Combinatorial Optimization Systems Theory Prospected from Rotational Symmetry

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Abstract: - Combinatorial optimization systems theory prospected from rotational symmetry involves techniques for improving the quality indices of engineering devices or systems with non-uniform structure (e.g., controllable cyber-physical objects) concerning transformation swiftness, position accuracy, and resolution, using designs based on extraordinary geometric properties and structural excellence of combinatorial conformations, namely the concept of Ideal Ring Bundles. Design techniques based on the underlying combinatorial theory provide configure one- and multidimensional systems with smaller amount elements than at present, while maintaining the other substantial operating characteristics of the systems.

Key-Words: - Galois field, cyclic group, Golomb ruler, Ideal Ring Bundle, torus reference system, GUS configuration, spatial perfection, harmony, vector data compression.

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1 Introduction

Combinatorial optimization theory of systems covers several allied fields of science and technology, including software engineering, algorithm theory, operations research, machine learning, and computational complexity theory, as well as applied mathematics and theoretical computer science. The classic combinatorial theory involves fundamental concepts of modern algebra and geometry, difference sets in a finite group, finite projective planes and Hadamard matrices theory, cyclic incidence matrices and problematic of certain symmetrical balanced incomplete block designs, orthomorphisms of groups and orthogonal Latin squares, [1]. In paper [2], the study of some permutations allows the discovery of unrelated classes. A multinomial function, describing a system with multi- input and multi-output systems, has the coefficients for parameters, [3]. The objective of the work is to test suitable sets of famous classes concerning a small subset of such functions. Many number of original models, conceptions, parallel algorithms, platforms, applications and processing gears, relate to improving the assessment of big data technology, [4], [5], [6], [7], [8], [9], [10]. This paper deals with techniques for improving the quality indices of controllable cyber-physical systems and vector processing, such as transformation speed, resolving ability, minimizing machinery memory and computing resources, using designs based on the combinatorial optimization systems theory, as well as structures created on difference families with applications a combinatorial approach, [11], Galois fields [12], and mathematical system's theory based on the idea of "perfect" multidimensional combinatorial construction, namely concept of Ideal Ring Bundles (IRBs), [13]. Theoretical research into the combinatorial configuration's properties led to better understanding of the role of rotational symmetries in the structurization of the IRBs. In mathematics, a finite field or Galois field is a field that contains a finite number of elements. As with any field, a finite field is a set on which the operations of multiplication, addition, subtraction and division. Modern combinatorial theory and system design connect with appropriate constructions, such as "Golomb rulers" [14], manifolds [15] and connecting algebra through geometry [16]. A new approach to a better understanding of remarkable properties of the "perfect" multidimensional structures based on the Golden ratio [17] and relationships of rotational symmetry spatial multidimensional configurations [13]. Symmetries and Curvature Structures are embedded in general relativity, [18].

2 Optimum Combinatorial Structures

2.1 Optimum Combinatorial Sequences

The "well-ordered" chain distributed elements in a sequence are known to be very profitable for finding optimal solutions for wide classes of technological problems.

2.1.1 Sums on a Chain- Ordered Numbers

Let us compute all L sums of ordered-chain numbers in the n-stage sequence of positive integers $\{k_1, k_2, \ldots, k_n\}$, where the sums of connected sub-sequences of the sequence enumerate the set of integers from 1 to L. The maximum number of distinct sums on ordered-chain numbers is

$$L = n \cdot (n+1)/2 \tag{1}$$

2.2 Ring Numerical Structures

Next we regard the n-stage ordered-chain sequence of positive integers $\{k_1, k_2, \ldots, k_n\}$, where k_n is followed by k_1, so that the sequence $\{k_1, k_2, \ldots, k_n\}$ turns from chain to n-stage ring numerical structure. Here unlike of ordered-chain a sum of connected sub-sequences on a ring numerical structure can have any length from 1 to n -1 as its starting point, adding the sum of all n integers. So, the maximum number of distinct sums S of the ring numerical structure is

$$S = n\,(n-1) + 1 \tag{2}$$

2.2.1 Ideal Ring Bundles

Ideal Ring Bundles are cyclic sequences of positive integers that form perfect partitions of a finite interval [1, S] of integers. The sums of consecutive sub-sequences of an Ideal Ring Bundle enumerate the set of integers [1, S] exactly once. Here is an example of an IRB with n=5 and S=5(5-1)+1=21, namely {1,5,2,10,3}. To see this, we observe:

