Fixed point theorems for a new nonlinear mapping in Hilbert Spaces

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Abstract: This study presents a new class of nonlinear mappings in Hilbert space. We establish demiclosed principles and fixed point theorems for this nonlinear mapping. Some works of literature are improved on and expanded upon by the results that this study presents.

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1 Introduction and Preliminaries

Throughout this paper, we denote by $\mathcal{N}$ the set of positive integers and by $\mathcal{R}$ the set of real numbers. Let $\mathcal{H}$ be a real Hilbert space with inner product $\langle .,. \rangle$ and norm $\|.\|$, respectively.

Let be $\mathcal{H}$ be a real Hilbert space and $\mathcal{M}$ be a nonempty closed convex subset of $\mathcal{H}$, and $\mathcal{F} : \mathcal{M} \to \mathcal{M}$ be a mapping. A point $z \in \mathcal{M}$ is called a fixed point of $\mathcal{F} : \mathcal{M} \to \mathcal{M}$ if $z = \mathcal{F}z$. We denote $Fix(\mathcal{F})$ the set of a fixed points of $\mathcal{F}$.

A mapping $\mathcal{F} : \mathcal{M} \to \mathcal{M}$ is called *nonexpansive* if

$$\|\mathcal{F}u - \mathcal{F}v\| \leq \|u - v\|$$

for all $u, v \in \mathcal{M}$. $\mathcal{F}$ is called quasi-nonexpansive if $Fix(\mathcal{F}) \neq \emptyset$ and

$$\|\mathcal{F}u - z\| \leq \|u - z\|$$

for all $u \in \mathcal{M}$ and $z \in Fix(\mathcal{F})$. If $\mathcal{F} : \mathcal{M} \to \mathcal{M}$ is nonexpansive mapping and the set $Fix(\mathcal{F}) \neq \emptyset$, then $\mathcal{F}$ is quasi-nonexpansive. It is well-known that the set $Fix(\mathcal{F})$ of fixed points of a quasi-nonexpansive mapping $\mathcal{F}$ is closed and convex, [1], [2].

The following demiclosed principle for nonexpansive mappings in Hilbert spaces was provided in 1965 by [3].

Theorem 1.1 *Let $\mathcal{M}$ be a nonempty closed convex subset of a real Hilbert space $\mathcal{H}$. Let $\mathcal{F}$ be a nonexpansive mapping of $\mathcal{M}$ into itself, and let $\{u_n\}$ be a sequence in $\mathcal{M}$. If $u_n \to z$ and $\|u_n - \mathcal{F}u_n\| = 0$, then $\mathcal{F}z = z$.*

The following fixed point theorem for nonexpansive mappings in Hilbert spaces was provided in 1971 by [4].

Theorem 1.2 *Let $\mathcal{M}$ be a nonempty closed convex subset of a real Hilbert space $\mathcal{H}$. Let T be a nonexpansive mapping of $\mathcal{M}$ into itself, Then $\{\mathcal{F}^n u\}$ is a bounded sequence for some $u \in \mathcal{M}$ iff $Fix(\mathcal{F}) \neq \emptyset$.*

In a real Hilbert space, the following result was provided in 1980 by [5].

Theorem 1.3 *Let $\mathcal{M}$ be a nonempty closed convex subset of a real Hilbert space $\mathcal{H}$.Then the following conditions are equivalent.*

(1) Every nonexpansive mapping of $\mathcal{M}$ into itself has a fixed point in $\mathcal{M}$;

(2) $\mathcal{M}$ is bounded.

A firmly nonexpansive mapping is an essential example of a nonexpansive mapping in a Hilbert space. A mapping $\mathcal{F} : \mathcal{M} \to \mathcal{M}$ is said to be firmly nonexpansive if

$$\|\mathcal{F}u - \mathcal{F}v\|^2 \leq \langle u - v, \mathcal{F}u - \mathcal{F}v \rangle$$

for all $u, v \in \mathcal{M}$, [3], [6], [7].

Nonspreading mapping was first introduced in 2008 by [8]. They also obtained a common fixed point theorem for a commutative family of nonspreading mappings in Banach spaces, as well as a fixed point theorem for a single nonspreading mapping. A mapping $\mathcal{F} : \mathcal{M} \to \mathcal{M}$ is called nonspreading, [8], if

$$2\|\mathcal{F}u - \mathcal{F}v\|^2 \leq \|\mathcal{F}u - v\|^2 + \|\mathcal{F}v - u\|^2$$

for all $u, v \in \mathcal{M}$.

