2. *Consider the equation formed, [50],*

$$ {}_0^C\mathscr{D}_x^{\frac{3}{2}}f(x,t)={}_0^C\mathscr{D}_t^{2}f(x,t)+\frac{\partial}{\partial t}f(x,t)+f(x,t), \quad (4.3)$$

satisfy the terminal condition $f(0,t)=e^{-t}, f_x(0,t)=e^{-t}, 0<\alpha\leq 1, 1<\beta\leq 2.$

Transform Eq. (4.3) : $\alpha=\frac{3}{2},\beta=2(p=3,q=\frac{1}{2}),$:
Apply Table 1, and using condition $f_x(0,t)=e^{-t}$, taking FRDTM, we have

$$H_{k+p}(t)=\frac{\Gamma(kq+1)}{\Gamma(kq+pq+1)}[\frac{\partial^2}{\partial t^2}H_k(t)+\frac{\partial}{\partial t}H_k(t)+H_k(t)].$$

We establish the inductive method to calculate the terms of the transform as follows

$$H_0(t)=e^{-t};H_1(t)=0;H_2(t)=e^{-t};$$

$$H_3(t)=\frac{e^{-t}}{\Gamma(\frac{5}{2})};H_4(t)=0;H_5(t)=\frac{e^{-t}}{\Gamma(\frac{7}{2})};\cdots.$$

The analytic solution will be gathered as the following

$$f(x,t) = \sum_{k=0}^{\infty} H_k(t)x^{kq} = e^{-t} + e^{-t}x + \frac{e^{-t}}{\Gamma(\frac{5}{2})}x^{1.5} + \frac{e^{-t}}{\Gamma(\frac{7}{2})}x^{2.5} + \cdots .$$

Example 3. *Consider the equation formed*

$$ {}_0^C\mathscr{D}_x^{\alpha} f(x,t) = {}_0^C\mathscr{D}_t^{\beta} f(x,t) + \frac{\partial}{\partial t} f(x,t) + f(x,t), \quad (4.4)$$

satisfy the terminal condition $f(0,t) = e^{-t}, 0 < \alpha \leq 1, 1 < \beta \leq 2$.

Transform the Eq. (4.4) by taking the J^{β} on both sides, we have

$$J_t^{\beta}\left({}_0^C\mathscr{D}_x^{\alpha}\right) f(x,t) = f(x,t) - f(x,0) - t f_t(x,0) + J_t^{\beta}[f_t(x,t) + f(x,t)]. \quad (4.5)$$

Apply the J_x^{β} transform formula on Table 1 for the Eq. (4.5), we have

$$J_t^{\beta} H_{k+1}(t) = \frac{\Gamma(k\alpha+1)}{\Gamma(\alpha(k+1)+1)}[H_k(t) - \frac{1}{k!} + \frac{t}{k!} + J_t^{\beta}[\frac{\partial}{\partial t}H_k(t) + H_k(t)]]. \quad (4.6)$$

From the initial conditions, using the inductive method and Eq. (2), Eq. (2.4), we have

$$H_0(t) = e^{-t}; H_1(t) = \frac{1}{\Gamma(\alpha+1)} {}_0^C\mathscr{D}_t^{\beta}[e^{-t} - 1 + t];$$
$$H_2(t) = \frac{\Gamma(\alpha+1)}{\Gamma(2\alpha+1)}[{}_0^C\mathscr{D}_t^{\beta}(H_1 - 1 - t) + \frac{\partial}{\partial t}H_1(t) + H_1(t)];$$
$$H_3(t) = \frac{\Gamma(2\alpha+1)}{\Gamma(3\alpha+1)}[{}_0^C\mathscr{D}_t^{\beta}(H_1 - \frac{1}{2!} - \frac{t}{2!}) + \frac{\partial}{\partial t}H_1(t) + H_1(t)];$$
$$H_4(t) = \frac{\Gamma(2\alpha+1)}{\Gamma(3\alpha+1)}[{}_0^C\mathscr{D}_t^{\beta}(H_3 - \frac{1}{3!} - \frac{t}{3!}) + \frac{\partial}{\partial t}H_3(t) + H_3(t)];$$
$$H_5(t) = \frac{\Gamma(2\alpha+1)}{\Gamma(3\alpha+1)}[{}_0^C\mathscr{D}_t^{\beta}(H_4 - \frac{1}{4!} - \frac{t}{4!}) + \frac{\partial}{\partial t}H_4(t) + H_4(t)]; \cdots .$$

The analytic solution is demonstrated as follows

$$f(x,t) = e^{-t} + H_1(t)x^{\alpha} + H_2(t)x^{2\alpha} + H_3(t)x^{3\alpha} + H_4(t)x^{4\alpha} + H_5(t)x^{5\alpha} + \cdots . \quad (4.7)$$

When $\alpha = 1, \beta = 2$, the exact solution is $f(x,t) = e^{x-t}$. The solutions are depicted in Figure 3, Figure 4, and Table 3 in Appendix. Table 3 (Appendix) shows the numerical solutions by comparing the values $\alpha_1 = 0.2, \alpha_2 = 0.5, \alpha = 1$. error_1 shows the difference between α_1, α, and error_2 shows the difference between α_2, α.