1=1	6=1+5	11=3+1+5+2	16=2+10+3+1
2=2	7=5+2	12=2+10	17=5+2+10
3=3	8=1+5+2	13=10+3	18=1+5+2+10
4=3+1	9=3+1+5	14=10+3+1	19=10+3+1+5
5=5	10=10	15=2+10+3	20=5+2+10+3
			21=1+5+2+10+3

2.2.2 Two-Dimensional Ideal Ring Bundles

Let's consider n -stage cyclic sequence $\{K_1, K_2, \ldots, K_i, \ldots, K_n\}$, $K_1=(k_{11}, k_{12})$, $K_2=(k_{21}, k_{22})$, ..., $K_i=(k_{i1}, k_{i2})$, …., $K_n=(k_{n1}, k_{n2})$ of 2-stage (t=2) sub-sequences of the sequence, where we require all two-dimensional modular (mod m_1, mod m_2) vector sums form two-dimensional coordinate grid of sizes $m_1 \times m_2$ over a toroidal surface, $m_1 \cdot m_2 = S - 1$. This combinatorial configuration is named the two-dimensional Ideal Ring Bundle (2-D IRB). Here are four variants of two-dimensional IRBs with S =7, n = 3, $m_1 = n - 1 = 2$, $m_2 = n = 3$:

(a) {(1,0),(1,1),(1,2)}; (b) {(0,1),(0,2),(1,0)};
(c) {(0,1),(0,2),(1,2)}; (d) {(0,1),(0,2),(1,1)}

The fact that addition and multiplication allow linear algebraic operations for such finite groups. For example, the variant (a) of two-dimensional IRB {(1,0),(1,1),(1,2)} forms following 2-D vector sums modulo m_1= 2, m_2 = 3:

$$\left.\begin{array}{l}(1,0)+(1,1)\equiv(0,1)\\(1,1)+(1,2)\equiv(0,0)\\(1,2)+(1,0)\equiv(0,2)\end{array}\right\}(\mathrm{mod}\,2,\mathrm{mod}\,3)$$

Therefore, two-dimensional IRB {(1,0),(1,1),(1,2)} generates a coordinate grid 2×3 over a toroidal surface with a common reference point (0,0):

(1,0)	(1,2)	(1,2)
(0,0)	(0,1)	(0,2)

The next we see result of multiplying IRB {(1,0),(1,1),(1,2)} by vector (1,2):

$$\left.\begin{array}{l}(1,0)\cdot(1,2)\equiv(1,0)\\(1,1)\cdot(1,2)\equiv(1,2)\\(1,2)\cdot(1,2)\equiv(1,1)\end{array}\right\}(\mathrm{mod}\,2,\mathrm{mod}\,3)$$

Here we see transformation IRB {(1,0),(1,1),(1,2)} into myself. Taking the same conversion for variants (b), (c), and (d), we obtain finally the next result: (a) × (1,2) ⇒ (a); (b) × (1,2) ⇒ (b); (c) × (1,2) ⇒ (d); (d) × (1,2) ⇒ (c).

Hence, the set of four 2-D IRBs {(a), (b), (c), (d)} form both two isomorphic (a, b), and two non-isomorphic (c, d) modifications of the 2-D IRB. We call this the cyclic two-dimensional IRB group. Note, that each of these variants makes it possible to obtain $m_1 \cdot m_2 = 6$ varied 2-D IRBs.

2.2.3 Multidimensional Ideal Ring Bundles

Multidimensional ideal ring bundles form a t-manifold coordinate system immersed in (t+1)-dimensional no real space without self-intersection of coordinate axes. A t-dimensional coordinate system (t > 2) with t axes is named the manifold coordinate system $m_1 \times m_2 \times \ldots \times m_t$. The principal property of coordinate grid $m_1 \times m_2 \times \ldots \times m_t$ over a t- manifold surface is n-stage sequence $\{K_1, K_2, \ldots, K_i, \ldots, K_n\}$, $K_1=(k_{11}, k_{12}, \ldots, k_{1t})$, $K_2=(k_{21}, k_{22}, \ldots,$

k_{2t}), ..., K_i=(k_{i1}, k_{i2}, …, k_{it}), …., K_n=(k_{n1}, k_{n2},…, k_{nt}) of *t*-stage sub-sequences of the sequence, where we require a set modulo sums taking *t*- modulo (m_1, m_2 ,…., m_t) enumerates all coordinates of the *t*-manifold surface. We call this perfect *t*-manifold coordinate system $m_1 \times m_2 \times\ldots\times m_t$ with information parameters *S*, *n*, m_i (i = 1, 2, …, *t*). It is a *t*-dimensional image surface involving spatially disjointed reference *t*-axes. A planar projection of *t*-dimensional manifold coordinate axes m_1, m_2, …, m_t for grid $m_1\times m_2\times\ldots\times m_t$ with common point "+" illustrates Figure 1.