An extension of Theorem 1.2 for nonspreading mapping in Hilbert spaces was made by [8]. An extension of Ray's type theorem for nonspreading mapping in Hilbert spaces was made in 2010 by [9]. Also, the demiclosed principles were extended for nonspreading mappings by [10]. In Hilbert spaces, the following two nonlinear mappings introduced by [11], are said to be $TJ-1, TJ-2$; see also, [12]. A mapping $\mathcal{F} : \mathcal{M} \to \mathcal{M}$ is called a $TJ-1$ mapping, [11], if

$$2\|\mathcal{F}u - \mathcal{F}v\|^2 \leq \|u - v\|^2 + \|\mathcal{F}u - v\|^2$$

for all $u, v \in \mathcal{M}$. A mapping $\mathcal{F}: \mathcal{M} \to \mathcal{M}$ is called a $TJ-2$, [11], mapping if

$$3\|\mathcal{F}u - \mathcal{F}v\|^2 \leq 2\|\mathcal{F}u - v\|^2 + \|\mathcal{F}v - u\|^2$$

for all $u, v \in \mathcal{M}$. Similar results to the above theorems were also obtained for $TJ-1$ and $TJ-2$ mappings, [11]. Motivated by the above works, we introduce a new nonlinear mappings in Hilbert spaces.

Definition 1.4 *Let $\mathcal{M}$ be a nonempty closed convex subset of a Hilbert space $\mathcal{H}$. We say $\mathcal{F}: \mathcal{M} \to \mathcal{M}$ is a new nonlinear mapping if there exists $\alpha, \beta \in \mathcal{R}$ with $0 \leq \alpha + \beta \leq 2$ such that*

$$\begin{aligned} 2\|\mathcal{F}u - \mathcal{F}v\|^2 &\leq \alpha\|\mathcal{F}u - v\|^2 + \beta\|\mathcal{F}v - u\|^2 \\ &+ (2-\alpha-\beta)\|u - v\|^2 \end{aligned}$$

for all $u, v \in \mathcal{M}$.

Remark 1.5 *Especially with special choices of α and β, the nonlinear mapping as defined in Definition 1.4 becomes nonspreading mapping, $TJ-1$ mapping, $TJ-2$ mapping and nonexpansive mapping. Indeed, in Definition 1.4, we know that if we choose*

(1) $\alpha = \beta = 1$ for all $u, v \in \mathcal{M}$, then $\mathcal{F}$ is a nonspreading mapping;

(2) $\alpha = 1, \beta = 0$ for all $u, v \in \mathcal{M}$, then $\mathcal{F}$ is a $TJ-1$ mapping;

(3) $\alpha = \frac{4}{3}, \beta = \frac{2}{3}$ for all $u, v \in \mathcal{M}$, then $\mathcal{F}$ is a $TJ-2$ mapping.

(4) $\alpha = 0, \beta = 0$ for all $u, v \in \mathcal{M}$, then $\mathcal{F}$ is a nonexpansive mapping.

We can also show that if $u = \mathcal{F}u$, then for any $v \in \mathcal{M}$,

$$\begin{aligned} 2\|u - \mathcal{F}v\|^2 &\leq \alpha\|u-v\|^2+\beta\|\mathcal{F}v-u\|^2 \\ &+ (2-\alpha-\beta)\|u-v\|^2 \\ (2-\beta)\|u - \mathcal{F}v\|^2 &\leq (2-\beta)\|u-v\|^2 \\ \|u - \mathcal{F}v\|^2 &\leq \|u-v\|^2. \end{aligned}$$

This means that the nonlinear mapping as defined in Definition 1.4 with a fixed point is quasi-nonexpansive mapping.

Let ℓ_∞ be the Banach space of bounded sequences with the supremum norm. A linear functional μ on ℓ_∞ is called a mean if $\mu(e) = ||\mu|| = 1$, where $e = (1, 1, 1, ...)$. For $u = (u_1, u_2, u_3, ...)$, the value μ is also denoted by $\mu_n(u_n)$. A Banach limit on ℓ_∞ is an invariant mean, that is, $\mu_n(u_n) = \mu_n(u_{n+1})$. If μ is a Banach limit on ℓ_∞, then for $u = (u_1, u_2, u_3, ...) \in \ell_\infty$, $liminf_{n\to\infty} u_n \leq \mu_n u_n \leq limsup_{n\to\infty} u_n$. In particular, if $u = (u_1, u_2, u_3, ...) \in \ell_\infty$ and $u_n \to a \in \mathcal{R}$, then we have $\mu_n(u_n) = \mu_n(u_{n+1}) = a$. For details, [7].

