New Common Uniqueness Results on Generalized Normed Space

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Abstract: - In this study, by evaluating two mappings that do not both exhibit direct continuity features, fresh results were found supporting the uniqueness of the solutions in generalized spaces.

Key-Words: - Contractive mapping, Fixed point, Banach space, Uniqueness.

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1 Introduction

One of the key areas in the development of functional analysis is fixed point theory. Additionally, it has been successfully applied in chemistry, biology, economics, computer science, engineering, and other scientific fields. Studying the fixed point and confirming the uniqueness of the solution is a very old study. Banach [1] is the first developer of this study and presented it in its basic form by presenting the Banach contraction principle, which is considered the raw material for all subsequent developments that appeared in this field. Banach proved his principle in a perfect one-dimensional space, and then more exciting results followed after that based on the development of space or the development of contraction, see ([5],[7],[14-18]). In our study, we dealt with the development in space and the contraction as well.

Gähler [2] introduced the idea of 2-Banach spaces, and these spaces have since been investigated by demonstrating the existence of fixed points of contractive mappings. The fixed point theory of mappings has been developed in such spaces as it has in other spaces. Iseki [8] also achieved the fundamental findings regarding the fixed points of mappings in 2-Banach spaces for the first time. After Iseki's works fixed point results in these spaces are found in [9].

Gähler [3], offered a fascinating n-norm theory on linear spaces then numerous authors, including Kim et al. [10], Malceski [12], Misiak [13], and Gunawan [4], have developed linear n-normed spaces systematically. In a linear n-Banach space current research on the functional analysis parts we're referring to [6].

2 Main Results

A new form of the common fixed points results in generalized Banach space will be discussed in this section.

Definition 2.1. T is a continuous mapping on an n–Banach space X at z if every sequence $\|\{x_k\}, y_2, \dots, y_n\| \subset X$, satisfied $\|\{x_k\}, y_2, \dots, y_n\| \to z$ implies that $\|T\{x_k\}, y_2, \dots, y_n\| \to \|Tx, y_2, \dots, y_n\|$ as $k \to \infty$, for all $z, y_2, \dots, y_n \in X$.

Theorem 2.1. Let T be a continuous mapping and F be a contractive self-mapping on an n–Banach space X and let $\emptyset_{y_2,\dots,y_n}: X \to [0,\infty)$ such that

$$\emptyset y \leq \|Fy - Ty, y_2, \dots, y_n\|, \tag{2.1}$$

for every $y_2, \dots, y_n$ in X, satisfied

$$\|Fx - Fy, y_2, \dots, y_n\| \leq \emptyset_{y_2,\dots,y_n}(Tx) + \emptyset_{y_2,\dots,y_n}(Ty) - \{\emptyset_{y_2,\dots,y_n}(Fx) + \emptyset_{y_2,\dots,y_n}(Fy)\}, \tag{2.2}$$

for all $x, y \in X$ and $x \neq y$, then F and T have a unique common fixed point in X.

Proof. Consider: $Tx_{m+1} = Fx_m = x_{m+1}$, $n \in N$. Now

$$\|x_m - x_{m+1}, y_2, \dots, y_n\| = \|Fx_{m-1} - Fx_m, y_2, \dots, y_n\|$$

$$\leq \emptyset_{y_2,\dots,y_n} Tx_{m-1} + \emptyset_{y_2,\dots,y_n} Tx_m - \emptyset_{y_2,\dots,y_n} Fx_{m-1} - \emptyset_{y_2,\dots,y_n} Fx_m$$

$$\leq \emptyset_{y_2,\dots,y_n} x_{m-1} + \emptyset_{y_2,\dots,y_n} x_m - \emptyset_{y_2,\dots,y_n} x_m - \emptyset_{y_2,\dots,y_n} x_{m+1}$$

$$= \emptyset_{y_2,\dots,y_n} x_{m-1} - \emptyset_{y_2,\dots,y_n} x_{m+1},$$

for all $y_2, \dots, y_n \in X$ and $m \in N$.

Hence, $\sum_{k=1}^{\infty} \|x_k - x_{k+1}, y_2, \dots, y_n\| < \infty$, implies $\|x_k - x_{k+1}, y_2, \dots, y_n\| \to 0$ as $k \to \infty$.