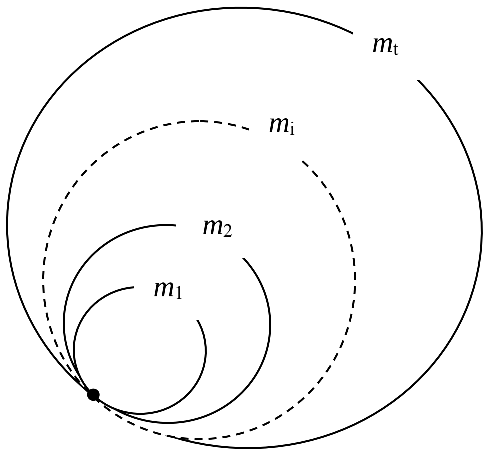

Fig. 1: A planar projection of *t*-dimensional manifold coordinate axes m_1, m_2, …, m_t for grid $m_1\times m_2\times\ldots\times m_t$ with common point " "

Here *S* is an order of spatial symmetry, *n*-number of *t*-stage sub-sequences of the *n*-sequence, and number of basic attribute-categories subset forming a complete set of *t*-dimensional vector data array. Hence, in each case, the *t*—dimensional IRB forms a manifold *t*-dimensional coordinate system. An *t*- dimensional perfect manifold coordinate system can be designed for configure *t*-dimensional optimized control systems or CAD. Therefore, all information about the *t*-dimensional vector data array of sizes $m_1 \times m_2 \times\ldots\times m_t$ is embedded into the coordinate system.

3 Glory to Ukraine Stars Ensembles

Of very exciting property has been discovered in "Glory to Ukraine Star" (GUSs) ensembles as a new type of spatial combinatorial configuration, [18].

Graphic representation of one paired seven-pointed (*n*=7) GUS-configurations {(4,2), (0,2), (1,2), (0,4), (2,2), (3,2), (5,2), (4,2)} (black ring line) and {(4,2), (1,2), (2,2), (5,2), (3,2), (0,4), (0,2), (4,2)} (colour broken line) are shown in Figure 2.

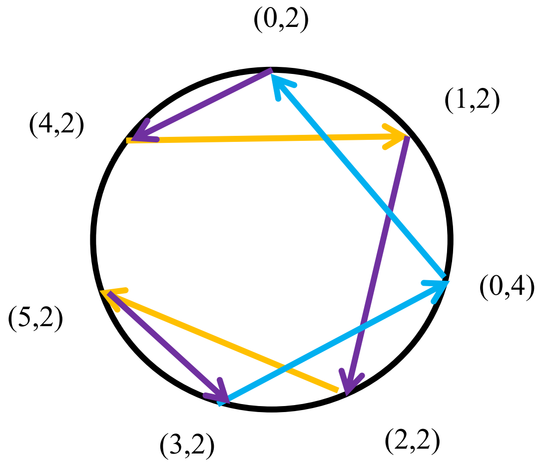

Fig. 2: Graphic representation of paired seven-pointed (*n*=7) GUS-configurations {(4,2), (0,2), (1,2), (0,4), (2,2), (3,2), (5,2)} (black ring line) and {(4,2), (1,2), (2,2), (5,2), (3,2), (0,4), (0,2)} (colour broken line)

The GUS-configuration {(4,2), (0,2), (1,2), (0,4), (2,2), (3,2), (5,2)} (black ring line) generates all *n* (*n* –1)= 42 two-dimensional ring sums, taking 2D modulo (mod 6, mod 7) as follows:

1. (0,0) ≡ ((5,2)+(4,2)+(0,2)+(1,2)+(0,4)+(2,2));
2. (0,1) ≡ ((3,2) +(5,2)+(4,2)+(0,2));
3. (0,2) ≡ (0,2);
4. (0,3) ≡ ((1,2)+(0,4)+(2,2) +(3,2));
5. (0,4) ≡ (0,4);
6. (0,5) ≡ ((0,2)+ (1,2)+ (0,4)+ (2,2)+ (3,2));
7. (0,6) ≡ ((3,2)+(5,2)+(4,2));
8. (1,0) ≡ ((3,2)+(5,2)+(4,2)+(0,2)+(1,2)+(0,4));
9. (1,1) ≡ ((0,2)+ (1,2)+ (0,4));

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42\. (5,6) ≡ ((4,2)+(0,2)+(1,2)).