Proposition 1.6 *Let $\mathcal{M}$ be a nonempty closed convex subset of a Hilbert space $\mathcal{H}$. Let α, β be the same as in Definition 1.4. Then, $\mathcal{F}: \mathcal{M} \to \mathcal{M}$ is a nonlinear mapping if and only if*

$$\begin{aligned} \|\mathcal{F}u - \mathcal{F}v\|^2 &\leq \frac{\alpha-\beta}{2-\beta}\|\mathcal{F}u - u\|^2 + \|u - v\|^2 \\ &+ \frac{2\langle \mathcal{F}u-u, \alpha(u-v) + \beta(\mathcal{F}v-u)\rangle}{2-\beta} \end{aligned}$$

for all $u, v \in \mathcal{M}$.

Proof. We have that for $u, v \in \mathcal{M}$,

$$\begin{aligned} 2\|\mathcal{F}u - \mathcal{F}v\|^2 &\leq \alpha\|\mathcal{F}u - v\|^2 + \beta\|\mathcal{F}v - u\|^2 \\ &+ (2-\alpha-\beta)\|u - v\|^2 \end{aligned}$$

$$\begin{aligned} &= \alpha\|\mathcal{F}u-u\|^2+2\alpha\langle \mathcal{F}u-u, u-v\rangle+\alpha\|u-v\|^2 \\ &\quad +\beta\|\mathcal{F}v - \mathcal{F}u\|^2 + 2\beta\langle \mathcal{F}v - \mathcal{F}u, \mathcal{F}u - u\rangle \\ &\quad +\beta\|\mathcal{F}u - u\|^2 + (2-\alpha-\beta)\|u - v\|^2 \\ &= (\alpha+\beta)\|\mathcal{F}u - u\|^2 + \beta\|\mathcal{F}u - \mathcal{F}v\|^2 \\ &\quad +(2-\beta)\|u - v\|^2 \\ &\quad +2\alpha\langle \mathcal{F}u-u, u-v\rangle+2\beta\langle \mathcal{F}v-\mathcal{F}u, \mathcal{F}u-u\rangle \\ &= (\alpha+\beta)\|\mathcal{F}u - u\|^2 + \beta\|\mathcal{F}v - \mathcal{F}u\|^2 \\ &\quad +(2-\beta)\|u - v\|^2 + 2\alpha\langle \mathcal{F}u - u, u - v\rangle \\ &\quad +2\beta\langle \mathcal{F}v-u, u-\mathcal{F}u\rangle-2\beta\langle \mathcal{F}u-u, u-\mathcal{F}u\rangle \\ &= (\alpha-\beta)\|\mathcal{F}u - u\|^2 + \beta\|\mathcal{F}u - \mathcal{F}v\|^2 \\ &\quad +(2-\beta)\|u - v\|^2 \\ &\quad +2\langle \mathcal{F}u - u, \alpha(u - v) + \beta(\mathcal{F}v - u)\rangle. \end{aligned}$$

We have

$$\begin{aligned} \|\mathcal{F}u-\mathcal{F}v\|^2 &\leq \frac{\alpha-\beta}{2-\beta}\|\mathcal{F}u-u\|^2 + \|u-v\|^2 \\ &+ \frac{2\langle \mathcal{F}u-u, \alpha(u-v)+\beta(\mathcal{F}v-u)\rangle}{2-\beta}. \end{aligned}$$

Hence, the proof is completed.

Remark 1.7 *If $\alpha = \beta = 1$, then Proposition 1.6 is reduced to Lemma 3.2 in [10]. In the sequel, we need the following lemmas as tools.*

Following the similar argument as in the proof of Theorem 3.1.5, [7], we get the following result.

Lemma 1.8 *Let $\mathcal{M}$ be a nonempty closed convex subset of a real Hilbert space $\mathcal{H}$, and let μ be a Banach limit. Let $\{u_n\}$ be a sequence with $u_n \rightharpoonup z$. If $u \neq z$, then $\mu_n\|u_n - z\| < \mu_n\|u_n - u\|$ and $\mu_n\|u_n - z\|^2 < \mu_n\|u_n - u\|^2$.*

The following fixed point theorem was proven using Banach limits.

