The Plancherel theory and the uncertainty principle

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Abstract: In this article, we are revising the Plancherel theory for a unimodular locally compact Hausdorff group with a Haar measure. Let be a connected semisimple real Lie group such that there exists an analytic diffeomorphism from the manifold to group according to the rule , decomposition is the Iwasawa decomposition of the group , dim(A) = rank(G). The group center(G) ⊂ K is closed, under the adjoint representation of is a maximal compact subgroup of the adjoint of ; subgroups and are simply connected. The associated minimal parabolic subgroup of is . Let and be Lies algebras of and , respectively, the norms correspond to the and dual algebra relative to the inner product induced by the Killing form of .Let an irreducible unitary representation of being presented as a left translation on where is finite-dimensional. Let be an element of the complexification of .Loosely said the Hardy uncertainty principle maintains that the function and its Fourier transform cannot be simultaneously both rapidly decreasing. The uncertainty principle is considered from several points of view: first, we consider the uncertainty principle in the case of a locally compact Hausdorff group G equipped with a probabilistic Haar measure μ_G and K be a maximal compact subgroup of G with a probabilistic Haar measure μ_K then we establish $(1 + \delta)^p \mu_G (T) \hat{\mu} (U) \geq (1 - \varepsilon - \delta)^p$, where T is ε-concentration for $\psi \in L^p (G)$; second, we establish several variants of the statement that a function and its Fourier transform cannot be too rapidly decreasing namely $|\psi (g)| \leq c_1 \exp (-c_2 \|g\|^2)$ and $\|\pi (\psi, u, \hat{u})\| \leq \tilde{c}_1 (u) \exp (-\tilde{c}_2 \| \hat{u}\|^2)$ for all g ∈ G, on semisimple Lie group with the finite center.

Keywords: Hausdorff groups, Heisenberg principle, uncertainty principle, Fourier transform, Wigner function, compact group, Peter-Weyl theorem.

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1. Introduction

In quantum physics, the concept of the matter wave provides an apparatus for the mathematical description of the position of the particle and its Fourier conjugation momentum characterizes the motion of the particle, position, and motion of the particle intricately undividedly entangled and cannot be sharply simultaneously known. According to Heisenberg, the standard deviation of position Δx and the standard deviation of momentum Δp cannot be known simultaneously more precisely than $\frac{\hbar}{2}$, where $\hbar$ is the reduced Planck constant, from an analytical perspective, it means that the pair position and momentum related via Pontryagin conjugation, and the wave functions in two dual orthonormal bases in the Hilbert space are Fourier transform of one another, more generally, the mathematical variant of the uncertainty principle states that a function ψ and its Fourier transform $F(\psi)$ cannot be highly concentrated at the same time [2, 5, 8, 12]. The uncertainty principle can be formulated in several different versions that were proved by Hardy, Narayanan [22, 23], Morgan, Cowling-Price [16], etc. see the references therein.

Since harmonic analysis is a classical brunch of mathematical science, there exists extensive literature on the subject [2, 5, 8], however, mathematical aspects of the uncertainty principle (the Heisenberg principle of quantum mechanics) are still demanding additional investigations [1-7].

Following notations of D.L. Donoho and P.B. Stark [2], we denote an integrable function on the locally compact Hausdorff group G by ψ and its Fourier transform $F(\psi)$ by $\hat{\psi}$. Assume that $\psi \in L^p(G)$ is ε-concentrated on a measurable set T and Fourier transform $\hat{\psi}$ δ-concentrated on W then we have obtained the uncertainty estimation in the form

$$\left(\frac{1+\delta}{1-\varepsilon-\delta}\right)^p \mu_G(T)\,\hat{\mu}(U) \geq 1$$

where μ_G is measure Haar on G and $\hat{\mu}$ is a Plancherel measure on the set of irreducible unitary representations of G.

To define the Fourier transform $F \;:\; L^2(G/K) \rightarrow \langle V(\cdot)\rangle_{(P,\,\hat{\mu})} = \Upsilon$ on G/K we need to define a direct integral decomposition $\langle V(\cdot)\rangle_{(P,\,\hat{\mu})} = \Upsilon = \int_{\oplus} V(\omega)\, d\hat{\mu}(\omega)$ of $L^2(G/K)$. The direct integral $\int_{\oplus} V(\omega)\, d\hat{\mu}(\omega)$ of the set $\{V(\omega)\}_{\omega}$ of Hilbert spaces with respect to the measure $\hat{\mu}$ is a space of measurable vector fields $v \in V(\omega)$ of the variable ω such that

