Comprehensive Results for the Error Functions in the Complex Plane and Some of Their Consequences

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Abstract: – In this extensive special note, various necessary information directly relating to the well-known error functions considered in certain domains of the complex plane, which are both in the family of classical special functions and important tools for nearly all sciences and technology, will be firstly introduced, and a number of our main results (consisting of various analytic-geometric properties of those error functions) will be also stated (and then proven) by an auxiliary theorem produced in recent studies. In addition, some special consequences of those main results, which are also associated with certain different types of special functions, be will pointed out to relevant researchers.

Key-Words: - Domains in the complex plane, the (complex) error functions, special functions, power series, series expansions, analytic functions

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1 Introduction

In this first section, various specific mathematical structures, which are expressed by certain types being of some integrals (or functions) and serving very comprehensive fields of science and technology, will be introduced. At the same time, certain special information in relation to historical dimensions, various possible applications, and comprehensive references will be introduced to our readers.

The first is directly related to the Gamma function, which is generally denoted by $\Gamma(\cdot)$ and also is defined by

$$\Gamma(\lambda) = \int_0^\infty v^{\lambda-1} e^{-v} dv \quad (\Re e(\lambda) > 0). \qquad (1)$$

The second is also related to the following significant tools given by

$$\kappa(\mu) = e^{-\mu^2} \text{ and } \int \kappa(\mu) d\mu \,, \qquad (2)$$

which are concerned with the well-known probability integral. In generally, it is also encountered as the normal distribution (or Gaussian distribution). For those, see the fundamental works, [1], [2], [3], [4], [5]. In particular, the well-known normal distribution describes its familiar form given by

$$\mathcal{N}_{v,\sigma}(\mu) = \frac{1}{\sigma\sqrt{2\pi}} \, e^{-\frac{(\mu-v)^2}{2\sigma}} \,, \qquad (3)$$

where $\sigma > 0$ and $|\mu| < \infty$.

Moreover, both its complex form and each one of its special forms can be considered for the theory of complex function. This status will also be taken into different importance for the literature. In special, as its familiar special form, when $v \coloneqq 0$ and $\sigma \coloneqq 1$, the standard normal distribution, which is called as

$$\varphi(\mu) = \frac{1}{\sqrt{2\pi}} \, e^{-\frac{\mu^2}{2}} \quad (|\mu| < \infty) \,, \qquad (4)$$

can be easily received. With the help of the function just above, Gauss considered the integral in the form given by

$$\int \varphi(\mu) d\mu \;, \qquad (5)$$

which is one of the special forms of the function-integral constituted as in (1).

In addition, for the other scientific fields, various effective-comprehensive roles of the forms, given in (2), continue to be undeniable. Specially, in light of the information in (2), Gauss considered the following-specific integral:

$$\frac{1}{\sqrt{\pi}} \int_{-\eta}^{\eta} e^{-\mu^2} d\mu = \frac{2}{\sqrt{\pi}} \int_{0}^{\eta} e^{-\mu^2} d\mu \qquad (6)$$

which also is appertaining to the error of normal distribution.

In the light of our classical analysis information and also by means of (2) and (6), it is known that the assertion:

$$\int_{-\infty}^{\infty} e^{-\mu^2} d\mu = 2 \int_{0}^{\infty} e^{-\mu^2} d\mu = \sqrt{\pi}\,. \qquad (7)$$

Of course, as is known, in the special forms given in (2), the related independent variable μ can be a real variable as well as a complex variable, which are generally considered as x and z, respectively. Naturally, it is clear that each of the mathematical expressions given in (2) to (5) will be also centered upon various roles for quite different dimensions in terms of the theory of the real-complex functions.

In general, as pointed out before, various types consisting of the different forms indicated in (2) are widely used in both normal and limit distributions in the theory of probability and statistics. See the main works given in, [1], [5], [6], [7], [8], [9]. At the same time, there are various types of those special functions and their inverse functions defined with the help of that mentioned integral. As some examples, the error function, the generalized error function, the complementary error function, the imaginary error function, and also their inverse functions, which are the inverse error function, the inverse of the generalized error function, and the inverse complementary error function, respectively, are also identified by the help of that integral in (2). Although the comprehensive roles are seen in very wide areas, in particular, these functions and their approximations are also used to predict various results that are valid with high probability (or low probability). In addition, they also appear in solutions to the heat equation when the boundary conditions are given by the Heaviside step function. For both those and some, their applications and their details, one may refer to the essential works, [1], [2], [5], [6], [8], [9], [10], [11], [12], [13], in the references.

