Neutrosophic Assessment and Decision Making

MICHAEL GR. VOSKOGLOU
School of Engineering - Department of Mathematical Sciences
University of Peloponnese (Ex Graduate TEI of Western Greece)
26334 Patras - GREECE

Abstract: - Smarandache's theory of neutrosophy is a generalization of Zadeh's fuzziness characterizing each element of the universal set with the membership degree, as in fuzzy sets, and in addition with the degrees of non-membership and indeterminacy with respect to the corresponding neutrosophic set. In the present work we use neutrosophic sets as tools for assessment and decision making. This turns out to be very useful when one is not sure about the correctness of the grades assigned to the elements of the universal set. Examples are also presented illustrating our results.

Key-Words: - Fuzzy Set (FS), Neutrosophic Set (NS), Neutrosophic Triplet (NT), Soft Set (SS), Fuzzy Assessment, Decision Making (DM).

Received: December 12, 2022. Revised: August 9, 2023. Accepted: September 5, 2023. Published: October 2, 2023.

1 Introduction

The theory of *Neutrosophy* introduced by Smarandache [1] in 1995, is an extension of Zadeh's *Fuzziness* [2]. The term neutrosophy was formed by the synthesis of the words neutral and the Greek "sofia", which means wisdom or complete knowledge. As we will see in detail in the next section, in a *neutrosophic set (NS)* all the elements of the universal set of the discourse are characterized by three parameters, which take values in the unit interval [0, 1].

NSs have found important applications to practical everyday life problems, in which the use of Zadeh's FSs has not been proved to be sufficient for obtaining the required results. The purpose of this work is to present applications of NSs to assessment [3] and decision making (DM) [4] processes.

The rest of the paper is organized as follows: Section 2 contains the mathematical background about NSs and *soft sets (SSs)* [5], needed for the understanding of the paper. The application of NSs to assessment processes is developed in section 3, whereas section 4 describes their application to DM. The article closes with the final conclusions and some hints for further research, contained in its last section 5.

2 Mathematical Background

2.1 Fuzzy Sets

Zadeh, in order to tackle mathematically the existing in real life partial truths, defined in 1965 the concept of *fuzzy set (FS)* as follows [2]:

Definition 1: Let U be the universal set of the discourse, then a FS A in U is defined with the help of its *membership function* m: $U \rightarrow [0,1]$ as the set of the ordered pairs

$$A = \{(x, m(x)): x \in U\} \quad (1)$$

The real number m(x) is called the *membership degree* of x in A**.** The greater m(x), the more x satisfies the characteristic property of A.

There is not any exact rule for defining the membership function of a FS. The methods used for this are usually empirical or statistical and its definition is not unique, depending on each observer's subjective criteria about the corresponding situation. Defining, for example, the FS of the tall men, one may consider all men with heights greater than 1.90 m as tall and another one all those with heights greater than 2 m. As a result, the first observer will attach membership degree 1 to all men between 1.90 m and 2 m, whereas the second one will attach membership degrees <1. Analogous differences will obviously appear to the membership degrees of the men with heights <1.90 m.

Consequently, the only restriction in the definition of the membership function is that it must be compatible with common sense; otherwise the resulting FS does not give a creditable description of the corresponding real situation. This could happen, for instance, if in the previous example men with heights <1.60 m possessed membership degrees $\geq$0.5.

In a later stage, when membership degrees were reinterpreted as *possibility* distributions, FSs were extensively used for managing the existing in real life *uncertainty*, which is connected with incomplete or vague information. Possibility is an alternative mathematical theory for tackling the uncertainty [6]. Zadeh articulated the relationship between possibility and probability by noticing that whatever is probable must be primarily possible [7].

All notions and operations defined on crisp sets are extended in a natural way to FSs For general facts on FSs and the connected to them uncertainty we refer to the book [8].

2.2 Neutrosophic Sets

Following the introduction of FSs, a series of extensions and related theories have been developed for tackling more effectively all the forms of the existing uncertainties [9].

Atanassov in 1986, added to Zadeh's membership degree the degree of *non-membership* and defined the concept of *intuitionistic FS (IFS)*, as an extension of FS [10].