$$\|v\|_{L^2}{}^2 = \int_P \|v(\omega)\|_{V(\omega)}{}^2 d\hat{\mu}(\omega).$$

Let G be a locally compact group and let $\pi(\omega)$ be a unitary representation G on a Hilbert space $V(\omega)$ defined for every ω, and mapping $\omega \mapsto \pi(g,\,\omega)$ be a measurable field of mapping for every $g \in G$, then a unitary representation of G on $\langle V(\cdot)\rangle_{(P,\,\hat{\mu})}$ is expressed by

$$\pi(\omega) = \int_{\oplus} (g,\ \omega)\, d\hat{\mu}(\omega).$$

The Fourier transform $F \;:\; L^2(G/K) \rightarrow \langle V(\cdot)\rangle_{(P,\,\hat{\mu})} = \Upsilon$ is defined by $F(\psi)(\omega) = \pi(\psi,\omega)\, v(\omega)$, where unit vector $v(\omega) \in V(\pi(\omega)) = V(\omega)$.

In its most general form, the Plancherel theory establishes that presume $(G,\,K)$ is communicative and $\hat{\mu}$ is its Plancherel measure then $\|\psi\|_{L^2(G/K)} = \|F()\|_{L^2(G/K)}$ and the inverse Fourier transform is given by $\psi(g) = \langle\langle F(\psi), \pi(g,\ \cdot)\, v(\cdot)\rangle_V\rangle_{(P,\,\hat{\mu})}$.

We are going to compare loosely the uncertainty principle for G/K with the classical uncertainty inequality of quantum mechanics. Let A and B be a pair of Hermitian operators and let Ψ be a physical state of the quantum system. The uncertainties of the operators A and B in the state Ψ is denoted by ΔA and ΔB, respectively. Then, the classical uncertainty inequality can be presented in the form

$$\left| \left\langle \Psi \left| \frac{1}{2i} [A, B] \right| \Psi \right\rangle \right| \leq \Delta A \cdot \Delta B$$

or if we take $A = \hat{x}$ and $B = \hat{p}$ then obtain

$$\left(\left\langle \Psi \left| \frac{1}{2i} [\hat{x}, \hat{p}] \right| \Psi \right\rangle \right)^2 \leq (\Delta x)^2 \cdot (\Delta p)^2$$

since $\frac{1}{2i} [\hat{x}, \hat{p}] = \frac{\hbar}{2}$ we have

$$\frac{\hbar}{2} \leq \Delta x \cdot \Delta p.$$

If $p = 2$ then the uncertainty principle for G/K can be rewritten as

$$1 - \varepsilon - \delta \leq \mu_G (T) \hat{\mu} (U).$$

So, loosely, since position and momentum are conjugate variables the right part of the classical equation is a special case of the general theory, and for the left side, we have that $1 - \varepsilon - \delta$ in the case of G/K corresponds to $\frac{\hbar}{2}$ of classical case.

In the simple case of R^n, the Hardy uncertainty principle can be considered for the Fourier transform given by

$$F (\psi) (\chi) = \hat{\psi} (\chi) = (2\pi)^{-\frac{n}{2}} \int_{R^n} \exp (-i\chi \cdot x) \psi (x) dx,$$

and if $\psi \in L^1 (R^n)$ and

$$\int_{R^n} \int_{R^n} \exp (|\chi \cdot x|) |F (\chi)| |\psi (x)| dx d\chi < \infty$$

then we have that necessary $\psi = 0$ almost everywhere.

The broader instance of Hardy's principle is given by the following statement: let G be a connected semisimple Lie group with a finite center having a uniquely defined class of Cartan subgroups, let K be a maximal compact subgroup of G. The centralizer of the exponent A of a maximal abelian subspace of positively defined Cartan-Killing form on the Lie algebras of G, in K is denoted by M. Let ψ be a measurable function on Gsuch that

$$|\psi (g)| \leq c_1 \exp \left(-c_2 \|g\|^2 \right)$$

for all $g \in G$, and the estimation

$$\|\pi (\psi, v, \tilde{v})\| \leq \tilde{c}_1 (v) \exp \left(-\tilde{c}_2 \|\tilde{v}\|^2 \right)$$

holds for all $v \in \hat{M}$ and $\tilde{v} \in \mathbf{a}^*$, where $c_1, \ c_2, \ \tilde{c}_1 (v), \ \tilde{c}_2$ are constants. If the product $c_2 \tilde{c}_2 > \frac{1}{4}$ then the function ψ equals zero almost everywhere.