Theorem 1.9 *, [11]. Let $\mathcal{H}$ be a Hilbert space, let $\mathcal{M}$ be a nonempty closed convex subset of $\mathcal{H}$, and let $\mathcal{F}$ be a nonlinear mapping of $\mathcal{M}$ into itself. Suppose that there exists an element $u \in \mathcal{M}$ such that $\mathcal{F}^n u$ is bounded and*

$\mu_n\|\mathcal{F}^n u - \mathcal{F}v\|^2 \le \mu_n\|\mathcal{F}^n u - v\|^2, \forall v \in \mathcal{M}$

for some Banach limit μ. Then , $\mathcal{F}$ has a fixed point in $\mathcal{M}$.

2 Main results

In this section, we study the fixed point theorems, demiclosed principles for nonlinear mappings in Hilbert spaces.

2.1 Fixed point theorems

Theorem 2.1 *Let $\mathcal{M}$ be a nonempty closed convex subset of a real Hilbert space $\mathcal{H}$, $\mathcal{F}$ be a nonlinear mappings defined in Definition 1.4 .Then, $Fix(\mathcal{F}) \neq \emptyset$ if and only if $\{\mathcal{F}^n z\}$ is bounded for some $z \in \mathcal{M}$.*

Proof. Since $\mathcal{F} : \mathcal{M} \to \mathcal{M}$ is a nonlinear mapping if there exist $\alpha, \beta \in \mathcal{R}$ with $0 \le \alpha + \beta \le 2$ such that

$2\|\mathcal{F}u - \mathcal{F}v\|^2 \le \alpha\|\mathcal{F}u - v\|^2 + \beta\|\mathcal{F}v - u\|^2 + (2 - \alpha - \beta)\|u - v\|^2$

for all $u, v \in \mathcal{M}$. If $Fix(\mathcal{F}) \neq \emptyset$, then $\mathcal{F}^n z = z$ for $z \in Fix(\mathcal{F})$. So $\{\mathcal{F}^n z\}$ is bounded. We show reverse. Take $z \in \mathcal{M}$ such that $\{\mathcal{F}^n z\}$ is bounded. Let μ be a Banach limit. Then for any $v \in \mathcal{M}$ and $n \in \mathcal{N} \cup \{0\}$, we have

$$\begin{aligned} 2\|\mathcal{F}^{n+1}z - \mathcal{F}v\|^2 \le\ & \alpha\|\mathcal{F}^{n+1}z - v\|^2 \\ & +\beta\|\mathcal{F}v - \mathcal{F}^n z\|^2 \\ & +(2 - \alpha - \beta)\|\mathcal{F}^n z - v\|^2 \end{aligned}$$

for any $v \in \mathcal{M}$. Since $\{\mathcal{F}^n z\}$ is bounded, we can apply a Banach limit μ to both sides of inequality. Then we have

$$\begin{aligned} \mu_n(2\|\mathcal{F}^{n+1}z - \mathcal{F}v\|^2) &\le \mu_n(\alpha\|\mathcal{F}^{n+1}z - v\|^2 \\ &+ \beta\|\mathcal{F}v - \mathcal{F}^n z\|^2 \\ &+ (2-\alpha-\beta)\|\mathcal{F}^n z - v\|^2) \\ \mu_n(2\|\mathcal{F}^{n+1}z - \mathcal{F}v\|^2) &\le \mu_n(\alpha\|\mathcal{F}^{n+1}z - v\|^2) \\ &+ \mu_n(\beta\|\mathcal{F}v - \mathcal{F}^n z\|^2) \\ &+ \mu_n((2-\alpha-\beta)\|\mathcal{F}^n z - v\|^2) \end{aligned}$$

$$\begin{aligned} (2-\beta)\mu_n\|\mathcal{F}^n z - \mathcal{F}v\|^2 &\le \alpha\mu_n\|\mathcal{F}^n z - v\|^2 \\ &+(2-\alpha-\beta)\mu_n\|\mathcal{F}^n z - v\|^2 \\ (2-\beta)\mu_n\|\mathcal{F}^n z - \mathcal{F}v\|^2 &\le (2-\beta)\mu_n\|\mathcal{F}^n z - v\|^2 \end{aligned}$$

for all $v \in \mathcal{M}$. By Theorem 1.9, then we have that $Fix(\mathcal{F})$ is nonempty.

