A Direct Relaxation-Iteration Technique for the Troesch Problem

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Abstract: In the work reported herein, we propose a direct relaxation-iteration technique for the solution of a highly unstable parameter-sensitive nonlinear boundary value differential equation known as the Troesch problem. It arises in the confinement of plasma column by radiation pressure. The proposed technique is based on an SOR successive relaxation method and guarantees that convergence is achieved straightforwardly. Numerical experiments conducted to investigate the overall influence of sensitivity parameter on the solution profiles confirm the relative advantage of the proposed numerical technique over previous methods.

Key-words: Troesch , relaxation-iteration, SOR, plasma column, sensitivity parameter, nonlinear, numerical, differential equation

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1. Introduction

Some models which are based on algorithms that seek equilibrium states sometimes do not converge. These include the Newton method and some of its variants. For convergence to be achieved, model parameters need to be adjusted or totally new approaches devised. This can pose a huge numerical challenge especially when large calculations need to be done. However, the Newton-Raphson iteration does converge easily when the initial guess is very close to a projected equilibrium but not very far away from it. Far from equilibrium, the algorithm can either get stuck in a local minimum or exhibit a chaotic behavior as the computation proceeds. The Jacob matrix which is simply the derivative of the function gets very small, yields a large inverse singular matrix which may be accompanied by imaginary eigenvalues and limit cycles.

We shall now turn our main focus to the theme of this work namely to seek equilibrium or bounded solutions for a parameter based nonlinear highly sensitive second order differential equation known as the Troesch problem. The motivation to avoid the computation of the Jacobian matrix together with its variants as well as other restrictions that are indicative of Newton-type approaches is deliberate. However, the major thrust here is to devise a simple iteration based numerical technique that can handle a range of sensitivity parameter values in such a way that computation initiated anywhere in the solution-space will eventually converge.

Throesch equation is a highly sensitive nonlinear boundary value problem given by

$$u'' = \lambda \sinh(\lambda u) \qquad 0 \le x \le 1 \quad (1a)$$
$$u(0) = 0, \qquad u(1) = 1 \qquad (1b)$$

where λ is highly sensitive and is known as the Troesch parameter, . Equation (1) is applicable to the investigation of the confinement of plasma column by radiation pressure [1] as well to the study of porous electrodes [2]. Robert and Shipman [3] obtained its closed form solution in terms of the Jacobi elliptic functions $sc(u|m)$; which is expressed as:

$$u(x) = {}^{2}\!/_{\lambda} \sinh^{-1}\left[\frac{u'(0)}{2} sc\left(\lambda x \middle| 1 - 0.25\left(u'(0)\right)^2\right)\right] \quad (2)$$

where $u'(0) = 2\sqrt{1-2m}$; m satisfies the transcendental equation

$$sc(\lambda|m) = \sinh(\lambda/\ 2)\Big/\sqrt{1-m} = \sin\phi/\cos\phi = \tan\phi \quad (3)$$

The variables ϕ, λ *and* m are related to each other by the integral.

$$\lambda = \int_0^{\phi} \frac{d\phi}{\sqrt{1-m\sin^2\phi}} d\phi \quad (4)$$

It was observed that the dependent variable $u(x)$ has a singularity $sc\left(\lambda x \middle| u^{'}(0)\right)$ which occurs at $x \approx 1/2\lambda \ln\left(16/(1-m)\right)$. This makes the solution of equation (1) a difficult task for numerical methods. Many attempts have been made to overcome this challenge. A good many of them can be found in [4-9]. In the present paper, we develop a direct and simple iteration numerical technique to handle the Troesch

2.Numerical formulation

The SOR method and its modifications is one of the most highly used iteration technique for handling a system of equations [10-12]. The main thrust of this method relies on the swapping of old and new values of the dependent variables at grid points as well as the application of an iteration parameter whose optimum value is determined. The number of iterations implemented within a certain error tolerance serve as criteria for determining the performance of the scheme. In addition, a finite difference discretization coupled with straightforward iteration simplify the whole procedure and provide a complexity reduction approach.

