# Rational solutions to the Gardner equation from particular polynomials

PIERRE GAILLARD
Institut de mathématiques de Bourgogne,
Université de Bourgogne Franche Comté,
9 avenue Alain Savary BP 47870
21078 Dijon Cedex, France :
FRANCE

*Abstract* Rational solutions to the Gardner (G) equation are constructed in terms of a quotient of determinants involving certain particular polynomials. This gives a very efficient method to construct solutions to this equation. We construct very easily explicit expressions of these rational solutions for the first orders for $n = 1$ until 8.

*Key-Words*: Gardner equation, rational solutions.

Received: October 23, 2022. Revised: February 20, 2023. Accepted: March 17, 2023. Published: May 17, 2023.

# 1 Introduction

We consider the Gardner equation in the following normalization

$$u_t+6u(u-2)u_x+u_{xxx}+6u_x=0, \quad (1)$$

where the subscripts $x$ and $t$ stand for partial derivatives.
This equation was introduced by Gardner [1] in 1968. He considered this equation as an auxiliary mathematical tool in the derivation of the infinite set of local conservation laws of the Korteweg de Vries equation. This equation is used to describe nonlinear wave effects in several physical contexts: for example, in plasma physics [2, 3], fluid flows [4], quantum fluid dynamics [5], in ocean and atmosphere [6].
A lot of research has been done to solve this equation. We can quote many methods as the Hirota method [8], the series expansion method [9], the mapping method [10] or the method of leading-order analysis [11].

Here, we used particular polynomials to construct rational solutions to the Gardner equation. We obtain rational solutions as a quotient of polynomials in $x$ and $t$. We give explicit solutions for the first orders.

# 2 Rational solutions to the Gardner equation

We consider the following polynomials $p_n(x,t)$ defined by

$$\left\{\begin{array}{l} p_n(x,t)=\sum_{l=0}^{n}\frac{x^k}{k!}\frac{t^{\left(\frac{n-k}{3}\right)}}{\left(\frac{n-k}{3}\right)!}\left(1-\left[\frac{1}{2}(n-k+1-3\left[\frac{n-k}{3}\right]\right]\right), \quad k\geq 0, \\ p_n(x,t)=0, \quad n<0, \end{array}\right. \quad (2)$$

with $[x]$ denoting the largest integer less or equal to $x$.
We denote $A_n(x,t)$ the determinant defined by

$$A_n(x,t)=\det(p_{n+1-2i+j}(x,t))_{\{1\leq i\leq n,\,1\leq j\leq n\}} \quad (3)$$

With these notations we have the following result

**Theorem 2.1** *The function $v_n(x,t)$ defined by*

$$v_n(x,t)=1-\partial_X\left(\ln\frac{A_{n+1}(X,T)}{A_n(X,T)}\right)_{|X=ix,T=4it} \quad (4)$$

*is a rational to the Gardner equation (1)*

$$u_t+6u(u-2)u_x+u_{xxx}=0,$$

**Proof :** We have proven in [12] that the function defined by

$$u_n(X,T)=\partial_X\left(\ln\frac{A_{n+1}(X,T)}{A_n(X,T)}\right)$$

is a solution the equation defined by

$$4u_T+6u^2u_X-u_{XXX}=0. \quad (5)$$

It is easy to verify that the function defined by $\tilde{u_n}$ where $T$ is replaced by $4it$ and $X$ replaced by $ix$ is a solution of the equation

$$u_t+6u^2u_x+u_{xxx}=0. \quad (6)$$

Then it is easy to check that the function defined by $v_n(x,t)=1-u_n\tilde{(x,t)}$ is a solution to the Gardner equation and we get the result.
□

# 3 Explicit construction of rational solutions to the Gardner equation for the first orders

The solutions we present in the following are singular. There is therefore no interest in constructing figures in the plan $(x, t)$ of the coordinates.

## 3.1 First order rational solution

The function $v_1(x, t)$ defined by

$$v_1(x,t) = -\frac{-x^4 - 12\,xt - 2\,ix^3 + 12\,it}{x\,(x^3 + 12\,t)} \quad (7)$$

is a rational to the Gardner equation (1).

## 3.2 Second order rational solution

The function $v_2(x, t)$ defined by

$$v_2(x,t) = \frac{-i\left(-72\,ix^6t + 8640\,it^3 - ix^9 + 72\,x^5t + 4320\,x^2t^2 + 3\,x^8\right)}{(-60\,tx^3 + 720\,t^2 - x^6)\,(x^3 + 12\,t)} \quad (8)$$

is a rational to the Gardner equation (1).

## 3.3 Third order rational solution

The function $v_3(x, t)$ defined by

$$v_3(x,t) = \frac{n(x,t)}{d(x,t)} \quad (9)$$

with
$n(x,t) = -x^{16} - 4\,ix^{15} - 240\,tx^{13} - 600\,ix^{12}t - 10080\,t^2x^{10} - 36000\,ix^9t^2 - 172800\,t^3x^7 + 2419200\,it^3x^6 - 18144000\,t^4x^4 + 36288000\,it^4x^3 + 217728000\,t^5x + 217728000\,it^5$
and
$d(x,t) = \left(x^9 + 180\,tx^6 + 302400\,t^3\right)\left(-x^6 - 60\,tx^3 + 720\,t^2\right)x$

is a rational to the Gardner equation (1).