The proof of the broader Hardy uncertainty principle employs that the Plancherel measure is supported on $\hat{M} \times \mathbf{a}^*$ then from the condition inequalities, we have $\pi (\psi, v, \tilde{v}) = 0$ on $\hat{M} \times \mathbf{a}^*$ therefore the Hardy principle is proven.

2. Fourier Transform

Let G be a locally compact Hausdorff group equipped with a Haar measure μ. We define the character χ of the group G as a topological continuous group homomorphism $\chi \ : \ G \to U (1)$. For instance, if we assume that G is an additive group then the character satisfies the condition $\chi (g - h) = \chi (g) (\chi (h))^{-1}$ for all $g, h \in G$.

Definition 1. ***The Fourier transform*** F ***of a function*** $\psi \in L^2 (G) \bigcap L^1 (G)$ ***is given by***

$$F (\psi) (\chi) = \hat{\psi} (\chi) = \int_G \psi (g) \overline{\chi (g)} \, d\mu (g). \qquad (1)$$

Definition 2. ***The topological group*** $\hat{G}$ ***consisting of all characters*** $\chi \ : \ G \to U (1)$ ***on*** G ***with its natural operation on multiplication is called a dual group of*** G***.***

Let G be a locally compact Hausdorff group equipped with a Haar measure μ then one can uniquely define a Haar measure $\hat{\mu}$ on the dual group $\hat{G}$. The measure $\hat{\mu}$ is defined by $\hat{\mu} (\chi) = \int_G \overline{\chi (g)} \ d\mu (g)$, this is a Fourier-Stieltjes transformation of the measure μ.

Definition 3. ***The Fourier inversion transform*** F^{-1} ***of a function*** $\hat{\psi} \in L^2 \left(\hat{G} \right) \bigcap L^1 \left(\hat{G} \right)$ ***is defined by***

$$\begin{aligned} \psi (g) = F^{-1} \left(\hat{\psi} \right) (g) = \left(\left(\hat{\psi} \right) \right)^{\vee} (g) = \\ = \textstyle\int_{\hat{G}} \hat{\psi} (\chi) \chi (g) \ d\hat{\mu} (\chi). \end{aligned} \qquad (2)$$

We denote $\mathrm{M} (G)$ the associative Banach algebra of all measures on the σ-algebra of all Borel sets of a Hausdorff topological locally compact group G. Let $\mu, \eta \in \mathrm{M} (G)$, the convolution of measures μ and η is defined by

$$(\mu * \eta) (D) = \int_{G \times G} \phi_D (g, h) \ d\mu (g) d\eta (h), \qquad (3)$$

where ϕ_D is an indicator of D, namely, $\phi_D (g, h) = \begin{cases} 1 & if \quad g, h \in D \\ 0 & if \quad g, h \in G \backslash D \end{cases}$.

Convolution of the elements of $L^1 (G)$ is defined by

$$(\psi * \varphi) (g) = \int_G \psi (h) \varphi \left(h^{-1} g \right) \ d\mu (h), \qquad (4)$$

and agrees with the convolution of the measures when $L^1 (G)$ is naturally embedded in $\mathrm{M} (G)$.

Straightforward calculation yields

$$F (\mu * \eta) (\chi) = F (\mu) (\chi) F (\eta) (\chi) = \hat{\mu} (\chi) \hat{\eta} (\chi). \qquad (5)$$

3. The Plancherel Theory

Let G be a unimodular locally compact Hausdorff group. If the continuous homomorphism π from G into the group $U (H)$ of the unitary operators on the separable Hilbert space H, such that mapping $\pi \ : \ G \to U (H)$ satisfies conditions:

$\pi (gh) = \pi (g) \pi (h)$ and $\pi \left(g^{-1} \right) = (\pi (g))^{-1} = (\pi (g))^*$, (**??**)

and the mapping $g \to \pi (g) x$ is continuous for all $x \in H$, then the homomorphism π is called a unitary representation of G in H.

Let us denote the right and left regular representations of G on $L^2(G)$ by

$$\rho(g)\psi(h) = \psi(hg) \quad (6)$$

and

$$\lambda(g)\psi(h) = \psi\left(g^{-1}h\right) \quad (7)$$

respectively.