In particular, we also note that, in this specific note, various types of forms specified by the integral with complex variables will be then concentrated on the complex forms of those functions as indicated just above. In general, each of these special functions is thought of as a typical probability integral and inverse. Especially in the literature, although definitions of integral forms are used for each of these special functions, their definitions in serial expansion form, which can be easily created with the help of classical analysis information, are also frequently used. For these complex special functions and their implications, one may look over the earlier studies given in, [3], [4], [14], [15], [16], [17], [18].

Let us first present some of the mentioned definitions concerning those error functions and then remind us of certain information relating to their relationships possessing infinite series forms in the next section.

2 The Power Series of some Basic Complex Error Functions

In this special section, the basic complex error functions will be introduced, which will have important roles for various special (complex) functions. The complex error function, the complex complementary error function, and the imaginary error function, which are designated by the help of the form presented in (6), are denoted by

$$\mathrm{erf}(z) \;,\; \mathrm{erfc}(z) \;\text{ and }\; \mathrm{erfi}(z) \qquad (8)$$

and also defined by

$$\mathrm{erf}(z) = \frac{2}{\sqrt{\pi}} \int_{0}^{z} e^{-\mu^2} d\mu \;, \qquad (9)$$

$$\mathrm{erfc}(z) = \frac{2}{\sqrt{\pi}} \int_{0}^{\infty} e^{-\mu^2} d\mu \;, \qquad (10)$$

and

$$\mathrm{erfi}(z) = \frac{2}{\sqrt{\pi}} \int_{0}^{z} e^{\mu^2} d\mu \;, \qquad (11)$$

respectively, where the mentioned parameter z is a member belonging to any domain of the complex plane $\mathbb{C}$, which consists of any path moving from the point origin of the set $\mathbb{C}$ to any point z.

There are a great number of relationships along with the error functions introduced as in (8)-(11) and also various special functions relating to those functions. For each of them and their details, the works given in, [1], [3], [4], [19], [20], [21], [22], [23], in the references can be also presented as main sources. It is only pertinent to remember the following special information between the error functions consisting of their complex forms given by (9) and (10).

As one of a large number of relationships between those special functions, by taking into account the elementary result presented in (7) and

$$1 = \frac{2}{\sqrt{\pi}}\int_0^{\infty} e^{-\mu^2} d\mu$$
$$= \frac{2}{\sqrt{\pi}}\left(\int_0^{z} e^{-\mu^2} d\mu + \int_z^{\infty} e^{-\mu^2} d\mu\right), \qquad (12)$$

one of those relations between the complex forms of the error functions given by 5

$$\mathrm{erf}(z) + \mathrm{erfc}(z) = 1 \qquad (13)$$

can be then received for all z in $\mathbb{C}$.

In addition, some of the extra specific results, which are directly interested in the mentioned fundamental information, are also presented just below. For further information, it can be checked the references given in, [7], [21], [24], [25], [26], [27], [28], [29], [30], [31].

Remarks 1. For the essential error functions, the following are satisfied:

- $\mathrm{erf}(0) = 0$
- $\mathrm{erf}(\infty) = 1$
- $\mathrm{erfc}(0) = 0$
- $\mathrm{erfc}(\infty) = 1$
- $\mathrm{erfi}(0) = 1$
- $\mathrm{erf}(-z) = -\mathrm{erf}(z)$
- $\mathrm{erfc}(-z) = -\mathrm{erfc}(z)$
- $\mathrm{erfi}(-z) = \mathrm{erfi}(z)$
- $\mathrm{erf}(\bar{z}) = \overline{\mathrm{erf}(z)}$
- $\mathrm{erfc}(\bar{z}) = \overline{\mathrm{erfc}(z)}$
- $\mathrm{erfi}(\bar{z}) = \overline{\mathrm{erf\imath}(z)}$
- $\mathrm{erf}(z) = 1 - \mathrm{erfc}(z)$
- $\mathrm{ercfc}(z) = 1 - \mathrm{erf}(z)$

and so on.

The accuracy of each of the fundamental assertions just above can be easily demonstrated by making use of their mentioned definitions of the mentioned complex functions. However, to shed light on the special result constituted as in (11), we would like to present some extra information for the second property.