Smarandache in 1995, motivated by the various neutral situations appearing in real life situations - like <positive, zero, negative>, <small, medium, high>, <win, draw, defeat>, etc. – introduced, in addition of the degrees of membership and non-membership, the degree of *indeterminacy* and extended the concept of IFS to the concept of NS [1]. In this work we will make use only of the simplest form of a NS, termed as *single valued NS (SVNS)* and defined as follows [11]:

Definition 2: A SVNS A in the universal set U is defined as the set of the ordered tumbles

$$A = \{(x,T(x),I(x),F(x)): x \in U,\ T(x),I(x),F(x) \in [0,1],\ 0 \leq T(x)+I(x)+F(x) \leq 3\} \quad (2).$$

In (2) T(x), I(x), F(x) are the degrees of *truth* (or membership), *indeterminacy* (or neutrality) and *falsity* (or non-membership) of x in A respectively, called the *neutrosophic components* of x. For simplicity, we write A<T, I, F> and the elements of A in the form (t, i, f) of *neutrosophic triplets (NTs)*, with t, i, f in [0, 1].

Example 1: Let U be the set of the players of a football team and let A be the SVNS of the good players of U. Then each player x of U is characterized by a NT (t, i, f), with t, i, f in [0, 1]. For instance, x(0.7, 0.1, 0.4) ∈ A means that there is a 70% belief that x is a good player, a 10% doubt about it and at the same time a 40% belief that x is not a good player. In particular, x(0,1,0) ∈ A means that we do not know absolutely nothing about x's affiliation with A.

When T(x)+I(x)+F(x)<1, it leaves room for *incomplete* information, when T(x)+I(x)+F(x)=1 for *complete* information, and when T(x)+I(x)+F(x)>1 for *inconsistent* information about x in A. A SVNS may contain simultaneously elements corresponding to all kinds of the previous information. All notions and operations defined on FSs are extended in a natural way to NSs [11]

Since the NTs of a SVNS A are ordered triplets, one may define addition among them and scalar multiplication of a positive number with a NT in the usual way, as follows:

Definition 3: Let (t_1, i_1, f_1), (t_2, i_2, f_2) be in A and let r be a positive number. Then:

- The *sum* $(t_1, i_1, f_1) + (t_2, i_2, f_2) = (t_1+ t_2,\ i_1+ i_2,\ f_1+ f_2)$ (3)
- The *scalar product* $r(t_1, i_1, f_1) = (rt_1, r i_1, f_1)$ (4)

The sum and the scalar product of the NTs of a SVNS A with respect to Definition 3 <u>need not,</u> be a NT of A, since it may happen that $(t_1+ t_2)+(i_1+ i_2)+(f_1+ f_2)>3$ or $rt_1+ri_1+ rf_1>3$. With the help of Definition 3, however, one can define the *mean value* of a finite number of NTs of A, which is always <u>in A.</u> as follows:

Definition 4: Let A be a SVNS and let (t_1, i_1, f_1), (t_2, i_2, f_2), …., (t_k, i_k, f_k) be a finite number of elements of A. Assume that (t_i, i_i, f_i) appears n_i times in an application, i = 1,2,…., k. Set $n = n_1+n_2+….+n_k$. Then the *mean value* of all these elements of A is defined to be the NT of A

$$(t_m,i_m,f_m) = \frac{1}{n}[n_1(t_1, i_1,f_1)+n_2(t_2,i_2,f_2)+….+n_k(t_k,i_k, f_k)] \quad (5)$$

2.3 Soft Sets

The disadvantage of FSs concerning the definition of their membership function remains obviously the same for all their extensions involving membership degrees, like the IFSs, the NSs, etc. To overcome this difficulty, the concept of *interval-valued FS (IVFS)* was introduced in 1975. An IVFS is defined by mapping the universe U to the set of the closed subintervals of [0, 1] [12]. Other theories related to FSs were also developed, in which the definition of a membership function is either not necessary (*grey systems and numbers* [13]), or it is overpassed by using a pair of crisp sets giving the lower and upper bound of the original set (*rough sets*) [14].

Molodstov introduced in 1999 the concept of SS for tackling the uncertainty in a parametric manner, which does not need the definition of a membership function. Namely, a SS is defined as follows [5]:

Definition 5: Let E be a set of parameters and let f be a map from E into the power set P(U) of the universe U. Then the SS (f, E) in U is defined as the set of the ordered pairs

$$(f, E) = \{(e, f(e)): e \in A\} \quad (6)$$

In other words, a SS is a parametrized family of subsets of U. The name "soft" is due to the fact that the form of (f, E) depends on the parameters of E.