Assume that the group G is such that every primary representation of the group G is a direct sum of copies of irreducible representations. The matrix coefficient Λ of the unitary representation π is a function $\Lambda(\psi,\varphi)(g)$ given by

$$\Lambda(\psi,\varphi)(g) = \langle \pi(g)\psi, \varphi\rangle \quad (8)$$

for $\psi,\varphi \in L^2(G) = H$ and $g \in G$. Let $\psi = e_j, \quad \varphi = e_k$ then denote

$$\pi_{jk}(g) = \Lambda(e_j,\ e_k)(g) = \langle \pi(g)e_j,\ e_k\rangle, \quad (9)$$

where the system $\{e_j\}$ is a basis of $L^2(G) = H$. Each element Λ_π is uniquely corresponded with a continuous function such that for each finite-dimensional representation π there exists a decomposition $\Lambda_\pi = \oplus_{1\le k\le n(\pi)}\Lambda_\pi * m_k$ where m_k is an irreducible idempotent, and so that $\phi_\pi = \sum_{k=1,..,n(\pi)} m_k$ and $\phi_\pi = \sum_{k=1,..,n(\pi)} e_k$. Let $\{a_k\}_{1\le k\le n(\pi)}$ be a Hilbert basis in $\Lambda_\pi * m_1$ such that the condition $a_k \in m_k * \Lambda_\pi * m_1$ holds.

For every finite-dimensional representation π, we define a matrix $M_\pi(g)$ of $n(\pi)\times n(\pi)$-dimension with coefficients

$$a_{ij}(g) = (n(\pi))^{-1}\left(a_i(g) * \overline{a_j\left(g^{-1}\right)}\right) \quad (10)$$

for $1 \le i \le n(\pi)$ and $1 \le j \le n(\pi)$. So, we have $a_{ii} = m_i$. We define a linear span θ_π of the matrix coefficients π_{jk}, which is a subspace of $L^2(G)$.

We denote the set of all finite linear combinations of the matrix coefficients of irreducible representations by θ, so θ is a linear span of Λ_π overall finite-dimensional representations of the group G.

Theorem 1. ***The set θ constitutes an algebra.***

Proof. Let $\pi,\ \tilde\pi \in \hat G$ as equivalence classes of irreducible representations then we have

$$\pi_{jk}(g) = \langle \pi(g)e_j,\ e_k\rangle \quad (11)$$

and

$$\tilde\pi_{mq}(g) = \langle \tilde\pi(g)\tilde e_m,\ \tilde e_q\rangle. \quad (12)$$

The spaces H and $\tilde H$ are defined by choices of bases $\{e_k\}$ and $\{\tilde e_m\}$ for π_{jk} and $\tilde\pi_{mq}$. The spaces H and $\tilde H$ can be identified with C^n and $C^{\tilde n}$, where $n = \dim(\pi)$ and $\tilde n = \dim(\tilde\pi)$. Let $C^{n,\tilde n}$ be a space of all matrices over C of $n\times\tilde n$ dimension. Let T be an operator of unitary equivalence of π and $\tilde\pi$ so that $\tilde\pi(g) = T\pi(g)T^{-1}$. The tensor product $\pi\otimes\tilde\pi$ of representations π and $\tilde\pi$, on $C^{n,\tilde n}$ is given by

$$(\pi\otimes\tilde\pi)(g)T = (\pi)(g)\ T\,\overline{\tilde\pi}\left(g^{-1}\right). \quad (13)$$

Since $\overline{\tilde\pi}_{mq}\left(g^{-1}\right) = \tilde\pi_{qm}(g)$ we obtain the statement of the theorem

$$\langle(\pi\otimes\tilde\pi)(g)e_{kq}, e_{jm}\rangle = \pi_{jk}(g)\,\tilde\pi_{mq}(g)(g).$$

Theorem 2. ***The algebra θ is uniformly dense in $L^p(G)$ in the L^p norm for $1 < p < \infty$.***

The proof of this theorem follows from the density of θ in $C(G)$.

The Peter-Weyl theorem states:

$$\begin{aligned}\psi(g) = F^{-1}\left(\hat\psi\right)(g) = \left(\left(\hat\psi\right)\right)^\vee(g) = \\ = \int_{\hat G}\hat\psi(\chi)\,\chi(g)\ d\hat\mu(\chi)\end{aligned}. \quad (14)$$

first statement. The mapping $F\ :\ L^2(G) \to L^2\left(\hat G\right)$ defined by

$$F(\psi)(\pi) = \int \psi(g)\,M_\pi\left(g^{-1}\right)\ d\mu(g) \quad (15)$$

is an isometric isomorphism. For each element $\psi \in L^2(G)$, we have a representation

$$\begin{aligned}\psi = \textstyle\sum_\pi n(\pi)\sum_{i,k=1,\dots,n(\pi)} \\ \langle\langle F(\psi)(\pi)(e_i(\pi)),(e_k(\pi))\rangle\rangle\,\phi_{ik}(\pi),\end{aligned} \quad (16)$$

where $\{e_i(\pi)\}_{i=1,\dots,n(\pi)}$ is an orthonormal basis in $C^{n(\pi)}$ and coordinate functions ϕ_{ik} are defined as

$$\phi_{ik}(\pi)(g) = \langle M_\pi(g)e_i(\pi), e_k(\pi)\rangle \quad (17)$$

for all $g \in G$ and $i,\ k = 1,\dots,n(\pi)$.