As a more special example, for the value of $\mathrm{erf}(z)$ at infinity (∞), by the definition given by (9) it will be in the form below:

$$\mathrm{erf}(\infty) = \frac{2}{\sqrt{\pi}}\int_0^{\infty} e^{-\mu^2} d\mu.$$

By considering the basic mathematical changes given by

$$\vartheta = \mu^2 \;\Rightarrow\; d\vartheta = 2\mu d\mu$$
$$\Leftrightarrow\; d\mu = \frac{d\vartheta}{2\sqrt{\vartheta}},$$

the following improper integral:

$$\mathrm{erf}(\infty) = \frac{2}{\sqrt{\pi}}\int_0^{\infty} e^{-\mu^2} d\mu$$
$$= \frac{1}{\sqrt{\pi}}\int_0^{\infty} \vartheta^{1/2-1} e^{-\vartheta} d\vartheta$$

is then received. At the same time, with the help of the well-known result given by

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

and the Gamma function defined by (1), can be also expressed as the following-special result:

$$\mathrm{erf}(\infty) = \frac{1}{\sqrt{\pi}}\Gamma\left(\frac{1}{2}\right) = 1.$$

As we have emphasized before, in the light of our classical analysis knowledge, we can easily determine the complex series expansions of those error functions between (9)-(11), which play very important roles in many research areas. Moreover, these series expansions are also considered as their second definitions of the complex function given in (8) in the literature. We want to present those remarks as follows.

Remark 2. The complex error function defined by the form in (9) possesses the series expansion given by

$$\mathrm{erf}(z) = \frac{2}{\sqrt{\pi}} \sum_{j=0}^{\infty} \frac{(-1)^j}{(2j+1)j!} z^{2j+1}, \qquad (14)$$

where

$$z \in \mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}.$$

Remark 3. The complex complementary error function defined by the form in (10) is of the series expansion presented by

$$\mathrm{erfc}(z) = 1 - \frac{2}{\sqrt{\pi}} \sum_{j=0}^{\infty} \frac{(-1)^j}{(2j+1)j!} z^{2j+1}, \qquad (15)$$

where $z \in \mathbb{U}$.

Remark 4. The imaginary error function defined by the complex form in (11) has the series expansion given by

$$\mathrm{erfi}(z) = \frac{2}{\sqrt{\pi}} \sum_{j=0}^{\infty} \frac{1}{(2j+1)j!} z^{2j+1}, \qquad (16)$$

where $z \in \mathbb{U}$.

As some of the possible main consequences of this scientific study, when focusing on the complex error functions introduced by the forms in (8) and their series expansions, all right, various new-classic results appertaining to those complex-power series determined in (14)-(16) can be also produced. For some of them, the auxiliary theorem given below will play a very important role in both their creations and their proofs. Although the earlier paper given in, [32], is the main reference for the related auxiliary theorem, and the extra works (or studies), [8], [11], [33], can be also presented as some extra references. In the same time, in particular, for the theory of complex functions, one may also center on, [22], [34], [35], [36], [37], the related references of this present paper.

Lemma 1. Let $z \in \mathbb{U}$ and also let a complex function $\xi(z)$ be both analytic in the domain $\mathbb{U}$ and satisfy the condition $s(0) = 1$. If there exists a point w_0 in $\mathbb{U}$ such that

$$\Re e\{\xi(z)\} > 0 \text{ for } |z| < |w_0| \qquad (17)$$

and

$$\Re e\{\xi(w_0)\} = 0, \qquad (18)$$

Then

$$w_0 \xi'(w_0) \leq -\frac{1}{2}\left[1 + |\xi(w_0)|^2\right]. \qquad (19)$$

3 The Main Results and Some of Their Implications

In this section, which is the main section of our investigation, the proofs of the first three theorems, which will be presented, can be easily composed with the help of fairly basic mathematical knowledge. The proof of the other three theorems can be accomplished by using the relations between the hypotheses and provisions of Lemma 1 in a logical and coherent way.

Let us now present the relevant theorems in relation to those complex error functions and then center on their proofs.