Example 2: Let U= $\{C_1, C_2, C_3\}$ be a set of cars and let E = $\{e_1, e_2, e_3\}$ be the set of parameters e_1=cheap, e_2=elegant and e_3= expensive. Let us further assume that C_1, C_2 are cheap, C_3 is expensive and C_2, C_3 are elegant cars. Then, a map f: E $\rightarrow$ P(U) is defined by $f(e_1)=\{C_1, C_2\}$, $f(e_2)=\{C_2, C_3\}$ and $f(e_3)=\{C_3\}$. Therefore, the SS (f, E) in U is the set of the ordered pairs

$$(f, E) = \{(e_1, \{C_1, C_2\}), (e_2, \{C_2, C_3\}, (e_3, \{C_3\}\} \quad (7)$$

Maji et al. [15] introduced a *tabular representation* of a SS in the form of a binary matrix in order to be stored easily in a computer's memory. For instance, the tabular representation of the soft set (f, E) of Example 2 is given by Table 1.

Table 1. Tabular representation of the SS of Example 2

	e_1	e_2	e_3
C_1	1	0	0
C_2	1	1	0
C_3	0	1	1

A FS in U with membership function y = m(x) is a SS in U of the form (f, [0, 1]), where $f(\alpha)=\{x \in U: m(x) \geq \alpha\}$ is the corresponding *a-cut* of the FS, for each α in [0, 1]. Consequently the concept of SS is a generalization of the concept of FS. All notions and operations defined on FSs are extended in a natural way to SSs [16].

3 Neutrosophic Assessment

The performance of the members of a group is assessed frequently by using qualitative grades (linguistic expressions) instead of numerical scores. This happens either because the existing data about their performance are not very clear, or for reasons of elasticity (e.g. from teacher to students). Obviously, in such cases the mean performance of the group cannot be assessed by calculating the mean value of the individual scores of its members. For tackling this situation, we have used in earlier works either *triangular fuzzy numbers* or *grey numbers* (closed real intervals) and we have shown that these two methods are equivalent [17] (sections 5.2 and 6.2).

Cases appear, however, in practice, in which one is not sure about the accuracy of the qualitative grades assigned to the objects under assessment (e.g. students). In such cases, the use of NSs is possibly the best way for evaluating a group's overall performance. The following example illustrates this situation.

Example 3: The new teacher of a student class is not sure yet about the quality of each of the students. He characterized, therefore, the set of the very good students by NTs as follows: $s_1(1, 0, 0)$, $s_2(0.9, 0.1, 0.1)$, $s_3(0.8, 0.2, 0.1)$, $s_4(0.4, 0.5, 0.8)$, $s_5(0.4, 0.5, 0.8)$, $s_6(0.3, 0.7, 0.8)$, $s_7(0.3, 0.7, 0.8)$, $s_8(0.2, 0.8, 0.9)$, $s_9(0.1, 0.9, 0.9)$, $s_{10}(0.1, 0.9, 0.9\}$ and the remaining 10 students of the class by (0, 0, 1). This means that the teacher is absolutely sure that s_1 is a very good student, 90% sure that s_2 is a very good student too, but at the same time he has a 10% doubt about it and a 10% belief that s_2 is not a very good student, etc. For the last 10 students the teacher is absolutely sure that they are not very good students. Evaluate the mean level of the student skills.

Solution: The mean level of the student skills can be estimated by the mean value of the corresponding NTs, i.e. by $\frac{1}{20}$[(1, 0, 0)+(0.9, 0.1, 0.1)+(0.8, 0.2, 0.1)+2(0.4, 0.5, 0.8)+2(0.3, 0.7, 0.8)+(0.2, 0.8, 0.9)+2(0.1, 0.9, 0.9)+10(0, 0, 1)], which by equations (3) and (4) is equal to $\frac{1}{20}$(4.5, 5.3, 16.3) = (0.225, 0.265, 0.815). This means that a random student of the class has a 22.5 % probability to be a very good student, however, there exists also a 26.5% doubt about it and an 81.5% probability to be not a very good student.

4 Neutrosophic Decision Making

Maji et al. [15] used the tabular form of a SS as a tool for DM in a parametric manner. The following example highlights their method:

Example 4: A person wants to buy one of the six cars C_1, C_2, C_3, C_4, C_5 and C_6. His ideal preference is a high-speed, automatic (gear-box), hybrid (petrol and electric power) and cheap car. Assume that C_1, C_2, C_6 are the high-speed, C_2, C_3, C_5, C_6 are the automatic, C_3, C_5 are the hybrid and C_4 is the unique cheap car. Which is the best choice for the candidate buyer?