second statement. Let G be a compact group then the inverse Fourier transform $F^{-1}\ :\ L^2\left(\hat G\right) \to L^2(G)$ is defined by

$$\psi(g) = \sum_\pi n(\pi)\,tr\left(F(\psi)(\pi)\,M_\pi(g)\right) \quad (18)$$

for any Fourier transform $F(\psi) \in L^2\left(\hat G\right)$ of $\psi \in L^2(G)$ and the series converges in L^2.

Now, we can formulate an analog of the Plancherel theory.

Theorem (analog of Plancherel) 3. ***Let G be a unimodular locally compact Hausdorff group with a Haar measure μ. Then a measure $\hat\mu$ on the dual group $\hat G$ is uniquely defined by the measure μ. The Fourier transform $F\ :\ L^2(G)\bigcap L^1(G) \to L^2(G)$ satisfies the equality***

$$\int_G \psi(g)\,\overline{\varphi(g)}dg = \int_G Tr\left(\hat\psi(\pi)\left(\hat\varphi(\pi)\right)^*\right)d\hat\mu(\pi) \quad (19)$$

for all $\psi,\ \varphi \in L^2(G)\bigcap L^1(G)$.

More precisely, let $\psi,\quad \varphi \in L^2(G)\bigcap L^1(G)$ then $F(\psi *\ \varphi)(\pi) = F(\varphi(\pi)\ )\,F(\psi(\pi))$ is a classical trace class for almost everywhere π so the trace of $F(\varphi(\pi)\ )\,F(\psi(\pi))$ is integrable on $\hat G$.

The equality (19) expresses the unitarity of the Fourier transform and can be rewritten in the form of the statement that for each $F(\zeta)$, the inverse Fourier transform is given by

$$\zeta(g) = \int_G Tr\left(\hat\pi(g)\,\hat\zeta(\pi)\right)d\hat\mu(\pi) \quad (20)$$

since if we take $\zeta = \varphi^* * \psi$ and $g = 1$ then we obtain (19).

Lemma. ***Let $\psi \in L^p(G)$ for all $1 < p < \infty$ then $F(\psi) \in L^{\frac{p}{p-1}}(G)$ so that $\|F(\psi)\|_{L^{\frac{p}{p-1}}} \le \|\psi\|_{L^p}$.***

4. The Generalized Heisenberg Principle

Below we will follow notations of the David L. Donoho and Philip B. Stark [2] when it is convenient.

Let G be a locally compact Hausdorff group equipped with a probabilistic Haar measure μ_G and K be a maximal compact subgroup of G with a probabilistic Haar measure μ_K.

For any $\psi \in L^p(G)$, we defined a measurable set T such that

$$\|\psi - \psi 1_T\| \le \varepsilon \|\psi\|, \tag{21}$$

where 1_T is a characteristic function of the set T. The set T is called ε-concentration set for the function $\psi \in L^p(G)$, loosely speaking it means that the support of the function $\psi \in L^p(G)$ is ε-close to the set T. Let the Fourier transform $F(\psi)$ be δ-concentrated on the measurable set W.

We define pair of operators

$$(P_T(\psi))(g) = \begin{cases} \psi(g), & g \in T, \\ 0, & g \notin T \end{cases} \tag{22}$$

and

$$(Q_W(\psi))(g) = F^{-1}\left(1_W \hat{\psi}(g)\right), \tag{23}$$

where F^{-1} denotes the inverse Fourier transform. The operator Q_W partially returns the function ψ neglecting all frequency information outside of the set W so that the function $Q_W(\psi)$ is the nearest function to ψ.

The convolution $\psi * \varphi$ of functions ψ and φ is given by

$$(\psi * \varphi)(g) = \int_G \psi(h)\varphi(h^{-1}g)\, d\mu_G(h). \tag{24}$$

Let $K\backslash G/K$ be a double coset of G then the convolution algebras $C_0(K\backslash G/K)$ and $L^1(K\backslash G/K)$ are subalgebras of algebras $C_0(G)$ and $L^1(G)$ respectively.

Definition 4. ***For $g \in G$, the measure μ_g given by***

$$\int_G \psi(h)\, d\mu_g(h) = \int_K \int_K \psi\left(kg\tilde{k}\right) d\mu_K(k)\, d\mu_K\left(\tilde{k}\right) \tag{25}$$

is called a K-dually invariant probability measure.