As a direct consequence of Theorem 2.1, we have the following

Theorem 2.2 *Let $\mathcal{M}$ be nonempty bounded closed convex subset of a Hilbert space $\mathcal{H}$ and let $\mathcal{F}$ be a nonlinear mapping from $\mathcal{M}$ to itself. Then T has a fixed point.*

Using Theorem 2.1, we can also prove the following well-known fixed point theorems. We first prove a fixed point theorem for nonexpansive mappings in a Hilbert space.

Corollary 2.3 *Let $\mathcal{M}$ be a nonempty closed convex subset of a real Hilbert space $\mathcal{H}$, $\mathcal{F}$ be a nonexpansive mappings. Then, $Fix(\mathcal{F}) \neq \emptyset$ if and only if $\{\mathcal{F}^n z\}$ is bounded for some $z \in \mathcal{M}$.*

Proof. If $\alpha = 0, \beta = 0$ in Theorem 2.1, $(0, 0)$-nonlinear mapping of $\mathcal{M}$ into itself is nonexpansive. By Theorem 2.1, $\mathcal{F}$ has a fixed in $\mathcal{M}$.

The following is a fixed point theorem for nonspreading mappings in a Hilbert space.

Corollary 2.4 *Let $\mathcal{M}$ be a nonempty closed convex subset of a real Hilbert space $\mathcal{H}$, $\mathcal{F}$ be a nonspreading mappings .Then, $Fix(\mathcal{F}) \neq \emptyset$ if and only if $\{\mathcal{F}^n z\}$ is bounded for some $z \in \mathcal{M}$.*

Proof. If $\alpha = 1, \beta = 1$ in Theorem 2.1, $(1, 1)$-nonlinear mapping of $\mathcal{M}$ into itself is nonspreading. By Theorem 2.1, $\mathcal{F}$ has a fixed in $\mathcal{M}$.

The following is a fixed point theorem for $TJ - 1$ mappings in a Hilbert space.

Corollary 2.5 *Let $\mathcal{M}$ be a nonempty closed convex subset of a real Hilbert space $\mathcal{H}$, $\mathcal{F}$ be a $TJ - 1$ mappings .Then, $Fix(\mathcal{F}) \neq \emptyset$ if and only if $\{\mathcal{F}^n z\}$ is bounded for some $z \in \mathcal{M}$.*

Proof. If $\alpha = 1, \beta = 0$ in Theorem 2.1, $(1, 0)$-nonlinear mapping of $\mathcal{M}$ into itself is $TJ - 1$. By Theorem 2.1, $\mathcal{F}$ has a fixed in $\mathcal{M}$.

The following is a fixed point theorem for $TJ - 2$ mappings in a Hilbert space.

Corollary 2.6 *Let $\mathcal{M}$ be a nonempty closed convex subset of a real Hilbert space $\mathcal{H}$, $\mathcal{F}$ be a $TJ - 2$ mappings .Then, $Fix(\mathcal{F}) \neq \emptyset$ if and only if $\{\mathcal{F}^n z\}$ is bounded for some $z \in \mathcal{M}$.*

Proof. If $\alpha = \frac{4}{3}, \beta = \frac{2}{3}$ in Theorem 2.1, $(\frac{4}{3}, \frac{2}{3})$-nonlinear mapping of $\mathcal{M}$ into itself is $TJ - 2$. By Theorem 2.1, $\mathcal{F}$ has a fixed in $\mathcal{M}$.

2.2 Demiclosed principles

Theorem 2.7 *Let $\mathcal{H}$ be a Hilbert space and let $\mathcal{M}$ be a nonempty closed convex subset of $\mathcal{H}$. Let $\mathcal{F}$ be a nonlinear mapping defined in Definition 1.4 of $\mathcal{M}$ into itself such that $Fix(\mathcal{F}) \neq \emptyset$. Then $\mathcal{F}$ is demiclosed,i.e.,$u_n \rightharpoonup z$ and $u_n - \mathcal{F}u_n \to 0$ imply $z \in Fix(\mathcal{F})$.*