Theorem 1. For the complex error function being of the form given by (14), the following assertion holds:

$$|\mathrm{erf}(z)| \leq \frac{2}{\sqrt{\pi}} \frac{|z|}{1-|z|^2} \quad (z \in \mathbb{U}). \qquad (20)$$

Theorem 2. For the complementary error function having the form given by (15), the following assertion holds:

$$\big||\mathrm{erfc}(z)| - 1\big| \leq \frac{2}{\sqrt{\pi}} \frac{|z|}{1-|z|^2} \quad (z \in \mathbb{U}). \qquad (21)$$

Theorem 3. For the imaginary error function possessing the form given by (16), the following assertion is true:

$$|\mathrm{erfi}(z)| \leq \frac{2}{\sqrt{\pi}} \frac{|z|}{1-|z|^2} \quad (z \in \mathbb{U}). \qquad (22)$$

Proofs of Theorems 1-3: For some $z \in \mathbb{U}$ and all $j \in \mathbf{N} := \{0,1,2,\cdots\}$, by taking cognizance of the modulus of the power series of those complex functions given by (14)-(16) and also making use of the incontrovertible equalities given by

$$\left|\frac{(-1)^j}{(2j+1)j!}\right| = \frac{1}{(2j+1)j!} \leq 1$$

and

$$|z| < 1,$$

their pending proofs can be easily completed. We think that they are easy. Therefore, their details are omitted here.

Theorem 4. For the complex error function being of the form presented by (14), if the statement given by

$$\Re e\left\{\frac{\mathrm{erf(z)}}{z}+z\frac{d}{dz}\left(\frac{\mathrm{erf(z)}}{z}\right)\right\}$$
$$>-\frac{\sqrt{\pi}}{4}\left(\left|\frac{\mathrm{erf(z)}}{z}\right|-\frac{2}{\sqrt{\pi}}\right)^2 \qquad (23)$$

is true, then the assertion given by

$$\Re e\left(\frac{\mathrm{erf(z)}}{z}\right)>0 \qquad (24)$$

is also true, where $z\in\mathbb{U}$ and, of course, the value of the complex power given in (23) is considered as its principal ones.

Proof. For the proof of Theorem 1, there is a need to consider Lemma 1. For it, when considering the mentioned function $\xi(z)$ being of the complex form given by

$$\xi(z)=\frac{\sqrt{\pi}}{2}\frac{\mathrm{erf(z)}}{z} \quad (z\in\mathbb{U}), \qquad (25)$$

It is easily seen that both $\xi(z)$ is an analytic function in $\mathbb{U}$ and it also satisfies the indicated condition $\xi(0)=1$, even although the critical point $z=0\in\mathbb{U}$ is a removable point for the function $\xi(z)$.

By virtue of differentiating the function in (25) with respect to the complex variable z, the following:

$$\frac{d}{dz}\{\xi(z)\}=\frac{\sqrt{\pi}}{2}\frac{d}{dz}\left(\frac{\mathrm{erf(z)}}{z}\right)$$

or, equivalently,

$$z\frac{d}{dz}\{\xi(z)\}=\frac{\sqrt{\pi}}{2}\mathrm{z}\frac{d}{dz}\left(\frac{\mathrm{erf(z)}}{z}\right) \qquad (26)$$

can be then propounded for some $z\in\mathbb{U}$.

Now, suppose that there exists a point z_0 in $\mathbb{U}$ satisfying the conditions given by (17) and (18) as it is pointed out in Lemma 1. Then, in the light of this special information and by making use of the assertion given in (26) together with (25), the following equality:

$$\Re e\left\{\frac{\sqrt{\pi}}{2}\left(\frac{\mathrm{erf(z)}}{z}+z\frac{d}{dz}\left(\frac{\mathrm{erf(z)}}{z}\right)\bigg|_{z:=z_0}\right)\right\}$$
$$=\Re e\{\xi(z_0)+z_0\xi'(z_0)\} \qquad (27)$$

can be easily represented for all z belonging to the complex domain $\mathbb{U}$.

Since the function with the complex variable $\xi(z)$ and the concerned point $z:=z_0$ are suitable for all conditions of Lemma 1, therefore, with the help of the information indicated in (17) and (18), the equation given by (27) gives us

$$\Re e\left\{\frac{\sqrt{\pi}}{2}\left(\frac{\mathrm{erf(z)}}{z}+z\frac{d}{dz}\left(\frac{\mathrm{erf(z)}}{z}\right)\bigg|_{z:=z_0}\right)\right\}$$
$$=\Re e\{\xi(z_0)\}+\Re e\{z_0\xi'(z_0)\}$$
$$\leq-\frac{1}{2}[1+|\xi(z_0)|^2]+\Re e\{\xi(z_0)\}$$
$$\leq-\frac{1}{2}[1+|\xi(z_0)|^2]+|\xi(z_0)|$$
$$\leq-\frac{1}{2}(\,|\xi(z_0)|-1\,)^2$$
$$\leq-\frac{1}{2}\left(\frac{\sqrt{\pi}}{2}\left|\frac{\mathrm{erf(z_0)}}{z_0}\right|-1\right)^2, \qquad (28)$$

where $z\in\mathbb{U}$.