Solution: Set V = $\{C_1, C_2, C_3, C_4, C_5, C_6\}$ and let E = $\{e_1, e_2, e_3, e_4\}$ be the set of the parameters

e_1=high-speed, e_2=automatic, e_3=hybrid and e_4=cheap. Consider the map g: E → Δ(V) defined by g(e_1) = { C_1, C_2, C_6}, g(e_2) = { C_2, C_3, C_5, C_6}, g(e_3) = { C_3, C_5}, g(e_4) = { C_4} and the corresponding SS

$$(g, E) = \{(e_1, \{C_1, C_2, C_6\}), (e_2, \{C_2, C_3, C_5, C_6\}), (e_3, \{C_3, C_5\}), (e_4, \{C_4\})\} \quad (7)$$

Table 2: Tabular representation of the SS of Example 4

	e_1	e_2	e_3	e_4
C_1	1	0	0	0
C_2	1	1	0	0
C_3	0	1	1	0
C_4	0	0	0	1
C_5	0	1	1	0
C_6	1	1	0	0

Forming the tabular representation of the SS (g, P) (Table 2) the *choice value* of each car is calculated by adding the binary elements of the corresponding row of it. The cars C_1 and C_4 have, therefore, choice value 1 and all the others have choice value 2. Consequently, the buyer must choose one of the cars C_2, C_3, C_5 or C_6.

The previous decision, obtained by applying the method of Maji et al. [15], is not very helpful for the candidate buyer, since it excludes only two among the six available cars. This is due to the fact that in the tabular form of the corresponding SS the characterizations of the elements of the universal set (cars in our example) by the corresponding parameters are replaced with the binary elements (truth values) 0, 1. In other words, although the method starts from a fuzzy basis (SS), it uses bivalent logic for making the required decision, e.g. cheap or not cheap! Consequently, this methodology could lead to a wrong decision, if some (or all) of the parameters have not a bivalent texture; e.g. the parameter “hybrid” has a bivalent texture, but not the parameter “cheap”.

In order to tackle this problem, we have used in earlier works grey numbers, instead of the binary elements 0, 1, in the tabular representation of the corresponding SS [18]. DM situations, however, appear frequently in reality, in which the decision maker has doubts about the correctness of the qualitative (fuzzy) parameters assigned to some or all of the elements of the set of the discourse. In such cases, the best way to perform the DM process is to use NSs. This is illustrated by the following example.

Example 5: Reconsider Example 4 and assume that the candidate buyer, being not sure about the correctness of the qualitative parameters p_1 and p_4 assigned to each of the six cars, decided to proceed by replacing the parameters by NTs. As a result, the tabular matrix of the DM process takes the form shown in Table 3. Which is the best decision for the candidate buyer in this case?

Table 3: Tabular representation of the SS of Example 5

	e_1	e_2	e_3	e_4
C_1	(1, 0,0)	(0,0,1)	(0,0,1)	(0.6,0.3,0.1)
C_2	(1,0,0)	(1,0,0)	(0,0,1)	(0.2,0.2,0.6)
C_3	(0.5,0.4, 0.1)	(1,0,0)	(1,0,0)	(0.6, 02,0.2)
C_4	(0.5, 0.2, 0.3)	(0,0,1)	(0,0,1)	(1, 0, 0)
C_5	(0.5, 0.1, 0.4)	(1,0,0)	(1,0,0)	(0.6,0.3,0.1)
C_6	(1, 0, 0)	(1,0,0)	(0,0,1)	(0.4,0.4,0.2)

Solution: The choice value of each car in this case is defined to be the mean value of the NTs of the line of Table 3 in which he belongs. Thus, by equation (5), the choice value of C_1 is equal to $\frac{1}{4}$[(1, 0, 0)+2(0, 0, 1)+(0.6, 0.3, 0.1)] = $\frac{1}{4}$(1.6, 0.3, 2.1) = (0.4, 0.075, 0.525). In the same way one finds that the choice values of C_2, C_3, C_4, C_5 and C_6 are (0.55, 0.005, 0.4), (0.775, 0.15, 0.075), (0.375, 0.05, 0.575), (0.775, 0.1, 0.125) and (0.6, 0.1, 0.3) respectively.