Fundamentally, let us consider a matrix :

$$Ax = b = (D-L-U)x \quad (5)$$

where D, L *and* U refer to the diagonal, lower and upper components of matrix A. The Jacobi iteration technique to solve a system of equations is given as :

$$x^{(k)} = D^{-1}(F+G)x^{(k-1)} + D^{(-1)}b \quad (6)$$

From equation (6), we obtain the Gauss-Seidel method which is written as :

$$x^k = (D-L)^{(-1)}Ux^{(k-1)} + (D-L)^{(-1)}b \quad (7)$$

for $k = 1,2,3.....n$ where k is the iteration counter.

Further modification of equation (7) by introducing a relaxation parameter, yields the SOR iteration method:

$$(D-\omega L)x^{(k)} = \left[\omega U + (1-\omega)D\right]x^{(k-1)} + \omega b \quad (8)$$

The vector based representation of the SOR technique is finally given as:

$$x^{(k)} = (D-\omega L)^{(-1)}\left[\omega U + (1-\omega)D\right]x^{(k-1)} + \omega(D-\omega U)^{(-1)}b \quad (9)$$

In this work the SOR is modified to deal with large sparse systems that come with the discretization of nonlinear differential equations represented compactly as $F(u) = 0$ *in* $\mathbf{R^n}$ The methodology is demonstrated by iteration on a discretized system of equations while achieving accurate convergent results and low computer memory. Applying the FDM discretization, equation (1) is written as

$$u_i^{(k+1)} = 1/2\left[u_{i-1}^{(k)} + u_{i+1}^{(k)} - \Delta x^2(\lambda\sinh\lambda u_i)\right] \quad (10)$$

We can now derive the nonlinear analog of the SOR method for nonlinear differential equations involving tridiagonal positive definite matrices . For any $u^0 = \left(u_1^0, u_2^0, u_3^0.........u_n^0\right)$. In the process, we generate a sequence for iteration step. At convergence or 'steady state'

$$u_i^{(k+1)} \approx u_i^{(k)} \quad (11a)$$

where $u_i^{(k+1)}$ is the solution of $F_i\left(u_1^{(k)},u_2^{(k)},u_3^{(k)}.......u_n^{(k)}\right)$.

Recast equation (11a) to read

$$u_i^{(k+1)} = u_i^{(k)} + \left(-u_i^{(k)} + u_i^{(k)}\right) \qquad (11b)$$

.

Simple algebraic manipulation yields :

$$u_i^{(k+1)} = u_i^{(k)} + \omega/2\left[u_{i-1}^{(k)} - 2u_i^{(k)} + u_{i-1}^{(k)} - \Delta x^2\left(\lambda \sinh \lambda u\right)\right] (12a)$$

where ω is a relaxation parameter. For replacement SOR, equation (12) is adjusted to read:

$$u_i^{(k+1)} = u_i^{(k)} + \omega/2\left[u_{i-1}^{(k)} - 2u_i^{(k)} + u_{i-1}^{(k+1)} - \Delta x^2\left(\lambda \sinh \lambda u\right)\right] (12b)$$

The algorithm for the computation of equation (12b) adopting the method developed in this work is given below.

i. Initialize u_i^0, $\varepsilon \leftarrow 10^{-6}$
ii. Set BCs, ω
iii. $for\ i = 2 : n-1$, iterate equation (12)
iv. Implement convergence test $\left|u_i^{(k+1)} - u_i^{(k)}\right| \le \varepsilon$
v. If iteration converges, break iteration and display numerical output
vi. λ Otherwise continue until convergence

Different values of the relaxation parameter are run to finally determine the optimal ω. For this problem, this was found to be 1.85.

3. Results and discussions

We verify the accuracy and convergence of the proposed numerical technique. Unless otherwise stated, a grid size of $h = .025$ and a convergence tolerance error of $e = 10^{-6}$ are employed. In Fig. 1 we display the graphical results for $\lambda = 1,3,5,6$.