Definition 5. ***If equality $\mu_g * \mu_h = \mu_h * \mu_g$ holds for all $g,\ h \in G$ then $(G,\ K)$ is called a Gelfand pair.***

Straightforward consideration shows that $(G,\ K)$ is a Gelfand pair if and only if equality $KgK \cdot KhK = KhK \cdot KgK$ holds for all $g,\ h \in G$.

The space $\mathrm{M}(G,\ K)$ is a Banach convolutive subalgebra of Radon measures on G that are dual K-invariant.

A measure $\mu \in \mathrm{M}(G,\ K)$ is a positive type relative K if and only if $\mu\left(\psi(g) * \overline{\psi(g^{-1})}\right) \ge 0$ for all $\psi \in C_0(G,\ K)$.

The projection of $C_0(G)$, $\mathrm{M}(G)$, and $L^p(G)$ onto its subspace $C_0(K\backslash G/K)$, $\mathrm{M}(K\backslash G/K)$, and $L^p(K\backslash G/K)$ of dual K-invariant functions and measure, respectively, we will denote by $\psi \mapsto \psi^{\#}$ for functions and $\mu \mapsto \mu^{\#}$ for measures.

Definition 6. ***Let G be a locally compact communicative group, the mapping $L^p(G) \mapsto L^q\left(\hat{G}\right)$ given by***

$$\hat{\psi}(\chi) = \int_G \psi(g)\chi(g^{-1})\ d\mu_G(g) \tag{26}$$

is called a spherical transport of the function ψ, here $\chi \in \hat{G}$, $p + q = pq, \quad p > 1$**.**

Now, we are going to define the class of continuous functions that are quasi-weights for spherical measures.

Definition 7. A continuous function $\omega\ :\ G \to C$ is called a zonal spherical function if the Radon measure $d\mu(g) = \omega(g^{-1})\, d\mu_G(g)$ satisfies the following conditions:

1. measure μ is dual K-invariant namely $\mu\left(kE\tilde{k}^{-1}\right) = \mu(E)$ for all measurable subsets $E \subseteq G$;

$$2. \mu(\psi * \varphi) = \mu(\varphi) \int_G \psi(g)\, d\mu(g).$$

The set of all zonal spherical functions is denoted by $S(G, K)$ and the subset positive functions of $S(G, K)$ by $P(G, K)$.

Such measures $d\mu(g) = \omega(g^{-1})\, d\mu_G(g)$ are called spherical.

For $\psi \in C_0(K\backslash G/K)$ we define $D(\psi) = \{z(\psi) \in C_\psi \ :\ |z(\psi)| \le \|\psi\|_{L^1}\}$. Since, for $\psi \in C_0(K\backslash G/K)$, the mapping

$$\omega \mapsto \int_G \psi(g)\,\omega(g^{-1})\ d\mu_G(g) = \hat{\psi}(\omega) \tag{27}$$

is an injection, we have $P(G, K) \subset \bigcap D(\psi)$.

Theorem (Godement) 4.

1. ***Let $\mu \in \mathrm{M}(G,\ K)$ be a positive type relative K measure then there exists a uniquely define positive Radon measure $\hat{\mu}$ that coincides with the spherical Fourier transform $F(\mu)$ of the measure μ.***

2. ***Let $\psi \in C_0(K\backslash G/K)$ then there exists a uniquely define positive Radon measure $\hat{\mu}$ on $P(G, K)$ such that $\|\psi\|_{L^2} = \left\|\hat{\psi}\right\|_{L^2}$.***

From definitions and simple considerations, we obtain the following statement.

Statement. 1. *Assume $\varphi \in P(G)$ such that $\varphi(1) = 1$ then there exist a uniquely define representation space $(V,\ \pi)$, π is a unitary irreducible representation of G, and a uniquely define unit cyclic vector $v \in V$ such that $\varphi(g) = \langle v,\ \pi(g)v\rangle$ holds for all $g \in G$. 2. Assume π is a unitary irreducible representation of Gand a unit vector $v \in V$ is spanned by $(V,\ K,\ \pi)$ then $\langle v,\ \pi(g)v\rangle \in P(G)$ and $\langle v,\ \pi(1)v\rangle = 1$.*

So, each density-function $\omega \in P(G)$ defines representation space $(V(\omega),\ \pi(\omega))$, and a vector $v(\omega) \in V(\omega)$ such that equality $\omega(g) = \langle v(\omega),\ \pi(g,\ \omega)v(\omega)\rangle_V$ holds for all $g \in G$.