Example 4. *We consider the equation formed [51]:*

$$ {}_0^C\mathscr{D}_x^{\alpha} f(x,t) = {}_0^C\mathscr{D}_t^{\beta} f(x,t) + \frac{\partial}{\partial t} f(x,t) - x^2 - t + 1, \quad (4.8)$$

satisfy the terminal condition

$$f(0,t) = t, f_x(0,t) = 0, f(x,0) = x^2,$$
$$f_t(x,0) = 1, 1 < \alpha \leq 2, 1 < \beta \leq 2.$$

Taking the J_x^{β} transformation on both sides using Eq. (2.3) for Eq. (4.8), we have:

$$J_t^{\beta} f(x,t) = f(x,t) - f(x,0) - t\frac{\partial}{\partial t} f(x,0) + J^{\beta}[\frac{\partial}{\partial t} f(x,t) - x^2 - t + 1].$$

Apply the properties in Table 1, we have

$$J_t^{\beta} H_{k+1}(t) = \frac{\Gamma(k\alpha+1)}{\Gamma(k\alpha+\alpha+1)}\{H_k(t) - \delta(k-2) + t\delta(k) + J_t^{\beta}[H_k(t) - \delta(k-2)] + (1-t)\delta(k)\}.$$

Using the inductive method, we calculate the following terms

$$H_0(t) = t; H_1(t) = \frac{1}{\Gamma(\alpha+1)}[{}_0^C\mathscr{D}_x^{\beta}(t+1) + 1];$$

$$H_2(t) = \frac{\Gamma(\alpha+1)}{\Gamma(2\alpha+1)}[{}_0^C\mathscr{D}_x^{\beta}(H_1) + H_1];$$
$$H_3(t) = \frac{\Gamma(2\alpha+1)}{\Gamma(3\alpha+1)}[{}_0^C\mathscr{D}_x^{\beta}(H_2 - 1) + H_2 - 1];$$
$$H_4(t) = \frac{\Gamma(3\alpha+1)}{\Gamma(4\alpha+1)}[{}_0^C\mathscr{D}_x^{\beta}(H_3) + H_3];$$
$$H_5(t) = \frac{\Gamma(4\alpha+1)}{\Gamma(5\alpha+1)}[{}_0^C\mathscr{D}_x^{\beta}(H_4) + H_4]; \cdots .$$

The analytics solution of the equation is

$$f(x,t) = t + H_1(t)x^{\alpha} + H_2(t)x^{2\alpha} + H_3(t)x^{3\alpha} + H_4(t)x^{4\alpha} + H_5(t)x^{5\alpha} + \cdots .$$

When $\alpha = 2, \beta = 2$, the exact solution becomes

$$f(x,t) = t + x^2.$$

The graphs of solutions are portrayed in Figure 5, Figure 6, and Table 4 in Appendix. Table 4 (Appendix) shows the solution by comparing the values $\alpha_1 = 1.2, \alpha_2 = 1.5, \alpha = 2$. error_1 shows the difference between α_1, α, and error_2 shows the difference between α_2, α.

Example 5. *We consider the equation formed, [50]:*

$$ {}_0^C\mathscr{D}_t^\beta f(x,t)+x\frac{\partial}{\partial x}f(x,t)+\frac{\partial^2}{\partial x^2}f(x,t)=2t^\beta+2x^2+2, $$

satisfy the terminal condition $f(x,0)=x^2, t>0, 0<\beta\leq 1$.