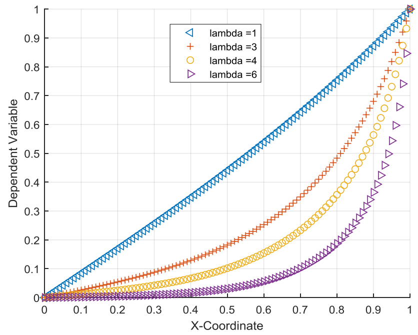

Fig.1 Numerical Solution Profiles

As λ increases, the profiles become flatter and tend towards initiating a boundary layer at the right end boundary. This the major reason why the Troesch problem poses a considerable numerical challenge fora certain range of λ values.

Fig. 2 shows the convergence profile for errors between current and previous iterations for $\lambda = 5$. The profile decreases remarkably and shows a downward linear trend .The small kink in the region of iteration equals 50 indicates the number of iterations it took the computation to stabilize.

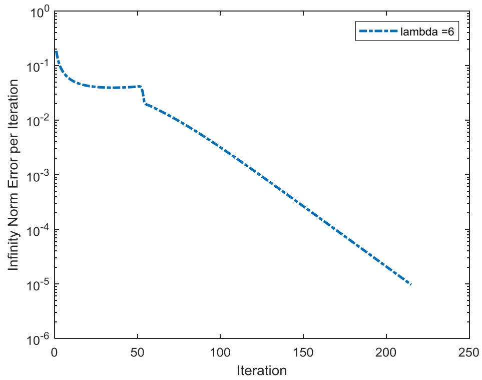

Fig. 2 Infinity norm of error between old and new values .

Next, the absolute errors obtained by comparing the numerical results with the semi-analytic homotopy perturbation results of [13]

for $\lambda = 3,5\ and\ 6$ are displayed in Figs. 3, 4 and 5

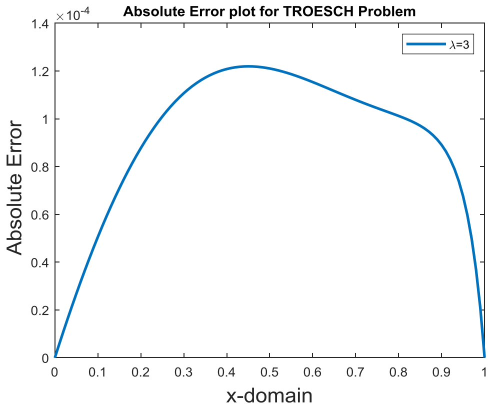

Fig. 3 :Absolute error for $\lambda = 3$

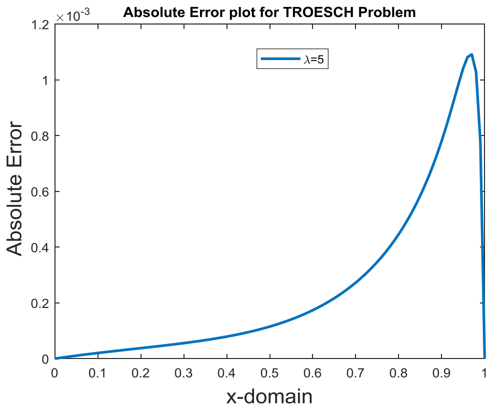

Fig. 4: Absolute error for $\lambda = 5$

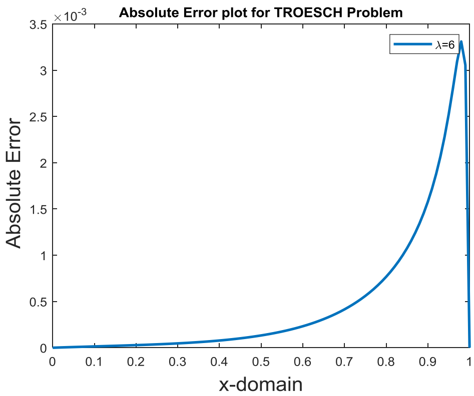

Fig. 5 Absolute error for $\lambda = 6$

It can be observed that as the value of the sensitivity parameter λ increases, the boundary layer effect and the accompanying high gradients become more prominent at the right endpoint $(x=1)$. While maximum absolute error is attained in the vicinity of the middle of the problem domain for $\lambda = 3$, The maximum absolute error is very much close to the rightside boundary for $\lambda = 5, 6$

Tables 1-4 compare the numerical results with those found in literature. The results displayed so far confirm that though the algorithm is direct, simple and straightforward, it is computationally appealing and also competitive with those reported in scientific literature.