### 3.4 Rational solution of order four

The function $v_4(x,t)$ defined by

$$v_4(x,t) = \frac{n(x,t)}{d(x,t)} \qquad (10)$$

with

$n(x,t) = -i(-ix^{25}+5\,x^{24}-600\,ix^{22}t+2280\,tx^{21}-100800\,ix^{19}t^2+352800\,t^2x^{18}-6955200\,ix^{16}t^3+4838400\,t^3x^{15}-254016000\,ix^{13}t^4+2794176000\,t^4x^{12}+39626496000\,it^5x^{10}+259096320000\,t^5x^9-365783040000\,ix^7t^6+5120962560000\,t^6x^6+76814438400000\,it^7x^4-153628876800000\,t^7x^3+460886630400000\,it^8x+460886630400000\,t^8)$

and

$d(x,t) = (-x^{15}-420\,tx^{12}-25200\,t^2x^9-2116800\,t^3x^6+254016000\,t^4x^3+1524096000\,t^5)(x^9+180\,tx^6+302400\,t^3)x$

is a rational to the Gardner equation (1).

### 3.5 Rational solution of order five

The function $v_5(x,t)$ defined by

$$v_5(x,t) = \frac{n(x,t)}{d(x,t)} \qquad (11)$$

with

$$n(x,t) = -i(ix^{36} - 6\,x^{35} + 1260\,itx^{33} - 6300\,tx^{32} + 544320\,it^2x^{30} - 2419200\,t^2x^{29} + 110073600\,it^3x^{27} - 381024000\,t^3x^{26} + 11430720000\,it^4x^{24} - 50295168000\,t^4x^{23} + 329204736000\,it^5x^{21} + 192036096000\,t^5x^{20} + 119062379520000\,it^6x^{18} + 1371137725440000\,t^6x^{17} + 7189831434240000\,it^7x^{15} + 129970029772800000\,t^7x^{14} + 1393721170329600000\,it^8x^{12} - 1161434308608000000\,t^8x^{11} - 59620294508544000000\,it^9x^9 + 319394434867200000000\,t^9x^8 - 4292661204615168000000\,it^{10}x^6 + 2146330602307584000000\,t^{10}x^5 + 128779836138455040000000\,t^{11}x^2 + 128779836138455040000000\,it^{12})$$

and

$$d(x,t) = (-x^{21} - 840\,tx^{18} - 166320\,t^2x^{15} - 16934400\,t^3x^{12} + 1397088000\,t^4x^9 - 352066176000\,t^5x^6 - 14082647040000\,t^6x^3 + 84495882240000\,t^7)(-x^{15} - 420\,tx^{12} - 25200\,t^2x^9 - 2116800\,t^3x^6 + 254016000\,t^4x^3 + 1524096000\,t^5)$$

is a rational to the Gardner equation (1).

### 3.6 Rational solution of order six

The function $v_6(x,t)$ defined by

$$v_6(x,t) = \frac{n(x,t)}{d(x,t)} \qquad (12)$$

with

$$\begin{aligned}
n(x,t) = {} & -i(ix^{49}-7\,x^{48}+2352\,itx^{46}-\\
& 14448\,tx^{45}+2116800\,it^2x^{43}-11733120\,t^2x^{42}+\\
& 979655040\,it^3x^{40}-4792435200\,t^3x^{39}+\\
& 259756761600\,it^4x^{37}-1190674598400\,t^4x^{36}+\\
& 40391592192000\,it^5x^{34}-148315878144000\,t^5x^{33}+\\
& 4855696699392000\,it^6x^{31}-8843902341120000\,t^6x^{30}+\\
& 61513002270720000\,it^7x^{28}-7215948343296000000\,t^7x^{27}-\\
& 865558918490112000000\,ix^{25}t^8-23299824023562240000000\,t^8x^{24}\\
& -291245138674237440000000\,ix^{22}t^9-19658770052000808960000000\,t^9x^{21}\\
& +1259180620020449280000000\,it^{10}x^{19}-\\
& 19426796003252994048000000\,t^{10}x^{18}-\\
& 26702499023308652544000000\,ix^{16}t^{11}+\\
& 573430854357312602112000000\,t^{11}x^{15}-\\
& 480928298059060346880000000000\,ix^{13}t^{12}+\\
& 899782139502662595379200000000\,t^{12}x^{12}-\\
& 1395306859308999990312960000000\,ix^{10}t^{13}+\\
& 6295896804199146297753600000000\,t^{13}x^9-\\
& 12251474862225365768601600000000\,ix^7t^{14}-\\
& 171520648071155120760422400000000\,t^{14}x^6-\\
& 857603240355577560380211200000000\,ix^4t^{15}-\\
& 1715206480711551207604224000000000\,t^{15}x^3+\\
& 5145619442134653622812672000000000\,it^{16}x-\\
& 5145619442134653622812672000000000\,t^{16})
\end{aligned}$$