Proof. Let u_n be a sequence in K with $u_n \rightharpoonup z$ and $u_n - \mathcal{F}u_n \to 0$ as $n \to \infty$. Then u_n and $\mathcal{F}u_n$ are bounded. Suppose $z \neq \mathcal{F}z$. From Opial's condition, [13], we have

$$\begin{aligned} & \liminf_{n\to\infty} \|u_n - z\|^2 < \liminf_{n\to\infty} \|u_n - \mathcal{F}z\|^2 \\ = & \liminf_{n\to\infty} (\|u_n - \mathcal{F}u_n\|^2 \\ & + \|\mathcal{F}u_n - \mathcal{F}z\|^2 + 2\langle u_n - \mathcal{F}u_n, \mathcal{F}u_n - \mathcal{F}z\rangle) \\ \leq & \liminf_{n\to\infty} (\|u_n - \mathcal{F}u_n\|^2 \\ & + \frac{\alpha-\beta}{2-\beta}\|\mathcal{F}u_n - u_n\|^2 + \|u_n - z\|^2 \\ & + \frac{2\langle u_n - \mathcal{F}u_n, \alpha(u_n - z) + \beta(\mathcal{F}z - u_n)\rangle}{2-\beta} \\ & + 2\langle u_n - \mathcal{F}u_n, \mathcal{F}u_n - \mathcal{F}z\rangle) \\ = & \liminf_{n\to\infty} \|u_n - z\|^2. \end{aligned}$$

This is a contradiction. Hence we get the conclusion.

We have the following results as special of Theorem 2.7.

Corollary 2.8 *, [10]. Let $\mathcal{H}$ be a Hilbert space and let $\mathcal{M}$ be a nonempty closed convex subset of $\mathcal{H}$. Let $\mathcal{F}$ be a nonspreading mapping of $\mathcal{M}$ into itself such that $Fix(\mathcal{F}) \neq \emptyset$. Then $\mathcal{F}$ is demiclosed,i.e.,$u_n \rightharpoonup z$ and $u_n - \mathcal{F}u_n \to 0$ imply $z \in Fix(\mathcal{F})$.*

Proof. If $\alpha = 1, \beta = 1$, $(1,1)$-nonlinear mapping of $\mathcal{M}$ into itself becomes nonspreading. By Theorem 2.7, we get the conclusion.

Corollary 2.9 *, [11]. Let $\mathcal{H}$ be a Hilbert space and let $\mathcal{M}$ be a nonempty closed convex subset of $\mathcal{H}$. Let $\mathcal{F}$ be a $TJ-1$ mapping of $\mathcal{M}$ into itself such that $Fix(\mathcal{F}) \neq \emptyset$. Then $\mathcal{F}$ is demiclosed,i.e.,$u_n \rightharpoonup z$ and $u_n - \mathcal{F}u_n \to 0$ imply $z \in Fix(\mathcal{F})$.*

Proof. If $\alpha = 1, \beta = 0$, $(1,0)$-nonlinear mapping of $\mathcal{M}$ into itself becomes $TJ-1$. By Theorem 2.7, we get the conclusion.

Corollary 2.10 *, [11]. Let $\mathcal{H}$ be a Hilbert space and let $\mathcal{M}$ be a nonempty closed convex subset of $\mathcal{H}$. Let $\mathcal{F}$ be a $TJ-2$ mapping of $\mathcal{M}$ into itself such that $Fix(\mathcal{F}) \neq \emptyset$. Then $\mathcal{F}$ is demiclosed,i.e.,$u_n \rightharpoonup z$ and $u_n - \mathcal{F}u_n \to 0$ imply $z \in Fix(\mathcal{F})$.*

Proof. If $\alpha = \frac{4}{3}, \beta = \frac{2}{3}$, $(\frac{4}{3}, \frac{2}{3})$-nonlinear mapping of $\mathcal{M}$ into itself becomes $TJ-2$. By Theorem 2.7, we get the conclusion.

2.3 Weak convergence theorem

Theorem 2.11 *Let $\mathcal{H}$ be a Hilbert space and let $\mathcal{M}$ be a nonempty closed convex subset of $\mathcal{H}$. Let $\mathcal{F}$ be a nonlinear mapping defined in Definition 1.4 of $\mathcal{M}$ into itself such that $Fix(\mathcal{F}) \neq \emptyset$. Let $\{\varsigma_n\}$ be a real sequence in $(0,1)$. Let $\{u_n\}$ be defined by*

$$\begin{cases} u \in \mathcal{M} \text{ chosen arbitrary}, \\ u_{n+1} = (1-\varsigma_n)u_n + \varsigma_n \mathcal{F}u_n, \forall n \in \mathcal{N}, \end{cases}$$

Assume that $\liminf_{n\to\infty} \varsigma_n(1-\varsigma_n) > 0$, then $u_n \rightharpoonup z$ for $z \in Fix(\mathcal{F})$.