The calculation procedures form $m_1 \times m_2$ = 6×7 grid, embracing a two-dimensional (*t*=2) toroid surface as being a coordinate system, where each point node from (0,0) to (5,6) occurs exactly once (*R*=1).

The second of the paired GUS-configurations {(4,2), (1,2), (2,2), (5,2), (3,2), (0,4), (0,2)} (colour broken line) forms the same set of sums, taking 2D modulo (mod 6, mod 7):

1. (0,0) ≡ ((0,4)+(0,2)+(4,2)+(1,2)+(2,2)+(5,2));
2. (0,1) ≡ ((4,2)+(1,2)+(2,2)+(5,2));
3. (0,2) ≡ (0,2);
4. (0,3) ≡ ((0,2)+(4,2)+(1,2)+(2,2)+(5,2));
5. (0,4) ≡ (0,4);
6. (0,5) ≡ ((5,2)+(3,2)+(0,4)+(0,2)+(4,2));
7. (0,6) ≡ ((0,4)+(0,2));

8. (1,0) ≡ ((5,2)+(3,2)+(0,4)+(0,2)+(4,2)+(1,2);
9. (1,1) ≡ ((0,2)+(4,2)+(1,2)+(2,2));
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42. (5,6) ≡ ((0,2)+(4,2)+(1,2)).

We observe either of the GUS-configurations {(4,2), (0,2), (1,2), (0,4), (2,2), (3,2), (5,2)} (black ring line) and {(4,2), (1,2), (2,2), (5,2), (3,2), (0,4), (0,2)} (colour broken line) forms $m_1 \times m_2$ = 6×7 grid, embracing toroid surface as 2D coordinate system.

Once more example of paired seven-pointed (n=7) GUS-configurations {(1,1), (1,3), (1,5), (1,0), (1,2), (1,4), (1,6)} (ring cycle), and {(1,1), (1,5), (1,2), (1,6), (1,3), (1,0), (1,4)} (star cycle) presents in Figure 3.

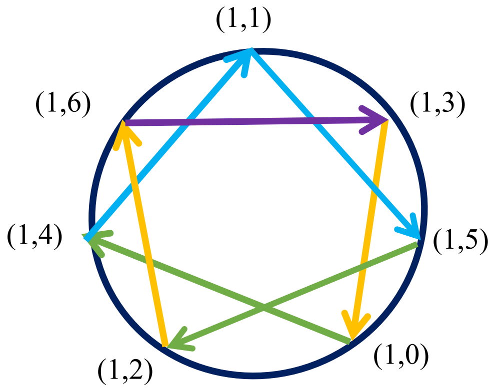

Fig. 3: Once more example of paired seven-pointed (n=7) GUS-configurations, namely the {(1,0), (1,2), (1,4), (1,6), (1,1), (1,3), (1,5)} (ring cycle), and {(1,0), (1,4), (1,1), (1,5), (1,2), (1,6), (1,3)} (star cycle)

GUS-configuration {(1,1), (1,3), (1,5), (1,0), (1,2), (1,4), (1,6)} (ring cycle) forms the set of 2D vector sums (clockwise), taking 2D modulo (6, 7):
1. (0,0) ≡ ((1,2)+(1,4)+(1,6)+(1,1)+(1,3)+(1,5));
2. (0,1) ≡ ((1,1)+(1,3)+ (1,5)+(1,0)+ (1,2)+(1,4));
3. (0,2) ≡ ((1,0)+(1,2)+(1,4)+(1,6)+(1,1)+(1,3));
4. (0,3) ≡ ((1,6)+ (1,1)+ (1,3)+(1,5)+(1,0)+ (1,2));
5. (0,4) ≡ ((1,5)+(1,0)+ (1,2)+(1,4)+ (1,6)+(1,1));
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41. (5,5) ≡ ((1,4)+(1,6)+ (1,1)+ (1,3)+(1,5);
42. (5,6) ≡ ((1,0)+ (1,2)+(1,4)+ (1,6)+(1,1)).