and

$$\begin{aligned}
d(x,t) = {} & (-x^{27}-1512\,tx^{24}-680400\,t^2x^{21}-\\
& 139708800\,t^3x^{18}-5029516800\,t^4x^{15}-\\
& 3168595584000\,t^5x^{12}+604145558016000\,t^6x^9+\\
& 108746200442880000\,t^7x^6\\
& +6089787224801280000\,t^9)(-x^{21}-840\,tx^{18}-\\
& 166320\,t^2x^{15}-16934400\,t^3x^{12}+1397088000\,t^4x^9-\\
& 352066176000\,t^5x^6-14082647040000\,t^6x^3+\\
& 84495882240000\,t^7)x
\end{aligned}$$

is a rational to the Gardner equation (1).

### 3.7 Rational solution of order seven

The function $v_7(x,t)$ defined by

$$v_7(x,t) = \frac{n(x,t)}{d(x,t)} \qquad (13)$$

with

$$n(x,t) = x^{64} + 8\,ix^{63} + 4032\,tx^{61} +$$
$$29232\,itx^{60}+6652800\,t^2x^{58}+44331840\,it^2x^{57}+$$
$$6008325120\,t^3x^{55}+36696844800\,it^3x^{54}+$$
$$3329540121600\,t^4x^{52}+18820451865600\,it^4x^{51}+$$
$$1197306651340800\,t^5x^{49}+6162707171174400\,it^5x^{48}+$$
$$293930755795353600\,t^6x^{46}+1373247357176217600\,it^6x^{45}+$$
$$46643754897653760000\,t^7x^{43}+260442131005292544000\,it^7x^{42}+$$
$$5156738132927496192000\,t^8x^{40}+7389254129539645440000\,it^8x^{39}$$
$$+\,1123297792337944903680000\,t^9x^{37} -$$
$$2861349861360019341312000\,it^9x^{36}$$
$$+621308836469927647641600000\,t^{10}x^{34}-$$
$$963993976521816655134720000\,it^{10}x^{33}+$$
$$96034709039084017916313600000\,t^{11}x^{31}-$$
$$176860411255154139306393600000\,it^{11}x^{30}+$$
$$12747529948908855776772096000000\,t^{12}x^{28}-$$
$$10326405154707176351072256000000\,it^{12}x^{27}+$$
$$300222391498832588159582208000000\,t^{13}x^{25}-$$
$$326501805078305997733232640000000\,it^{13}x^{24}-$$
$$49835324297074120336940728320000000\,t^{14}x^{22}$$
$$-206661308388978892887186997248000000\,it^{14}x^{21}$$
$$-1049284425401214297456714448896000000\,t^{15}x^{19}$$
$$-4645990085542263025344807174144000000\,it^{15}x^{18}$$
$$-5587991432946635075558051099443200000\,t^{16}x^{16}$$
$$+11814383404096701795206877413376000000\,it^{16}x^{15}$$
$$-211387398127111390864912290938880000000\,t^{17}x^{13}$$
$$-853337549281549667333619616579584000000\,it^{17}x^{12}$$
$$+1251561738946272845422642104316723200000000\,t^{18}x^{10}$$
$$-774271245280321336575024352670515200000000\,it^{18}x^{9}$$
$$-7636647898655224141561884026339328000000000\,t^{19}x^{7}$$
$$-106913070581173137981866376368750592000000000\,it^{19}x^{6}$$
$$+801848029358798534863997822765629440000000000\,t^{20}x^{4}$$
$$+1603696058717597069727995645531258880000000000\,it^{20}x^{3}$$
$$+3207392117435194139455991291062517760000000000\,t^{21}x$$
$$-3207392117435194139455991291062517760000000000\,it^{21}$$

and

$$d(x,t) = (-x^{36} - 2520\,tx^{33} - 2162160\,t^2x^{30} - 884822400\,t^3x^{27} - 163459296000\,t^4x^{24} - 30207277900800\,t^5x^{21} + 4833164464128000\,t^6x^{18} - 891718843631616000\,t^7x^{15} - 83125595618537472000\,t^8x^{12} - 3483358292586332160000\,t^9x^9 - 219451572432938926080000\,t^{10}x^6 + 13167094345976335564800000\,t^{11}x^3 + 526683773839053422592000000\,t^{12})(-x^{27} - 1512\,tx^{24} - 680400\,t^2x^{21} - 139708800\,t^3x^{18} - 5029516800\,t^4x^{15} - 3168595584000\,t^5x^{12} + 604145558016000\,t^6x^9 + 108746200442880000\,t^7x^6 + 6089787224801280000\,t^9)x$$

is a rational to the Gardner equation (1).