Taking the J_x^β transformation on both sides, we have the expression

$$ f(x,t)-f(x,0)-J_t^\beta[x\frac{\partial}{\partial x}f(x,t)+\frac{\partial^2}{\partial x^2}]=J_t^\beta[2t^\beta+2x^2+2], $$

Applying the Table 1, we have the following equation

$$ H_k(t)-\delta(k-2)+J_t^\beta[\sum_{r=0}^{k}\delta(r-1)H_{k+1-r}(t) + (k+1)(k+2)H_{k+2}(t)] $$

$$ =2\frac{\Gamma(\beta+1)}{\Gamma(2\beta+1)}t^{2\beta}+J_t^\beta[2\delta(k-2)+2\delta(k)]. $$

Using the identity method and solving simultaneous equations, we have the following terms

$$ H_0(t)=2\frac{\Gamma(\beta+1)}{\Gamma(2\beta+1)}t^{2\beta}; H_1(t)=0; $$

$$ H_2(t)=1; H_3(t)=0; H_4(t)=0; H_5(t)=0; \cdots. $$

The solution is formed to the exact solution

$$ f(x,t)=2\frac{\Gamma(\beta+1)}{\Gamma(2\beta+1)}t^{2\beta}+x^2. $$

The analytic solutions are illustrated in Figure 7, Figure 8, and Table 5 in Appendix. Table 5 (Appendix) shows the solutions by comparing the values $\beta_1=0.4, \beta_2=0.5, \beta=1$ where error_1 shows the difference between β_1, β, and error_2 shows the difference between β_2, β.

5 Discussion

This study introduces an enhanced version of the FRDTM method to solve a class of multi-fractional space-time telegraph equations. The results show that by eliminating one of the fractional derivative terms, the modified FRDTM can effectively simplify the problem and lead to accurate analytical solutions. Compared to existing methods, such as the Adomian Decomposition Method and the Laplace Transform Method, the modified FRDTM offers a more direct and computationally efficient approach to solving fractional differential equations . However, the method also has some limitations. In cases where the fractional derivatives involve transcendental terms or anomalous series, the FRDTM's performance can be hindered by difficulties in approximating the fractional series. This suggests that while the FRDTM is effective for many types of fractional telegraph equations, further refinement may be needed when applied to more complex or highly nonlinear equations. Future research could explore hybrid methods or extensions of the FRDTM to better address these challenges. The versatility of the FRDTM in fractional calculus applications, particularly in engineering and physics, highlights its potential for broader applications. Specifically, the methods ability to handle fractional time-space equations suggests promising opportunities in fields requiring long-term behavior predictions, such as finance, electrical transmission systems, and fluid dynamics.

6 Conclusions

In conclusion, this paper demonstrates the effectiveness of a modified Fractional Reduced Differential Transform Method (FRDTM) for solving multi-fractional telegraph equations. The method simplifies complex fractional differential equations by isolating one variable and transforming the remaining terms. The resulting analytical solutions highlight the potential of FRDTM for solving large-scale fractional partial differential equations. Although this method provides a more efficient route to exact solutions, its limitations in handling anomalous series and transcendental terms suggest that further work is needed. Future research should focus on enhancing the FRDTM for more complex fractional problems and exploring its applications in diverse fields such as economics, physics, and engineering. This study contributes to the ongoing development of fractional calculus and provides a strong foundation for future work in solving space-time fractional equations in various scientific and industrial applications.

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APPENDIX

Table 2: Numerical comparison of the example 1.

t	$\alpha_1 = 0.5$	$\alpha_2 = 0.7$	$\alpha = 1$
0.0	1.4794	1.4794	1.4794
0.1	1.9363	1.7347	1.5846
0.2	2.1846	1.9362	1.7008
0.3	2.4008	2.1387	1.8293
0.4	2.6053	2.3486	1.9713
0.5	2.8106	2.5681	2.1281
0.6	3.0290	2.7985	2.3015
0.7	3.2746	3.0406	2.4932
0.8	3.5643	3.2947	2.7050
0.9	3.9181	3.5611	2.9390
1.0	4.3601	3.8401	3.1977

Table 3: Numerical comparison of the example 3.

t	$\alpha = 0.2$	$\alpha = 0.5$	$\alpha = 1$
0.0	1.0000	1.0000	2.0138
0.1	1.4078	1.3567	1.8221
0.2	1.6494	1.5651	1.6487
0.3	1.8734	1.7585	1.4918
0.4	2.0884	1.9445	1.3499
0.5	2.2955	2.1241	1.2214
0.6	2.4937	2.2963	1.1052
0.7	2.6815	2.4598	1.0000
0.8	2.8573	2.6130	0.9048
0.9	3.0192	2.7541	0.8187
1.0	3.1655	2.8816	0.7408

Table 4: Numerical comparison of the example 4.

t	$\alpha = 1.2$	$\alpha = 1.5$	$\alpha = 2$
0.1	0.4260	1.2508	0.3500
0.2	0.4089	0.5918	0.4500
0.3	0.4722	0.5340	0.5500
0.4	0.5535	0.5715	0.6500
0.5	0.6419	0.6387	0.7500
0.6	0.7338	0.7186	0.8500
0.7	0.8277	0.8049	0.9500
0.8	0.9229	0.8949	1.0500
0.9	1.0190	0.9873	1.1500
1.0	1.1157	1.0812	1.2500