The second of the paired GUS-configuration {(1,1), (1,5), (1,2), (1,6), (1,3), (1,0), (1,4)} (star cycle) forms the set of 2D vector sums, taking 2D modulo (mod 6, mod 7):
1. (0,0) ≡ ((1,4)+(1,1)+(1,5)+ (1,2)+(1,6)+(1,3));
2. (0,1) ≡ ((1,3)+ (1,0)+(1,4)+(1,1)+ (1,5)+(1,2));
3. (0,2) ≡ ((1,2)+(1,6)+(1,3) +(1,0)+(1,4)+(1,1));
4. (0,3) ≡ ((1,1)+(1,5)+ (1,2)+(1,6)+(1,3)+(1,0));
5. (0,4) ≡ ((1,0)+(1,4)+ (1,1)+(1,5)+ (1,2)+(1,6));
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………………………………………………
41. (5,5) ≡ ((1,0)+(1,4)+ (1,1)+ (1,5)+(1,2);
42. (5,6) ≡ ((1,3)+ (1,0)+(1,4)+ (1,1)+(1,5)).

Each of the paired seven-pointed (n=7) GUS-configurations, {(1,1), (1,3), (1,5), (1,0), (1,2), (1,4), (1,6)} (ring cycle), and {(1,1), (1,5), (1,2), (1,6), (1,3), (1,0), (1,4)} (star cycle) forms $m_1 \times m_2$ = 6×7 grid, embracing two-dimensional (t=2) toroid surface as coordinate system exactly once (R=1).

The underlying examples of paired seven-pointed (n=7) GUS-configurations evident that either of the combinatorial configurations {(4,2), (0,2), (1,2), (0,4), (2,2), (3,2), (5,2)}, {(4,2), (1,2), (2,2), (5,2), (3,2), (0,4), (0,2)}, {(1,1), (1,3), (1,5), (1,0), (1,2), (1,4), (1,6)}, {(1,1), (1,5), (1,2), (1,6), (1,3), (1,0), (1,4)} forms complete coordinate system $m_1 \times m_2$ = 6×7 over toroid surface.

Graphic representation of a set of paired seven-pointed (n=7) GUS ensembles is illustrated Figure 4.

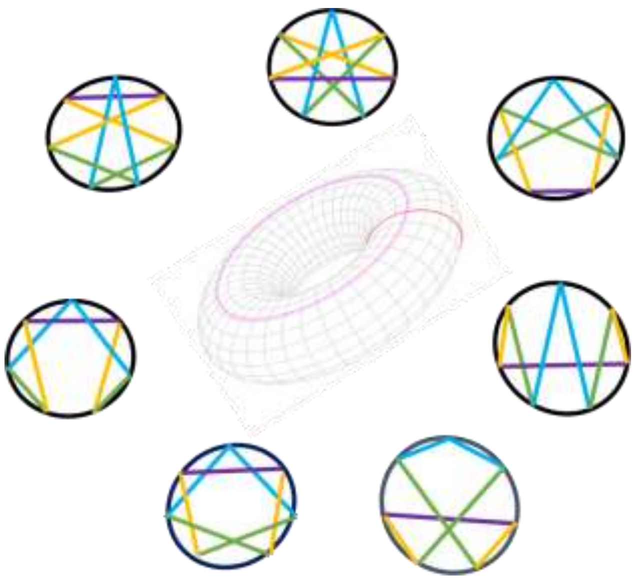

Fig. 4: Graphic representation of a set of paired seven-pointed (n=7) GUS ensembles

Scientifically, the GUSs are the most perfect combinatorial configurations with favorable qualities of the structures and remarkable properties.

4 Conclusion

Combinatorial optimization systems theory prospected from rotational symmetry provides a conceptual model of technical systems. Moreover, the optimization is embedded into the underlying model, which make it possible to configure systems with fewer elements than at present, while maintaining or improving on the other characteristics of the system. The theoretical connection between the theory of cyclic groups, and IRBs offers great opportunities for the development of advanced systems theory for configuring innovative devices and process engineering based on the remarkable mathematical properties and structural perfection of the IRBs. Application of optimized perfect manifold coordinate systems for information technologies provides new conceptual techniques for improving the quality indices of the technologies and management systems concerning transmission content, and compression of vector data, embodying reliability of vector data coding and processing under the minimized basis of manifold coordinate system. The essence of the technology is processing vector information in the database of manifold coordinate systems, where the basis is a set of coordinates smaller than the total number of coordinates of this coordinate system, which generates it by adding the latter. The theoretical connection between the rotational symmetry - asymmetry ensembles, and combinatorial optimization systems theory offers the great opportunities for the development of advanced systems engineering and information technologies under minimized manifold coordinate systems. Combinatorial optimization systems theory prospected from rotational symmetry allows expanding the applicability of the systems theory for the development of new mathematical methods and models in systems and control.

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Conflict of Interest

The authors have no conflicts of interest to declare.