Now the candidate buyer may use either an optimistic criterion by choosing the car with the greatest truth degree, or a conservative criterion by choosing the car with the lower falsity degree. Consequently, using the optimistic criterion he must choose one of the cars C_3 and C_5, whereas using the conservative criterion he must choose the car C_3. A combination, therefore, of the two criteria leads to the final choice of the car C_3. Observe, however, that, since the indeterminacy degree of C_3 is 0.15 and of C_5 is 0.1, there is a slightly greater doubt about the suitability of C_3 with respect to C_5. In other words, the choice of C_3 is connected with a slightly greater risk. In final analysis, therefore, all the neutrosophic components assigned to each car give useful information about its suitability.

4. Discussion and Conclusions

In this work NSs were used as tools in assessment and DM processes. This is a very useful approach when one is not sure about the correctness of the parameters / qualitative grades assigned to the elements of the universal set.

Our DM method in particular, was obtained by adapting a parametric DM method of Maji et al. [15] using SSs as tools. In our method the binary elements 0, 1 of the tabular representation of the corresponding SS are replaced by NTs.

The combination of two or more of the extensions of FSs that have been developed for tackling efficiently the various forms of the existing in real world uncertainty (SSs and NSs in this paper), appears in general to be an effective way for obtaining better results, not only for DM, but also for assessment and for various other human activities. Consequently this is a promising area for further research.

References:

[1] Smarandache, F., *Neutrosophy/ Neutrosophic probability, set, and logic*, Proquest, Michigan, USA, 1998.

[2] Zadeh, L.A., Fuzzy Sets, *Information and Control*, 8, 1965, pp. 338-353.

[3] Voskoglou, M.Gr., Neutrosophic Assessment of Student Mathematical Skills, *Physical and Mathematical Education*, 38(2), 22-26, 2023.

[4] Voskoglou, M.Gr., An Application of Neutrosophic Sets to Decision Making, *Neutrosophic Sets and Systems*, 53, 1-9, 2023.

[5] Molodtsov, D. , Soft Set Theory-First Results, *Computers and Mathematics with Applications*, 37(4-5), 1999, pp. 19-31.

[6[Dubois, D., Prade, H. Possibility theory, probability theory and multiple-valued logics: A clarification, *Ann. Math. Artif. Intell.*, 32, 2001, pp. 35-66.

[7] Zadeh, L.A. Fuzzy Sets as a basis for a theory of possibility, *Fuzzy Sets Syst.*, 1, 1978, pp. 3–28.

[8] Klir, G. J. & Folger, T. A., *Fuzzy Sets, Uncertainty and Information*, Prentice-Hall, London, 1988

[9] Voskoglou, M.Gr., Fuzzy Systems, Extensions and Relative Theories, *WSEAS Transactions on Advances in Engineering Education*, 16, 2019, pp. 63-69

[10] Atanassov, K.T., Intuitionistic Fuzzy Sets, *Fuzzy Sets and Systems*, 20(1), 1986, pp.87-96.

[11] Wang, H., Smarandanche, F., Zhang, Y. and Sunderraman, R., Single Valued Neutrosophic Sets, *Review of the Air Force Academy (Brasov)*, 1(16), 2010, pp. 10-14.

[12] Dubois, D., Prade, H., Interval-Valued Fuzzy Sets, Possibility Theory and Imprecise Probability, *Proceedings EUSFLAT-LFA*, 2005, pp. 314-319.

[13] Deng, J., Control Problems of Grey Systems, *Systems and Control Letters*, 1982, pp. 288-294.

[14] Pawlak, Z., *Rough Sets: Aspects of Reasoning about Data*, Kluer Academic Publishers, Dordrecht, 1991.

[15] Maji, P.K., Roy, A.R., Biswas, R., An Application of Soft Sets in a Decision Making Problem, *Computers and Mathematics with Applications*, 44, 2002, pp. 1077-1083.

[16] Maji, P.K., Biswas, R., & Ray, A.R., Soft Set Theory, *Computers and Mathematics with Applications*, 45, 2003, pp. 555-562

[17] Voskoglou, M.Gr., Assessing Human-Machine Performance under Fuzzy Conditions, *Mathematics*, 7, article 230, 2019.

[18] Voskoglou, M.Gr., A Hybrid Method for Assessing Student Mathematical Modelling Skills under Fuzzy Conditions, *International Journal of Computational and Applied Mathematics & Computer Science*, 2, 2022, pp. 106-114.

Contribution of Individual Authors to the Creation of a Scientific Article (Ghostwriting Policy)

The author contributed in the present research, at all stages from the formulation of the problem to the final findings and solution.

Sources of Funding for Research Presented in a Scientific Article or Scientific Article Itself

No funding was received for conducting this study.

Conflict of Interest

The author has no conflict of interest to declare that is relevant to the content of this article.