Definition 8. ***The mapping $F\ :\ L^1(G/K) \to \langle V(\cdot)\rangle_{(P,\ \hat{\mu})} = \Upsilon$ given by $F(\psi)(\omega) = \pi(\psi,\ \omega)v(\omega)$ is called the Fourier transform of the function $\psi \in L^1(G/K)$ on G/K.***

In the last definition, $\langle V(\cdot)\rangle_{(P,\ \hat{\mu})}$ is understood as a direct integral with the Plancherel measure $\hat{\mu}$.

Theorem (Plancherel-Godement) **5.** ***Let $(G,\ K)$ be Gelfand and Abelian then the Fourier transform satisfies the equalities***

$$\psi(g) = \left\langle \langle F(\psi), \pi(g,\ \cdot)v(\cdot)\rangle_V \right\rangle_{(P,\ \hat{\mu})} \tag{28}$$

and

$$\|F(\psi)\|_{\Upsilon} = \|\psi\|_{L^2(G/K)} \tag{29}$$

for all $\psi \in L^2(G/K)$.

Definition 9. ***The mapping*** $F^{-1} : \Upsilon \to L^1(G/K)$ ***given by*** $F^{-1}(\zeta)(g) = \langle\langle\zeta(\cdot), \pi(g, \cdot) v(\cdot)\rangle_V\rangle_{(P, \hat{\mu})}$ ***is called the inverse Fourier transform of the function*** ζ ***on*** G/K.

Theorem (Heisenberg principle) **6.**

Let $\varepsilon, \delta \geq 0$. ***Let function*** $\psi \in L^p(G/K)$ ***be*** ε***-concentrated on*** $T = TK \subset G$ ***in*** L^p***-norm and satisfies the condition there exists a function*** $\psi_U \in L^p(G/K)$ ***such that*** $\sup p(F(\psi_U)) \subset U$ ***and*** $\|\psi - \psi_U\|_{L^p} \leq \delta \|\psi\|_{L^p}$. ***Then***

$$\left(\frac{1+\delta}{1-\varepsilon-\delta}\right)^p \mu_G(T)\hat{\mu}(U) \geq 1. \tag{30}$$

Proof. First, we show that

$$\|PQ\|_{L^p}^p \leq \mu_G(T)\hat{\mu}(U)$$

where operators are defined by

$$(P_T(\psi))(g) = \begin{cases} \psi(g), & g \in T, \\ 0, & g \notin T \end{cases}$$

and

$$(Q_W(\psi))(g) = F^{-1}(1_W(F\psi(g))).$$

Indeed, we have

$$\begin{aligned} &PQ\psi(g) = 1_T(g) F^{-1}(1_W(F\psi)(g)) = \\ &= 1_T(g) \int_G \int_P \psi(h) 1_W(\omega) \\ &\langle v(\omega), \pi(h^{-1}g, \omega) v(\omega)\rangle_{V(\omega)} d\hat{\mu}(\omega) d\mu_G(h) = \\ &= \left\langle \psi(\cdot), \overline{1_T(g) F^{-1}((\omega \mapsto 1_W(\omega) v(\omega))(h^{-1}g))} \right\rangle_{L^2}, \end{aligned}$$

since, by Riesz-Thorin theorem, for all $\psi \in L^p(G/K)$ and $v \in V^p$, we have $\|F(\psi)\|_{L^q} \leq \|\psi\|_{L^p}$ and $\left\|F^{-1}(v)\right\|_{L^q} \leq \|v\|_{L^p}$, thus Holder and Titchmarsh inequalities yield

$$\begin{aligned} &\|PQ\psi\|_{L^p}^p \leq \\ &\leq \|\psi\|_{L^p}^p \left\|F^{-1}((\omega \mapsto 1_W(\omega) v(\omega)))\right\|_{L^q}^q \mu_G(T) \leq \\ &\leq \mu_G(T)\hat{\mu}(U)\|\psi\|_{L^p}^p. \end{aligned}$$

Second, there is the estimation

$$\frac{1-\varepsilon-\delta}{1+\delta} \leq \|PQ\|_{L^p}.$$

Indeed, using the conditions, we estimate

$$\begin{aligned} &\|\psi\|_{L^p} - \|PQ\psi\|_{L^p} \leq \|\psi - PQ\psi\|_{L^p} \leq \\ &\leq \|\psi - PQ\psi\|_{L^p} + \|P\psi - P\psi_U\|_{L^p} + \\ &+ \|PQ\psi_U - PQ\psi\|_{L^p} \leq \\ &\leq \varepsilon\|\psi\|_{L^p} + \delta\|\psi\|_{L^p} + \delta\|\psi\|_{L^p}\|PQ\psi\|_{L^p}, \end{aligned}$$

so, we obtain $1 - \varepsilon - \delta \leq (1+\delta)\|PQ\|_{L^p}$ that proves the Heisenberg principle.