### 3.8 Rational solution of order eight

The function $v_8(x,t)$ defined by

$$v_8(x,t) = \frac{n(x,t)}{d(x,t)} \qquad (14)$$

with

$$n(x,t) = -i(-ix^{81}+9\,x^{80}-6480\,itx^{78}+
54000\,tx^{77}-17962560\,it^2x^{75}+139708800\,t^2x^{74}-
28440720000\,it^3x^{72}+206689190400\,t^3x^{71}-
28894574016000\,it^4x^{69}+196729549632000\,t^4x^{68}-
20027877904742400\,it^5x^{66}+127548405221529600\,t^5x^{65}-
9845156013428736000\,it^6x^{63}+58650450772193280000\,t^6x^{62}-
3496233842718769152000\,it^7x^{60}+19742220213202206720000\,t^7x^{59}-
917522577057504337920000\,it^8x^{57}+4584204344380840058880000\,t^8x^{56}
-186047506160225456947200000\,it^9x^{54}+
705970885405282482585600000\,t^9x^{53}-
25933156618443791880683520000\,it^{10}x^{51}+
379336840256361840468295680000\,t^{10}x^{50}+
6395652736671085473890304000000\,it^{11}x^{48}+
297650067947402651243642880000000\,t^{11}x^{47}+
5207428862069036677958467584000000\,it^{12}x^{45}+
129905635334268499051254644736000000\,t^{12}x^{44}+
1702586892784983334684867952640000000\,it^{13}x^{42}
+29620639540599344304800019775488000000\,t^{13}x^{41}
+396994691532759366938073508085760000000\,it^{14}x^{39}
+4232360525213696990146612942602240000000\,t^{14}x^{38}
-43491550690438395253575322089553920000000\,it^{15}x^{36}
+583120649740812817456115098224427008000000\,t^{15}x^{35}
-2024662141745739755496885865100083200000000\,it^{16}x^{33}
+28159381522332939098979530493552230400000000\,t^{16}x^{32}
+551861269446619964700806468894264918016000000\,it^{17}x^{30}
-3171092584423586307369767135657318154240000000\,t^{17}x^{29}
+125379579786320277204956635814482862407680000000\,it^{18}x^{27}
-377931213353733185825377312862110954291200000000\,t^{18}x^{26}
+12720585867579067854099735517310182490112000000000\,it^{19}x^{24}
-45469832616234467760991077268529404811673600000000\,t^{19}x^{23}
+1234128242496442531104836094950737862197248000000000\,it^{20}x^{21}
-2854755006196727627570205593847694317374668800000000\,t^{20}x^{20}
+446177931156674484955283193242186295080386560000000000\,it^{21}x^{18}
+3908572835159623913132810908575137703584071680000000000\,t^{21}x^{17}
+136621393127592416934216653425069481541186355200000000000\,it^{22}x^{15}
+3707909998886926345196885899662635238672039936000000000000\,t^{22}x^{14}
+26483531590887145436478920510090391806445355008000000000000\,it^{23}x^{12}$$

$-1471307310604841413137717806116132878135853056000000000000\,t^{23}x^{11}$
$-4248399859371479580435160165160333685617275699200000000000000\,it^{24}x^{9}$
$+333802846093473395605619727262597646727071662080000000000000\,t^{24}x^{8}$
$-3058847898747465297913315318915440253644438503424000000000000\,it^{25}x^{6}$
$+1019615966249155099304438439638480084548146167808000000000000\,t^{25}x^{5}$
$+61176957974949305958266306378308805072888770068480000000000000\,t^{26}x^{2}$
$+4078463864996620397217753758553920338192584671232000000000000\,it^{27})$
and
$d(x,t)=(x^{45}+3960\,tx^{42}+5821200\,t^{2}x^{39}+$
$4324320000\,t^{3}x^{36}+1743565824000\,t^{4}x^{33}+$
$455986052121600\,t^{5}x^{30}+33630769396224000\,t^{6}x^{27}+$
$1837810787484672000\,t^{7}x^{24}+31723441593196953600000\,t^{8}x^{21}-$
$6261336530923932057600000\,t^{9}x^{18}$
$-108829169791957741142016000\,t^{10}x^{15}-$
$9893754691566618543390720000\,t^{11}x^{12}+$
$512147301681095548128460800000\,t^{12}x^{9}-$
$6453056001181803906418606080000\,t^{13}x^{6}-$
$193591680035454117192558182400000\,t^{14}x^{3}+$
$774366720141816468770232729600000\,t^{15})$
$(-x^{36}-2520\,tx^{33}-2162160\,t^{2}x^{30}-884822400\,t^{3}x^{27}-$
$163459296000\,t^{4}x^{24}-30207277900800\,t^{5}x^{21}+$
$4833164464128000\,t^{6}x^{18}-891718843631616000\,t^{7}x^{15}-$
$83125595618537472000\,t^{8}x^{12}-348335829258633216000000\,t^{9}x^{9}$
$-\,219451572432938926080000\,t^{10}x^{6}+$
$1316709434597633556480000\,t^{11}x^{3}+$
$5266837738390534225920000\,t^{12})$

is a rational to the Gardner equation
(1).