Proof. For any $z \in Fix(\mathcal{F})$ and all $u \in \mathcal{M}$, we consider $\mathcal{F}$ nonlinear mapping defined Definition 1.4, i.e., $2\|\mathcal{F}u - \mathcal{F}z\|^2 \leq \alpha\|\mathcal{F}u - z\|^2 + \beta\|\mathcal{F}z - u\|^2 + (2-\alpha-\beta)\|u - z\|^2$, then we have $\|\mathcal{F}u - z\|^2 \leq \|u - z\|^2$.

Therefore we get for each $n \in \mathcal{N}$, $\|\mathcal{F}u_n - z\| \leq \|u_n - z\|$. Now

$$\begin{aligned} \|u_{n+1} - z\|^2 &= \|((1-\varsigma_n)u_n + \varsigma_n\mathcal{F}u_n) - z\|^2 \\ &= (1-\varsigma_n)\|u_n - z\|^2 + \varsigma_n\|\mathcal{F}u_n - z\|^2 \\ &\quad -\varsigma_n(1-\varsigma_n)\|\mathcal{F}u_n - u_n\|^2 \\ &\leq (1-\varsigma_n)\|u_n - z\|^2 + \varsigma_n\|u_n - z\|^2 \\ &\quad -\varsigma_n(1-\varsigma_n)\|\mathcal{F}u_n - u_n\|^2 \\ &= \|u_n - z\|^2 - \varsigma_n(1-\varsigma_n)\|\mathcal{F}u_n - u_n\|^2. \end{aligned}$$

Hence $\{\|u_n - z\|\}$ is a nonincreasing sequence and $\lim_{n\to\infty} \|u_n - z\|$ exists. Besides, we know that

$$\varsigma_n(1-\varsigma_n)\|\mathcal{F}u_n - u_n\|^2 \leq \|u_n - z\|^2 - \|u_{n+1} - z\|^2.$$

This implies that $\lim_{n\to\infty} \|\mathcal{F}u_n - u_n\| = 0$. Now, it is enough to show that $\{u_n\}$ has a unique weak subsequential limit in $Fix(\mathcal{F})$. Let $\{u_{n_j}\}$ and $\{u_{n_k}\}$ be two subsequences of $\{u_n\}$, converge weakly to z and w, respectively. Then $\lim_{n\to\infty} \|\mathcal{F}u_n - u_n\| = 0$ and $\mathcal{I}-\mathcal{F}$ is demiclosed at zero by Theorem 2.7, where $\mathcal{I}$ is identity mapping. This implies that $(\mathcal{I}-\mathcal{F})z = 0$. That is, $\mathcal{F}(z) = z$; similarly $\mathcal{F}(w) = w$. Next we prove the uniqueness. Suppose that $z \neq w$. Then by the Opial's condition, [13], we have

$$\begin{aligned} \lim_{n\to\infty} \|u_n - z\| &= \lim_{j\to\infty} \|u_{n_j} - z\| < \lim_{j\to\infty} \|u_{n_j} - w\| \\ &= \lim_{n\to\infty} \|u_n - w\| = \lim_{k\to\infty} \|u_{n_k} - w\| \\ &< \lim_{k\to\infty} \|u_{n_k} - z\| = \lim_{n\to\infty} \|u_n - z\|. \end{aligned}$$

This leads to a contradiction. So, $z = w$. Therefore $u_n \rightharpoonup z$. This completes the proof.

2.4 Example for nonlinear mapping as defined in Definition 1.4.

Example 2.12 *The following example shows that $\mathcal{F}$ is a nonlinear mapping as defined in Definition 1.4.*

Define

$$\mathcal{F} : [0,10] \to [0,10]$$

by $\mathcal{F}u = \begin{cases} 0, & 0 \leq u < 1 \\ 0.005, & u \geq 1 \end{cases}$

Indeed, $\mathcal{F}$ is a nonlinear mapping as defined in Definition 1.4. Especially with special choices of α and β, the nonlinear mapping as defined in Definition 1.4 becomes nonspreading mapping, $TJ-1$ mapping, $TJ-2$ mapping and nonexpansive mapping. If we choose

(1) $\alpha = \beta = 1$ for all $u, v \in \mathcal{M}$, then $\mathcal{F}$ is a nonspreading mapping;

(2) $\alpha = 1, \beta = 0$ for all $u, v \in \mathcal{M}$, then $\mathcal{F}$ is a $TJ-1$ mapping;

(3) $\alpha = \frac{4}{3}, \beta = \frac{2}{3}$ for all $u, v \in \mathcal{M}$, then $\mathcal{F}$ is a $TJ-2$ mapping;

(4) $\alpha = 0, \beta = 0$ for all $u, v \in \mathcal{M}$, then $\mathcal{F}$ is a nonexpansive mapping.