Thus for any two positive integers s, r with $s > r$ and for all $y_2, \dots, y_n \in X$, we get as $s, r \to \infty$.

$$\|x_r - x_s, y_2, \dots, y_n\|$$

$$\leq \|x_r - x_{r+1}, y_2, \dots, y_n\| + \|x_{r+1} - x_{r+2}, y_2, \dots, y_n\| + \cdots + \|x_{s-1} - x_s, y_2, \dots, y_n\| \to 0.$$

Treating the case $r > s$ in a similar, we obtain that $\{x_m\}$ is a Cauchy sequence in X, and since X is complete, there exists a point $z \in X$ such that $\{x_m\} \to z$ as $m \to \infty$.

Since F is a contractive mapping thus it is a continuous, $\{Fx_m\} \to F(z)$ as $m \to \infty$.

Hence, z is a common fixed point of F and T. Then the existence of (2.2) has been proven.

Now for any $y_2, \dots, y_n \in X$, suppose that $z_1 \in X$ be another fixed point of F and T then $Tz_1 = Fz_1 = z_1$, where $z_1 \neq z$. Hence, for all $y_2, \dots, y_n \in X$,

$$\|z - z_1, y_2, \dots, y_n\| = \|Fz - Fz_1, y_2, \dots, y_n\|$$

$$\leq \emptyset_{y_2,\dots,y_n}(Tz) + \emptyset_{y_2,\dots,y_n}(Tz_1) - \emptyset_{y_2,\dots,y_n}(Fz) - \emptyset_{y_2,\dots,y_n}(Fz_1)$$

$$= \emptyset_{y_2,\dots,y_n} z + \emptyset_{y_2,\dots,y_n} z_1 - \emptyset_{y_2,\dots,y_n} z - \emptyset_{y_2,\dots,y_n} z_1 = 0.$$

This implies $z_1 = z$. Uniqueness has been proven.

Theorem 2.2. Let F and T are contractive mappings on an n–Banach spaces X, suppose for $w_2, \dots, w_n \in X$, there exists a function: $\emptyset_{w_2,\dots,w_n}: X \to [0, \infty)$ such that

$$\|Tx - Fy, w_2, \dots, w_n\|$$

$$\leq \emptyset_{w_2,\dots,w_n} x + \emptyset_{w_2,\dots,w_n} y - \{\emptyset_{w_2,\dots,w_n}(Tx) + \emptyset_{w_2,\dots,w_n}(Fy)\},$$

for all $x, y \in X$. Then T and F have a unique common fixed point in X.

Proof. For $x_0 \in X$, $y_0 \in X$, take $x_1 = T(x_0), x_2 = T(x_1), \dots, x_{m+1} = T(x_m)$

and $y_1 = F(y_0), y_2 = F(y_1), \dots, y_{m+1} = F(y_m)$, $m = 1,2,\dots$

Now ,

$$\sum_{k=1}^{m} \|x_k - y_k, w_2, \dots, w_n\| = \sum_{k=1}^{m} \|T(x_{k-1}) - F(y_{k-1}), w_2, \dots, w_n\|$$

$$\leq \sum_{k=1}^{m} \{\emptyset_{w_2,\dots,w_n}(x_{k-1}) + \emptyset_{w_2,\dots,w_n}(y_{k-1}) - \emptyset_{w_2,\dots,w_n}(x_k) - \emptyset_{w_2,\dots,w_n}(y_k)\}$$

$\leq \emptyset_{w_2,\dots,w_n}(x_0) + \emptyset_{w_2,\dots,w_n}(y_0)$. Also, for all $w_2, \dots, w_n \in X$, and $k \in N$

$$\|x_{k+1} - y_k, w_2, \dots, w_n\| = \|T(x_k) - F(y_{k-1}), w_2, \dots, w_n\|$$

$$\leq \emptyset_{w_2,\dots,w_n}(x_k) - \emptyset_{w_2,\dots,w_n}(x_{k+1}) + \emptyset_{w_2,\dots,w_n}(y_{k-1}) - \emptyset_{w_2,\dots,w_n}(y_k).$$

Therefore

$$\sum_{k=1}^{m} \|x_{k+1} - y_k, w_2, \dots, w_n\| \leq \emptyset_{w_2,\dots,w_n}(x_1) + \emptyset_{w_2,\dots,w_n}(y_0).$$

Thus,

$$\sum_{k=1}^{m} \|x_k - x_{k+1}, w_2, \dots, w_n\|$$

$$\leq \sum_{k=1}^{m} \|x_k - y_k, w_2, \dots, w_n\| + \sum_{k=1}^{m} \|x_{k+1} - y_k, w_2, \dots, w_n\|$$

$$\leq \emptyset_{w_2,\dots,w_n}(x_0) + \emptyset_{w_2,\dots,w_n}(x_1) + 2\emptyset_{w_2,\dots,w_n}(y_0)$$