### 3.9 Rational solution of order nine

The function $v_9(x,t)$ defined by

$$v_9(x,t) = \frac{n(x,t)}{d(x,t)} \qquad (15)$$

with

$$n(x,t) = -i(-ix^{100}+10\,x^{99}-9900\,ix^{97}t+$$
$$93060\,tx^{96}-43243200\,ix^{94}t^2+383961600\,t^2x^{93}-$$
$$111198700800\,ix^{91}t^3+934226092800\,t^3x^{90}-$$
$$189254729664000\,ix^{88}t^4+1506596547456000\,t^4x^{87}-$$
$$227051874137088000\,ix^{85}t^5+1713219123103488000\,t^5x^{84}-$$
$$199652557934269440000\,ix^{82}t^6+1428145956200291328000\,t^6x^{81}-$$
$$131959599374218936320000\,ix^{79}t^7$$
$$+895581632952719032320000\,t^7x^{78}-66760303356563349012480000\,ix^{76}t^8$$
$$+428022377616371217039360000\,t^8x^{75}-$$
$$26201771314573067845632000000\,ix^{73}t^9+$$
$$158789776023912561028300800000\,t^9x^{72}-$$
$$7982051135095428620156928000000\,ix^{70}t^{10}+$$
$$48867778365532304849672601600000\,t^{10}x^{69}-$$
$$1845794510242697463355146240000000\,ix^{67}t^{11}+$$
$$10500563590228982713863477657600000\,t^{11}x^{66}-$$
$$436938674857771455558389071872000000\,ix^{64}t^{12}-$$
$$3509125889298278795399059734528000000\,t^{12}x^{63}$$
$$-22976775057846926955527860440268800000\,ix^{61}t^{13}$$
$$-615334935724324545071070178600550400000\,t^{13}x^{60}$$
$$-1449815926554684377327575785642393600000\,ix^{58}t^{14}$$
$$-36652935408290927099033675692297420800000\,t^{14}x^{57}$$
$$-654718262158702066893896307662703820800000\,ix^{55}t^{15}$$
$$-1304243412541510083295718221838737735680000\,t^{15}x^{54}$$
$$-15948204848317269786001172326369984512000000\,ix^{52}t^{16}$$
$$-32147023187668013135063133423374117634048000000\,t^{16}x^{51}$$
$$-326340962792421276911736711427565971046400000000\,ix^{49}t^{17}$$
$$-502266661319331641970710360305345033666560000000\,t^{17}x^{48}$$
$$-1109034109153602138195454830018333524361216000000000\,ix^{46}t^{18}$$
$$-16487342297386300709571923264331815111884800000000000\,t^{18}x^{45}$$
$$+2776803054157053440103861018138541088047104000000000000\,it^{19}x^{43}$$
$$-174439194796998080212212405200122029953305804800000000000\,t^{19}x^{42}$$
$$+1763536190793961642216834855538817392014818017280000000000\,it^{20}x^{40}$$
$$-9847385594579806531352586987180097475728858152960000000000\,t^{20}x^{39}$$
$$+60347123414229766954512942201852809682107215380480000000000\,it^{21}x^{37}$$
$$-27630869414226911515554150803354701184994842876313600000000\,t^{21}x^{36}$$
$$+79171393629687382405000302947769898890755836792012800000000000\,it^{22}x^{34}$$
$$-39520943879481443272459668542895966001157408009748480000000000\,t^{22}x^{33}$$

$+86191846162210839375903238715922309125212199706427392000000000\,it^{23}x^{31}$
$-20181270380241253061429567546404341511648356738269184000000000\,t^{23}x^{30}$
$+371146844998608938745409479633584353495168217851625472000000000\,it^{24}x^{28}$
$-111009714025675678910786955318908309101987512458202316800000000\,t^{24}x^{27}$
$-790292787560605227039974112295286693718824812235383111680000000000\,it^{25}x^{25}$
$-128446740342524481121949490167970826891662327175606330982400000000000\,t^{25}x^{24}$
$-140297230639136534260525403388901069495050322358460796108800000000000\,ix^{22}t^{26}$
$-845178584186676456943949690755485502260287071230371498557440000000000\,t^{26}x^{21}$
$-1814073251355131043450164895241263140377527325959580444262400000000000\,ix^{19}t^{27}$
$-6967217887900169109541511035767629813488068102366890809098240000000000\,t^{27}x^{18}$
$+2034153160898066379192666232190869484574785313405769869164544000000000000\,it^{28}x^{16}$
$+4912219331191612575642775040874437032348812776784915131793408000000000000\,t^{28}x^{15}$
$-6073960125327728835552583422677937010934348746300175946665164800000000000000\,ix^{13}t^{29}$
$+9014097711503587372659988280976759837018003528469712645337907200000000000000\,t^{29}x^{12}$
$-13516110758478004753195200305107052180199062273973580301724549120000000000000\,ix^{10}t^{30}$
$+65364797930344449216271870327976727756700383128232888344405606400000000000000\,t^{30}x^{9}$
$-79767211033640683789348723112107193194617416698860473911817011200000000000000\,ix^{7}t^{31}$
$-111674095447096957305088212356950070472464383378404663476543815680000000000000\,t^{31}x^{6}$
$-41877785792661358989408079633856276427174143766901748803703930880000000000000000\,ix^{4}t^{32}$
$-83755571585322717978816159267712552854348287533803497607407861760000000000000000\,t^{32}x^{3}$
$+16751114317064543595763231853542510570869657506760699521481572352000000000000000\,it^{33}x$
$-16751114317064543595763231853542510570869657506760699521481572352000000000000000\,t^{33})$
and
$d(x,t)=(-x^{54}-5940\,tx^{51}-13899600\,t^{2}x^{48}-$
$17254036800\,t^{3}x^{45}-12586365792000\,t^{4}x^{42}-$
$5851581261926400\,t^{5}x^{39}-1623511727405568000\,t^{6}x^{36}-$
$400077271429355520000\,t^{7}x^{33}-28551097433877258240000\,t^{8}x^{30}+$
$5092669823761217617920000\,t^{9}x^{27}$
$-423186650252723431813939200000\,t^{10}x^{24}-$
$24094202601815176923316224000000\,t^{11}x^{21}-$
$486667973422461044609069875200000\,t^{12}x^{18}-$
$8932873235921668550456613273600000\,t^{13}x^{15}-$
$56277101386306511867876663623680000000\,t^{14}x^{12}$
$+42920669323956433051233935456993280000000\,t^{15}x^{9}$
$+77257204783121579492221083822587904000000000\,t^{16}x^{6}$
$+2163201733927404225782190347032461312000000000\,t^{18})(x^{45}+$
$3960\,tx^{42}+5821200\,t^{2}x^{39}+4324320000\,t^{3}x^{36}+$
$1743565824000\,t^{4}x^{33}+455986052121600\,t^{5}x^{30}+$
$33630769396224000\,t^{6}x^{27}+18378107874846720000\,t^{7}x^{24}+$
$3172344159319695360000\,t^{8}x^{21}-626133653092393205760000\,t^{9}x^{18}-$
$108829169791957741142016000000\,t^{10}x^{15}-$
$9893754691566618543390720000000\,t^{11}x^{12}+$
$512147301681095548128460800000000\,t^{12}x^{9}-$
$645305600118180390641860608000000\,t^{13}x^{6}-$
$19359168003545411719255818240000000\,t^{14}x^{3}+$
$774366720141816468770232729600000000\,t^{15})x$