4. Conclusion

In the work reported herein, we provided an SOR based solution for the Troesch problem. Besides, we presented a comparison between the numerical solution and similar work reported in scientific literature. Both the graphical and tabular illustrations of the solutions show that the method can be relied on to provide good results for a modest range of the Troesch sensitivity parameter λ . Moreover it has been shown [13] that due to the nonlinear term $\sinh(\lambda u(x))$ which is not analytic, some techniques like Laplace, homotopy, variation methods can not handle the Troesch problem for values of $\lambda \geq 5$. This however, is below the sensitivity values considered in this study.

On the whole we have provided an algorithm that comes with a simple formulation, and is free from restrictive numerical requirements. There is practically no need for the computation of the Jacobian matrix. In addition the requirement for first guesses to be close the to equilibrium domain is relaxed. While it is not the aim of this work to replace the more conventional approaches with the algorithm developed herein, It can be used as the starting point or predictor for Newton's based methods where the initial values may start far away from the solution and lead to unphysical results.

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Table 1. Troesch problem $\lambda = 0.5$

X	Exact	HPM [13]	ADM [14]	Current work
0.1	0.0959443493	0.0959443155	0.0959383534	0.0959489113
0.2	0.1921287477	0.1921286848	0.1921180592	0.1921369383
0.3	0.2887944009	0.2887943176	0.2887803297	0.2888055809
0.4	0.3861848464	0.3861847539	0.3861687095	0.3861979059
0.5	0.4845471647	0.4845470753	0.4845302901	0.4845611944
0.6	0.5841332484	0.5841331729	0.5841169798	o.5841466643
0.7	0.6852011483	0.685201943	0.6851868451	0.6852120551
0.8	0.7880165227	0.7880164925	0.7880055691	0.7880225823
0.9	0.8928542161	0.8928542059	0.8928480234	0.8928526032

Table 2. Results of Troesch problem $\lambda = 1.0$

X	Exact	HPM [13]	ADM [14]	Current work
0.1	0.0846612565	0.0846607585	0.084248760	0.0844894176
0.2	0.1701713582	0.1701704581	0.169430700	0.1701788828
0.3	0.2573939080	0.2573927827	0.256414500	0.2581006216
0.4	0.3472226551	0.3472217324	0.346085720	0.3479004515
0.5	0.4405998351	0.4405989511	0.439401885	0.4411998845
0.6	0.5385343980	0.5385339413	0.537365700	0.5380350371
0.7	0.6421286091	0.6421286573	0.641083800	0.6421286957
0.8	0.7526080939	0.7526085475	0.751788000	0.7526085475
0.9	0.8713625196	0.8713630450	0.870908700	0.8713634502

Table 3. Results of Troesch problem $\lambda = 5.0$

x	Doha [15]	Collocation [14]	B-spline [16]	This work
0.1	---------------	-----------------	-------------------	.00478801573
0.2	0.01078872	0.00762552	0.01002027	0.0107890233
0.3	---------------	---------------	---------------------	0.0195483593
0.4	0.03338672	0.03817903	0.03099793	0.0332747853
0.5	---------------	----------------	--------------------	0.0555475126
0.6	---------------	-----------------	-------------------	0.0922139919
0.7	---------------	------------------	-------------------	0.1534122341
0.8	0.25956596	0.23252435	0.24170496	0.2586613278
0.9	0.45706638	0.44624551	0.42461830	0.4558409106

Table 4. Results of Troesch problem $\lambda = 6.0$

x	HPM [13]	Current Work
0.1	0.0090475673	0.0019069772
0.2	0.0045760732	0.0045284096
0.3	0.0088028222	0.0083469019
0.4	0.0163568079	0.0164225025
0.5	0.0299907268	0.0301148429
0.6	0.0548272882	0.0550539664
0.7	0.1004821099	0.1008938682
0.8	0.1864571511	0.1872231560
0.9	0.3633244286	0.3649032771