Hence,

$$\sum_{k=1}^{\infty} \|x_k - x_{k+1}, w_2, \dots, w_n\| < \infty, \quad \|x_m - x_{m+1}, w_2, \dots, w_n\| \to 0, \quad as \quad m \to \infty.$$

Thus for any two positive integers s and r with $s > r$ and for all $w_2, \dots, w_n \in X$,

$$\|x_r - x_s, w_2, \dots, w_n\|$$

$$\leq \|x_r - x_{r+1}, w_2, \dots, w_n\| + \|x_{r+1} - x_{r+2}, w_2, \dots, w_n\| + \dots + \|x_{s-1} - x_s, w_2, \dots, w_n\|$$

$$\to 0, \quad as \quad s, r \to \infty.$$

Treating the case $r > s$ in a similar we conclude that $\{x_m\}$ is a Cauchy sequence in X, similarly we can show that $\{y_m\}$ is a Cauchy sequence in X, and since X is complete, there exists a common fixed point in X.

Let $z = \lim_{m\to\infty} x_m$ and $z_1 = \lim_{m\to\infty} y_m$, $z, z_1 \in X$, that is for all $w_2, \dots, w_n \in X$,

$$\|x_m - z, w_2, \dots, w_n\| \to 0 \quad as \quad m \to \infty$$

and

$$\|y_m - z_1, w_2, \dots, w_n\| \to 0 \quad as \quad m \to \infty.$$

As T and F are continuous,

$$\|T(x_m) - T(z), w_2, \dots, w_n\| \to 0 \quad as \quad m \to \infty,$$

and

$$\|F(y_m) - F(z_1), w_2, \dots, w_n\| \to 0 \quad as \quad m \to \infty.$$

Therefore,

$$\|z - T(z), w_2, \dots, w_n\| \leq \|z - x_{m+1}, w_2, \dots, w_n\| + \|x_{m+1} - T(z), w_2, \dots, w_n\|$$

$$= \|z - x_{m+1}, w_2, \dots, w_n\| + \|T(x_m) - T(z), w_2, \dots, w_n\|$$

$$\to 0 \quad as \quad m \to \infty.$$

This implies

$$z = T(z).$$

Similarly it can be show that $z_1 = F(z_1)$. Now for all $w_2, \dots, w_n \in X$,

$$\|z - z_1, w_2, \dots, w_n\| = \|T(z) - F(z_1), w_2, \dots, w_n\| \leq$$

$$\leq \emptyset_{w_2,\dots,w_n}(z) + \emptyset_{w_2,\dots,w_n}(z_1) - \emptyset_{w_2,\dots,w_n}\big(T(z)\big) - \emptyset_{w_2,\dots,w_n}(F(z_1)) = 0.$$

Hence, z is a common fixed point of T and F.

Let $\bar{z} \in X$ be an another common fixed point of F and G, then $F(\bar{z}) = G(\bar{z}) = \bar{z}$.

So,

$$\|z - \bar{z}, w_2, \dots, w_n\| = \|T(z) - F(\bar{z}), w_2, \dots, w_n\|$$

$$\leq \emptyset_{w_2,\dots,w_n}(z) + \emptyset_{w_2,\dots,w_n}(\bar{z}) - \emptyset_{w_2,\dots,w_n}\big(T(z)\big) - \emptyset_{w_2,\dots,w_n}(F(\bar{z})) = 0 .$$

Hence, z is the unique common fixed point of T and F in X.

3 Conclusion

The above results can be generalized to obtain a conclusion whose proof is left to the reader

Theorem 3.1 Let $\{T_\alpha\}_{\alpha\in\Delta}$ be a family of continuous self-mapping of an $n-$Banach spaces X, suppose that for any $w_2, \dots, w_n \in X$, there exists a function

$\emptyset_{w_2,\dots,w_n}: X \to [0,\infty)$, such that for all $x, y \in X$,

$$\left\|T_\alpha(x) - T_\beta(y), w_2, \dots, w_n\right\| \leq \emptyset_{w_2,\dots,w_n}(x) + \emptyset_{w_2,\dots,w_n}(y) - \left\{\emptyset_{w_2,\dots,w_n}\left(T_\alpha(x)\right) + \emptyset_{w_2,\dots,w_n}\left(T_\beta(y)\right)\right\}.$$

Then there exists a unique $z \in X$ satisfying $T_\alpha(z) = z$ for all $\alpha \in \Delta$.

4 Recommendation

The n^{th} spaces are possible, wide, sizes spaces, so it's really interesting to develop this study to search at specific conditions and contractions for the existence of double or triple fixed points.

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