is a rational to the Gardner equation (1).

## 3.10 Rational solution of order ten

The function $v_{10}(x,t)$ defined by

$$v_{10}(x,t) = \frac{n(x,t)}{d(x,t)} \tag{16}$$

with

$n(x,t) = -i(-ix^{121}+11\,x^{120}-14520\,ix^{118}t+$
$151800\,tx^{117}-95135040\,ix^{115}t^2+948261600\,t^2x^{114}-$
$375171825600\,ix^{112}t^3 + 3570850483200\,t^3x^{111} -$
$1001486566080000\,ix^{109}t^4+9111786780096000\,t^4x^{108}-$
$1928677978260633600\,ix^{106}t^5+16783754184791193600\,t^5x^{105}-$
$2788145214059132928000\,ix^{103}t^6+23214265448682147840000\,t^6x^{102}-$
$310738774009028895129600\,ix^{100}t^7+2475889510577320341504000\,t^7x^{99}-$
$27209570404362661066752000\,ix^{97}t^8+207395681734001780206387200\,t^8x^{96}$
$-1897998004600345592443699200000\,ix^{94}t^9+13838512826996381618660966400000\,t^9x^{93}$
$-10650314276701941373724550758400\,ix^{91}t^{10}+$
$74418322354209485544634476134400\,t^{10}x^{90} -$
$4843598262529197425137812504576000\,ix^{88}t^{11}$
$+32162852236335200930285774438400000\,t^{11}x^{87}$
$-1809734776713707822539975648346112000\,ix^{85}t^{12}$
$+10940075510863621828209488009625600000\,t^{12}x^{84}$
$-55662624659826717571311231389663232000\,ix^{82}t^{13}$
$+366409302919024594307060726369353728000\,t^{13}x^{81}$
$-10629210508837371167598914804499087360000\,ix^{79}t^{14}$
$+28771961208260528013652105080525029376000\,t^{14}x^{78}$
$+428003080736150916954161438964414676992000\,it^{15}x^{76}$
$+3045389769829638777026994310508134898073600\,t^{15}x^{75}$
$+68856701093559453890142786087968393134080000\,it^{16}x^{73}$
$+228041011956866466611552032321683792592896000\,t^{16}x^{72}$
$+445372152027470864098747300927900024307712000000\,it^{17}x^{70}$
$+1177628053649144662576947782988763859339182080000\,t^{17}x^{69}$
$+19287676949611900512916934123001252380304998400000\,it^{18}x^{67}$
$+42939910195222368442373794245841509282389950464000000\,t^{18}x^{66}$
$+654735071078952292481291250564426986112984023040000000\,it^{19}x^{64}$
$+1110064072526528900597633317868514157559165013196800000\,t^{19}x^{63}$
$+11898388510029268300683286437262873928059593739468800000\,it^{20}x^{61}$
$+25480586894270823871865727615638392620721670140723200000\,t^{20}x^{60}$
$-715336762093535316523488958451303113361066947510272000000\,ix^{58}t^{21}$
$+5013428121652009915249481460644157043637089555238092800000\,t^{21}x^{57}$