Table 1: To verify that $\mathcal{F}$ is a nonlinear mapping defined in Definition 1.4, consider the following cases:

Cases	α, β	u, v	Definition 1.4
1	$\frac{11}{10}, \frac{9}{10}$	$0.3, 10$	$0.00005 \leq 110.07832$
2	$\frac{11}{10}, \frac{9}{10}$	$10, 0.5$	$0.00005 \leq 90.26952$
3	$\frac{1}{100}, \frac{199}{100}$	$1.2, 0.2$	$0.00005 \leq 2.85698$
4	$\frac{1}{200}, \frac{99}{100}$	$0.9, 1.1$	$0.00005 \leq 0.83926$
5	0 , 2	$0.9, 1.1$	$0.00005 \leq 1.60205$
6	2 , 0	$0.9, 1.1$	$0.00005 \leq 2.42000$
7	$\frac{1}{200}, \frac{99}{100}$	$1, 0.2$	$0.00005 \leq 1.63339$
8	2 , 0	$1, 0.2$	$0.00005 \leq 0.07605$
9	0 , 2	$1, 0.2$	$0.00005 \leq 2$
10	$\frac{1}{200}, \frac{99}{100}$	$0.9, 10$	$0.00005 \leq 84.51706$
11	$\frac{1}{200}, \frac{99}{100}$	$0.99, 9.99$	$0.00005 \leq 82.86452$
12	$\frac{1}{200}, \frac{99}{100}$	$1.01, 0.99$	$0.00005 \leq 1.01515$

Table 2: To verify that $\mathcal{F}$ is a nonspreading mapping for $\alpha = 1, \beta = 1$, consider the following cases:

Cases	α, β	u, v	Nonspreading
1	1, 1	1.01 , 0.99	$0.00005 \leq 1.99032$
2	1, 1	0.99 , 1.01	$0.00005 \leq 1.99032$
3	1, 1	1.2 , 0.2	$0.00005 \leq 1.47802$

Table 3: To verify that $\mathcal{F}$ is a $TJ-1$ mapping for $\alpha = 1, \beta = 0$, consider the following cases:

Cases	α, β	u, v	$TJ-1$
1	1, 0	1.01 , 0.99	$0.00005 \leq 0.97062$
2	1, 0	1.2 , 0.2	$0.00005 \leq 1.03802$
3	1, 0	0.99 , 1.01	$0.00005 \leq 1.02050$

Table 4: To verify that $\mathcal{F}$ is a $TJ-2$ mapping for $\alpha = \frac{4}{3}, \beta = \frac{2}{3}$, consider the following cases:

Cases	α, β	u, v	$TJ-2$
1	$\frac{4}{3}, \frac{2}{3}$	1.01 , 0.99	$0.000075 \leq 2.96055$
2	$\frac{4}{3}, \frac{2}{3}$	1.2 , 0.2	$0.000075 \leq 1.51605$
3	$\frac{4}{3}, \frac{2}{3}$	0.99 , 1.01	$0.000075 \leq 3.01042$

Table 5: To verify that $\mathcal{F}$ is a nonexpansive mapping for $\alpha = 0, \beta = 0$, consider the following cases:

Cases	α, β	u, v	Nonexpansive
1	0, 0	1.01 , 0.99	$0.005000 \leq 0.02000$
2	0, 0	1.2 , 0.2	$0.005000 \leq 1.00000$
3	0, 0	0.99 , 1.01	$0.005000 \leq 0.02000$

3 Conclusion

In this paper we introduce a new class of nonlinear mappings in Hilbert space. We prove fixed point theorems and demiclosed principles for this nonlinear mapping. Furthermore, we have given an example of a nonlinear mapping as defined in Definition 1.4. It is shown that for various α and β values, this mapping satisfies the inequality given in Definition 1.4 as mentioned in Table 1. Especially with special choices of α and β, the nonlinear mapping as defined in Definition 1.4 becomes nonspreading mapping(Table 2), $TJ-1$ mapping(Table 3), $TJ-2$ mapping(Table 4) and nonexpansive mapping(Table 5).

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