As a result of various mathematical operations, it is easily seen the inequality determined in (28) is a contradiction with the inequality presented in (23), which is the hypothesis of Theorem 1. Therefore, the desired proof is completed.

Theorem 5. For the error function being of the complex form given as in (15), if the inequality given by

$$\Re e\left\{\frac{1-\mathrm{erfc(z)}}{z}+z\frac{d}{dz}\left(\frac{1-\mathrm{erfc(z)}}{z}\right)\right\}$$
$$>-\frac{\sqrt{\pi}}{4}\left(\left|\frac{1-\mathrm{erfc(z)}}{z}\right|-\frac{2}{\sqrt{\pi}}\right)^2 \qquad (29)$$

is supplied, then the assertion is given by

$$\Re e\left(\frac{1-\mathrm{erfc(z)}}{z}\right)>0$$

is also supplied for $z\in\mathbb{U}$ and the value of the complex power given in (29) is considered as its principal one.

Theorem 6. For the error function being of the complex form given as in (16), if the inequality given by

$$\Re e\left\{\frac{\mathrm{erfi(z)}}{z}+z\frac{d}{dz}\left(\frac{\mathrm{erfi(z)}}{z}\right)\right\}$$
$$>-\frac{\sqrt{\pi}}{4}\left(\left|\frac{\mathrm{erfi(z)}}{z}\right|-\frac{2}{\sqrt{\pi}}\right)^2 \qquad (30)$$

is provided, then the assertion is given by

$$\Re e\left(\frac{\mathrm{erfi(z)}}{z}\right) > 0$$

is also provided for some z in the set $\mathbb{U}$ and the value of the complex power presented in (30) is considered as its principal one.

Proofs of Theorems 5 and 6. For each one of those proofs, with similar thought, it will be enough to follow all of the similar steps used in the proof of Theorem 4. For them, when one reconsiders the mentioned-analytic function $\xi(z)$ as the complex forms designed by

$$\xi(z) := \frac{\sqrt{\pi}}{2}\frac{1-\mathrm{erfc}(z)}{z}$$

and

$$\xi(z) := \frac{\sqrt{\pi}}{2}\frac{\mathrm{erf}\,i(z)}{z} \ ,$$

respectively, where $z \in \mathbb{U}$, and then applies the same ways, as we did in the proof of Theorem 4, to those two theorems, the pending proofs of Theorems 5 and 6 can be easily completed. The related details of those theorems are also omitted again.

4 Conclusions and Recommendations

As is known, in the first part of this special research, some special mathematical forms have been mentioned, various information has been given about the (complex) error functions in the second part, and in the third part, some extensive theories (together with their possible implications) in relation to those complex error functions have been put forward and some of them have been also demonstrated. In this section relating to the conclusion and recommendations, in this note, we would like to elaborate on the scope of our scientific research with various information for our readers. In such cases, we have to focus on extensive research or information.

As we have emphasized in the previous sections, error functions are special functions that will appeal to almost all fields of science, which are both with real variables and with complex variables. As it is known, we have only concentrated on three basic types of all error functions encountered in the literature. We have stated that each of them has many basic features unique to them, and we have also presented some of them. In specially, the diverse studies given in, [29], [30], [33], can be reviewed for other special properties as well as for other type error functions. In addition, the relationships related to the three related error functions and their other types with other special functions are quite extensive.

It even differs considerably in its theoretical applications together with its inverses in special functions, serial forms, transformation theory, approximation theory, and measurement theory. Each result, which may be unusual, can have quite different meanings in terms of error functions with complex parameters. For those, the special information given in, [1], [3], [4], [5], [10], [14], [17], [18], [22], [23], [26], [28], [33], can be also examined.

From various aspects, since our study includes various theoretical results, for researchers, it would be appropriate to mention only some special implications of the related theories that we obtained. Some of those possible suggestions can be listed below.