5. The Hardy Uncertainty Principle

Let G be a connected semisimple real Lie group such that there exists an analytic diffeomorphism from the manifold $K \times A \times N$ to group G according to the rule $(k, a, n) \mapsto kan$, decomposition KAN is called the Iwasawa decomposition of the group G, where the dimension of A is equal to the real rank of G. The group K is closed and contains the center of G, $Imag(K)$ under the adjoint representation of $G$is a maximal compact subgroup of the adjoint of G; subgroups A and N are simply connected. The associated minimal parabolic subgroup of G is MAN . Let $\mathbf{g}$ and $\mathbf{a}$ be Lies algebras of G and A, respectively, the norms $\|\cdot\|$ correspond to the $\mathbf{a}$ and dual algebra $\mathbf{a}^*$ relative to the inner product induced by the Killing form of $\mathbf{g}$.

Let an irreducible unitary representation v of M being presented as a left translation on $V_v \subset C(M)$ where V_v is finite-dimensional. Let $\tilde{v}$ be an element of the complexification ${\mathbf{a}^*}_C$ of $\mathbf{a}^*$.

Loosely said the Hardy uncertainty principle maintains that the function ψ and its Fourier transform $F(\psi)$ cannot be simultaneously both rapidly decreasing.

Theorem 7. ***Let a measurable on*** G ***function*** ψ ***satisfies the conditions:***

$$|\psi(k_1 a k_2)| \leq c_1 \exp\left(-c_2 |\log(a)|^2\right) \tag{31}$$

for all $k_1, k_2 \in K$ ***and*** $a \in A$***, and the following estimation***

$$\|\pi(\psi, v, \tilde{v})\| \leq \tilde{c}_1(v) \exp\left(-\tilde{c}_2 |\tilde{v}|^2\right) \tag{32}$$

for all $v \in \hat{M}$ ***and*** $\tilde{v} \in \mathbf{a}^*$***, where*** $c_1, c_2, \tilde{c}_1(v), \tilde{c}_2$ ***are constants. If the product*** $c_2\tilde{c}_2 > \frac{1}{4}$ ***then the function*** ψ ***equals identically to zero.***

Proof. We denote

$$\psi_{\rho,\tilde{\rho}}(g) = \dim(\rho)\dim(\tilde{\rho})$$

$$\int_K \int_K \overline{\chi}_\rho(k_1)\overline{\chi}_{\tilde{\rho}}(k_2)\psi(k_1 g k_2) d\mu_K(k_1) d\mu_K(k_2),$$

where ρ and $\tilde{\rho}$ are irreducible representations of K.

Employing conditions and remarks $\pi(\psi_{\rho,\tilde{\rho}}, v, \tilde{v}) = P_\rho \pi(\psi, v, \tilde{v}) P_{\tilde{\rho}}$ here P_ρ and $P_{\tilde{\rho}}$ are the projections of $L^2(K)$ on the sum of all submodules, which are isomorphic to the ρ_1 and ρ_2 weight modules. Let $0 < c_2(v) < c_2$and take $\tilde{c}_2$ so that $c_2(v)\tilde{c}_2 > \frac{1}{4}$, we have

$$\|\pi(\psi_{\rho,\tilde{\rho}}, v, \tilde{v})\| \leq c_5 \exp\left(\frac{|\tilde{v}|^2}{4c_2(v)}\right)$$

for $v \in \hat{M}$ and $\tilde{v} \in {\mathbf{a}^*}_C$.

Applying the Naimark equivalent, there is a quotient representation $\tilde{\pi}(v, \tilde{v})$ of $\pi(v, \tilde{v})$ to close quotient subset V_1/V_0, there exists a densely define intertwining operator $(\pi, V_\pi) \mapsto (\tilde{\pi}(v, \tilde{v}), V_1/V_0)$ on the domain of which $\pi(\psi_{\rho,\tilde{\rho}}) = 0$, from the properties of quotient representation, by continuity, we have $\pi(\psi_{\rho,\tilde{\rho}}) = 0$ on V_π. By summing over all ρ and $\tilde{\rho}$, we have $\pi(\psi) = 0$ for all representations of G, applying Plancherel theory, we obtain $\psi = 0$ [19].

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