$+1122098111116143218387479137255124173623861285996199936000000000000\, it^{22}x^{55}$
$-532806316664358395685058887047235855636333952229179392000000000000\, t^{22}x^{54}$
$+18666731499151103582697435263135280572769467169281723596800000000000\, it^{23}x^{52}$
$-111900720524349972607748089158539708346975177833478881280000000000000\, t^{23}x^{51}$
$+821160492098703652494243618508146724991341109688407792025600000000000\, it^{24}x^{49}$
$-472356267985166166429107110009412445346002202412648836890624000000000\, t^{24}x^{48}$
$+20745768903383088336797180423440390760579470053263315194557235200000000\, it^{25}x^{46}$
$-88712810537698606548709845510973812347610188041362448225350451200000000\, t^{25}x^{45}$
$+3109826288710026717097043900440441727905550505973786876537995264000000000\, it^{26}x^{43}$
$-13102264646488046071756374704260548274170124938337651562423255040000000000\, t^{26}x^{42}$
$+275930186232651073615404657875346905303154392267848934894157692928000000000\, it^{27}x^{40}$
$-270396210866183602782620361062750448925713674136174283687987249152000000000\, t^{27}x^{39}$
$+34612681307949155341638944237023314224694049592402208339361215283200000000\, it^{28}x^{37}$
$+16944349808274198207103097908284912672054542890962731378372411129856000000000\, t^{28}x^{36}$
$+867875156870281716096455706687283790071157369998346682307347008716800000000000\, it^{29}x^{34}$
$+121522198287550125954273147621213608370048486986445078771138702606336000000000000\, t^{29}x^{33}$
$+142716414561877226702889120299857500886877537625672562581184735988416512000000000000\, it^{30}x^{31}$
$+24687175735912096653186956638362780594794115399596167209163392620251054080000000000000\, t^{30}x^{30}$
$+72065896246651341144726283758009193947220836774448444763669676376562073600000000000000\, it^{31}x^{28}$
$+14138837524386502842316547022970338182746739417942914661323932982711418880000000000000\, t^{31}x^{27}$
$+277998787379363399431485808097106509165514597039715322194987956096573898752000000000000\, it^{32}x^{25}$
$+218634700695580885432316065747322255972847842407755634772472860088226283520000000000000\, t^{32}x^{24}$
$-90840229532843846042412137018193738079990797936534294316472658867592888320000000000000000\, ix^{22}t^{33}$
$+401214645216852730269554616818765276029818689646851534038793223244097362329600000000000000\, t^{33}x^{21}$
$-306852163904191313442393275460829953540472987251272901063075672251181469532160000000000000000\, ix^{19}t^{34}$
$+157627886390273508530607586958466768093765832388950910966517947023627068112896000000000000000\, t^{34}x^{18}$
$-131976341024753878974187543561506727128141848344430806743002070279306174026219520000000000000000\, ix^{16}t^{35}$
$-591801242565108195890928339048751084245952648706547487191551445014106805685452800000000000000\, t^{35}x^{15}$
$-447754816361067412308486674736797339601264991451325668870175277472582093116538880000000000000000\, ix^{13}t^{36}$
$+1092483058553130993844768324668367259869681122809270164256557826617547721120153600000000000000000\, t^{36}x^{12}$
$+1639196163250654676807661020092335796535400683259722197534875541865868724581433344000000000000000000\, it^{37}x^{10}$
$+985592629802608824662834157650455067537108005757427903581096053653528663514152960000000000000000000\, t^{37}x^{9}$
$-746975466797766688165516414219292261712334488574050621661462272242674355505463296000000000000000000\, ix^{7}t^{38}$
$+104576565351687336343172297990700916639726828400367087032604718113974409770764861440000000000000000000\, t^{38}x^{6}$
$+522882826758436681715861489953504583198634142001835435163023590569872048853824307200000000000000000000\, it^{39}x^{4}$
$-1045765653516873363431722979907009166397268284003670870326047181139744097707648614400000000000000000000\, t^{39}x^{3}$
$+156864848027531004514758446986051374959590242600550630548907077170961614656147292160000000000000000000\, it^{40}x$
$+156864848027531004514758446986051374959590242600550630548907077170961614656147292160000000000000000000\, t^{40})$

and

$d(x,t) = (x^{66} + 8580\, tx^{63} + 30270240\, t^2x^{60} +$
$58853995200\, t^3x^{57} + 70523605152000\, t^4x^{54} + 55594337313715200\, t^5x^{51} +$
$29373557030737920000\, t^6x^{48} + 11136392862316001280000\, t^7x^{45} +$
$2669301014056143667200000\, t^8x^{42}$
$+278616413032517777817600000\, t^9x^{39} + 59371963969867191762223104000\, t^{10}x^{36}$
$-14313436251642163712702545920000\, t^{11}x^{33} -$
$4375571809516075503861889499136000000\, t^{12}x^{30}$