1. Special relationships that we have not highlighted with Remarks 1 can be taken into account in the theories that we have obtained before or in any specific results regarding them. For example, considering the fundamental relation given by (13) for Theorem 5 and Theorem 6, it is easily seen that the related theorems are equivalent to each other.

However, given the special relations between the various special functions of error functions, each of my theorems presented will also require various special theories (or consequences) that may be related to those special functions.

However, the mentioned special relations between various special functions and error functions, each of our main results will also require various special theories (or consequences), which are related to those special functions. As extra information, those relationships concerning some special functions can be checked in the paper in, [1], [21], [29]. Moreover, as a simple example in relation to the well-known incomplete Gamma function, the special relations are given by

$$\begin{aligned}\mathrm{erf(z)} &= \frac{1}{\sqrt{\pi}}\gamma\left(\frac{1}{2}, z^2\right)\\ &= \frac{2}{\sqrt{\pi}}\sum_{j=0}^{\infty}\frac{(-1)^j}{(2j+1)j!}\ z^{2j+1}\end{aligned}$$

can be also re-evaluated in the scope mentioned just above, where $z \in \mathbb{U}$.

2. As is known, Lemma 1 has been used for the proofs of all of our theorems. Naturally, it is essential to satisfy both the condition and the differentiability condition for all points in the set U for the function, which have been used in the relevant lemma, Lemma 1.

Therefore, provided that these fundamental conditions are taken into account, the relevant complex function used in the proofs of all theorems can be defined in various different types and all of our main results can be also reconstructed according to these new functions. For instance, by considering the series expansion belonging to the complex error function given in (15), the corresponding function can be then selected as follows:

$$\xi(z) = \mathrm{erfc}(z),$$

which explicitly satisfies the mentioned conditions in the lemma, *where* the variable z belongs to $\mathbb{U}$ of the complex plane $\mathbb{C}$.

3. By taking cognizance of various inequalities frequently used in classical analysis, various special results can be reconstituted with the help of the mentioned theories (or their special results). As an example, the following proposition can be easily derived with the help of Theorem 1**.**

Proposition 1. For the complex error function being of the form presented by (14), if any one of the statements given by

$$\left|\Re e\left\{\frac{\mathrm{erf(z)}}{z} + z\frac{d}{dz}\left(\frac{\mathrm{erf(z)}}{z}\right)\right\}\right| < \frac{\sqrt{\pi}}{4}\left(\left|\frac{\mathrm{erf(z)}}{z}\right| - \frac{2}{\sqrt{\pi}}\right)^2$$

and

$$\sqrt{\left|\Re e\left\{\frac{\mathrm{erf(z)}}{z} + z\frac{d}{dz}\left(\frac{\mathrm{erf(z)}}{z}\right)\right\}\right|} < \frac{\sqrt[4]{\pi}}{2}\left|\left|\frac{\mathrm{erf(z)}}{z}\right| - \frac{2}{\sqrt{\pi}}\right|$$

is satisfied, then the inequality given by (24) is also satisfied.

4. Lastly, some complex forms of various types of real functions that are frequently used in the literature can be also used. As novel research, by using those possible forms of all those complex functions that can be created with different logic, together with our main results, various logical results can be also reproduced. For example, the complex forms of the normal distribution function in (3) (or all their possible special cases), which have important roles in statistics, can be reconsidered. In such cases, various new results can be also reproduced with the help of those special complex functions with our main results. For instance, by considering the special function $\varphi(\mu)$ defined as in (4), the following special integral, which was used by K. Gauss for this study, given by

$$\int \varphi(\mu)\, d\mu$$

can be then reused for our main results. In this case, for the indicated complex function, when it is evaluated in the information of Theorem 1, only two of the pending propositions can be easily reconstructed with the help of the complex functions given by

$$\Phi_1(z) = \frac{1}{2}\left(1 + \mathrm{erf}\left(\frac{z}{\sqrt{2}}\right)\right)$$

or

$$\Phi_2(z) = \frac{1}{2}\left(2 - \mathrm{erfc}\left(\frac{z}{\sqrt{2}}\right)\right),$$

where $z \in \mathbb{U}$.

Especially, noting here that, clearly, if these special-complex functions:

$$\Phi_1(z) \quad \text{and} \quad \Phi_2(z)$$

are taken into account together with the complex function $\xi(z)$ in the mentioned lemma, i.e., Lemma 1, those complex functions (just above), which are also suitable for all the hypotheses of the lemma, can be also re-arranged for each one of our fundamental results constituted as in the third section.

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