Table 5: Numerical comparison of the example 5.

t	$\beta = 0.4$	$\beta = 0.5$	$\beta = 1$
0.1	0.5520	0.4272	0.2600
0.2	0.7757	0.6045	0.2900
0.3	0.9772	0.7817	0.3400
0.4	1.1654	0.9590	0.4100
0.5	1.3443	1.1362	0.5000
0.6	1.5161	1.3135	0.6100
0.7	1.6823	1.4907	0.7400
0.8	1.8438	1.6680	0.8900
0.9	2.0012	1.8452	1.0600
1.0	2.1553	2.0225	1.2500
1.1	2.3062	2.1997	1.4600

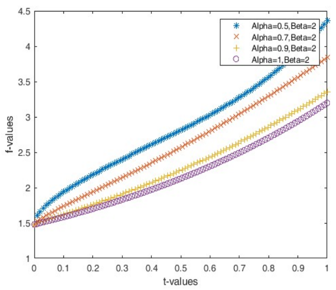

Figure 1: Solution performance of example 1 for $x = 0.5$.

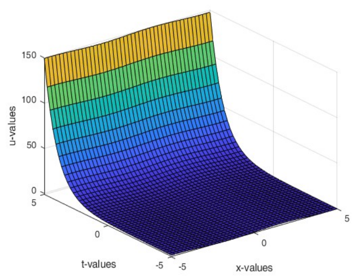

Figure 2: Exact solution of example 1 using FRDTM, $\alpha = 1, \beta = 2$.

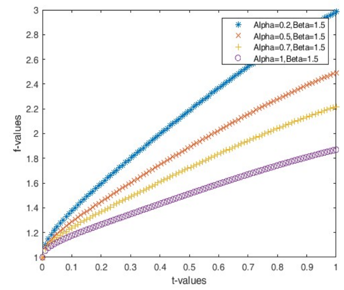

Figure 3: Solution performance of example 3 for $x = 0.5$.

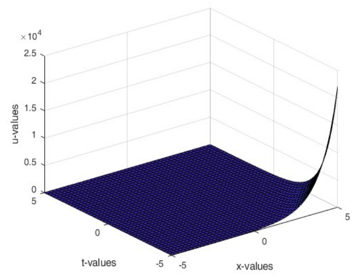

Figure 4: Exact solution of example 3 using FRDTM, $\alpha = 1, \beta = 2$.

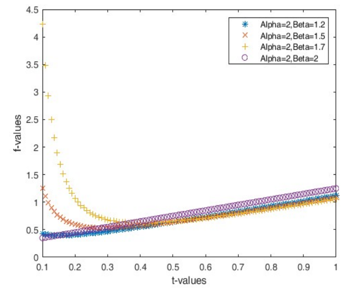

Figure 5: Solution performance of example 4 for $x = 0.5$.

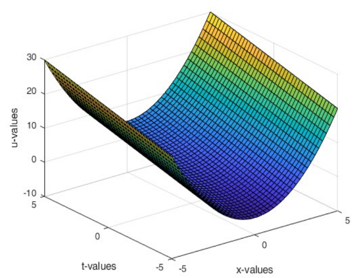

Figure 6: Approximate solution of example 4, $\alpha = \frac{3}{2}, \beta = 2$.

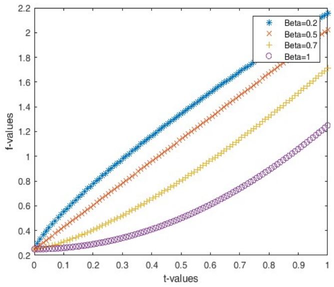

Figure 7: Solution performance of example 5 for $x = 0.5$, $\beta = 1$.

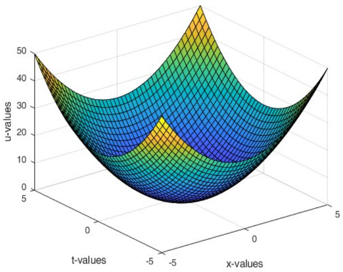

Figure 8: Exact solution of example 5 using FRDTM, $\beta = 1$.

Contribution of Individual Authors to the Creation of a Scientific Article (Ghostwriting Policy)

Nguyen Minh Tuan: Conceptualization, data curation, investigation, methodology, software, visualization, writing-original draft and writing-review and editing, validation, visualization, writing-original draft and writing-review and editing. Phayung Meesad, Piwan Wongsashinchai, Nguyen Hong Son: methodology, resources, supervision, validation, visualization, and writing review and editing.

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