$$-2085428883999446421262710016573440000000\,t^{13}x^{27}-$$
$$3076213556254344998720743913029632000000\,t^{14}x^{24}$$
$$-54864878444394594706063036352739409920000000\,t^{15}x^{21}$$
$$+384097069780086119375492488404632862720000000\,t^{16}x^{18}$$
$$+1197795702957516968447395683585402863616000000\,t^{17}x^{15}$$
$$-3724146473112079841064361079547615070126080000000\,t^{18}x^{12}$$
$$-119899349866047448541584307926898826647961600000000\,t^{19}x^{9}$$
$$-604292723324879140649584911951570086305726464000000000\,t^{20}x^{6}$$
$$+24171708932995165625983396478062803452229058560000000000\,t^{21}x^{3}$$
$$+72515126798985496877950189434188410356687175680000000000\,t^{22})$$
$$(-x^{54}-5940\,tx^{51}-13899600\,t^{2}x^{48}-17254036800\,t^{3}x^{45}-$$
$$12586365792000\,t^{4}x^{42}-5851581261926400\,t^{5}x^{39}-$$
$$1623511727405568000\,t^{6}x^{36}-400077271429355520000\,t^{7}x^{33}-$$
$$28551097433877258240000\,t^{8}x^{30}+50926698237612176179200000\,t^{9}x^{27}-$$
$$4231866502527234318139392000\,t^{10}x^{24}-2409420260181517692331622400000\,t^{11}x^{21}$$
$$-48666797342246104460906987520000\,t^{12}x^{18}-$$
$$8932873235921668550456613273600000\,t^{13}x^{15}-$$
$$56277101386306511867876663623680000000\,t^{14}x^{12}+$$
$$42920669323956433051233935456993280000000\,t^{15}x^{9}$$
$$+77257204783121579492221083822587904000000000\,t^{16}x^{6}+$$
$$2163201733927404225782190347032461312000000000\,t^{18})x$$

is a rational to the Gardner equation (1).

We could go on for greater orders but the expressions of the solutions become too long to be given in this text.

# 4 Conclusion

We have constructed rational solutions to the Gardner equation and we obtain an infinite hierarchy of families of rational solutions of this equation as a quotient of a polynomials in $x$ and $t$.
At order $k$ the numerator is a polynomial of degree $(k+1)^2$ en $x$ and of degree $\left[\frac{(k+1)^2}{3}\right]$ in $t$, where as usual, $[x]$ denotes the bigger integer less or equal to $x$.
The denominator is also a polynomial of degree $(k+1)^2$ en $x$ and of degree $\left[\frac{(k+1)^2}{3}\right]$ in $t$ at order $k$, with the same notation as previously.
It will relevant to study in details the structure of these polynomials.

# References


[1] R.M. Miura, C.S. Gardner and M.D. Kruskal, Korteweg-deVries equation andgeneralizations. II. Existence of conservation laws and constants of motion,, J. Math. Phys., V. 9, 1204-1209, 1968

[2] S. Watanabe, Ion Acoustic Soliton in Plasma with Negative Ion, J. Phys. Soc. Japan, V. 53, 950-956, 1984

[3] M.S. Ruderman, T. Talipova, and E. Pelinovsky, Dynamics of modulationally unstable ion-acoustic wavepackets in plasmas with negative ions, J. Plasma Phys., V. 74, 639-656, 2008

[4] R. Grimshaw, Environmental Stratified Flows, Topics in Environmental Fluid Mechanics, Kluwer, 2002

[5] E. Demler and A. Maltsev, Semiclassical solitons in strongly correlated systems of ultracold bosonic atoms in optical lattices, Ann. Phys., V. 326, 1775-1805, 2011

[6] R. Grimshaw, D. Pelinovsky, E. Pelinovsky, T. Talipova, Wave Group Dynamics in Weakly Nonlinear Long-Wave Models, Phys. D, V. 159, 35-37, 2001

[7] M.H. Alford, T. Peacock, T.Y. Tang, The formation and fate of internal waves in the South China Sea, Nature, V 521, 65-69, 2015

[8] M. Wadati, Wave propagation in nonlinear lattice III, J. Phys. Soc. Jpn., V. 38, 681-686, 1975

[9] M. Coffey On series expansions giving closed-form solutions of Korteweg-de Vries-like equations, J. Appll. Math., V. 50, N. 6, 1580-1592, 1990

[10] S.Y. Lou, L.L Chen, Solitary wave solutions and cnoidal wave solutions to the combined KdV and mKdV equation, Math Meth. Appl. Sci., V. 17, 339-347, 1994

[11] J Zhang, New Solitary Wave Solution of the Combined KdV and mKdV Equation, Int. Jour. Of Theo. Phys., V. 37, 1541-1546, 1998

[12] P. Gaillard, Rational solutions to the mKdV equation associated to particular polynomials, Wave Motion, V. **107**, 102824-1-11, 2021



**Contribution of Individual Authors to the Creation of a Scientific Article (Ghostwriting Policy)**

The author contributed in the present research, at all stages from the formulation of the problem to the final findings and solution.

**Sources of Funding for Research Presented in a Scientific Article or Scientific Article Itself**

No funding was received for conducting this study.

**Conflicts of Interest**

The author has no conflict of interest to declare that is relevant to the content of this article.