On one series of the reciprocals of the product of two Fibonacci numbers whose indices differ by an even number

POTŮČEK R.

Department of Mathematics and Physics, University of Defence,
Kounicova 65, 662 10 Brno, CZECH REPUBLIC
https://fvt.unob.cz/fakulta/struktura/katedra-matematiky-a-fyziky-k-215/struktura-k-215/
ORCID iD: 0000-0003-4385-691X

Abstract: This paper is inspired by a very interesting YouTube video by Michael Penn, professor of mathematics at Randolph College in Virginia, USA. He dedicated himself to the popularization of mathematics on his website in addition to his teaching and scientific work at the university and in addition to his scientific work. First, we deal with four specific series of the reciprocals of the product of two Fibonacci numbers whose indices differ by 2, 4, 6, and 8. Then, we generalize these four results to the series of the reciprocals of the product of two Fibonacci numbers whose indices differ by an even number. Finally, we perform a numerical verification of the derived formula using Maple 2020 software. Based on the derived formula, it can be concluded that the series we are dealing with belong to infinite series whose sum can be expressed in closed form.

Key-Words: Fibonacci numbers, partial fractions decomposition, telescoping series, Maple 2020

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1 Introduction

This paper, inspired by the YouTube video [1], has been written to popularize Fibonacci numbers and, in particular, the sums of the reciprocals of the expressions with Fibonacci numbers. Articles dealing with a similar topic include e.g. articles [2] and [3]. Some results can also be found on Wikipedia [4]. All the series discussed deal with in the paper are examples of more complex numerical series with very simple sums.

The Fibonacci numbers appeared for the first time in ancient Indian mathematics before the beginning of the era. In Europe, they were first described by Italian mathematician Leonardo Pisano (Leonardo of Pisa), also known as Fibonacci (about 1175–1250), in the year 1202 in his book Liber Abaci. The Fibonacci numbers include the rule of golden proportions. In essence, this is an observation that the ratio of any two sequential Fibonacci numbers approximates to the value of the golden ratio.

The Fibonacci numbers, or the Fibonacci sequence, are one of the most famous and most widely studied numbers in modern mathematics, with interesting and amazing properties. The Fibonacci numbers continue to be very popular and offer provide many numerous new topics for further research work. For instance, several following articles have been published in recent years, including [5], [6], [7], and [8]. The Fibonacci numbers are a central topic in various monographs, such as [9], [10], [11], [12], [13], and [14].

2 Basic definitions and properties of Fibonacci numbers

The *Fibonacci numbers* F_k are defined for an arbitrary integer $k \geq 2$ by the recursive formula

$$F_k = F_{k-1} + F_{k-2}. \tag{1}$$

The two initial values are $F_0 = 0$ and $F_1 = 1$.

Using formula (1), we obtain the well-known *Fibonacci sequence*

$$\{F_k\} = \{0,\ 1,\ 1,\ 2,\ 3,\ 5,\ 8,\ 13,\ 21,\ 34,\ 55,\ 89,\ 144,\ 233,\ 377,\ 610,\ 987,\ \ldots\}.$$

The ratio of two consecutive Fibonacci numbers converges and approaches the *golden ratio*

$$\varphi = \lim_{k\to\infty} \frac{F_{k+1}}{F_k}.$$

This limit has the value

$$\varphi = \frac{1+\sqrt{5}}{2} \approx 1.618033988.$$

The Fibonacci numbers have a closed-form expression which is known as *Binet's formula*

$$F_k = \frac{\varphi^k - \psi^k}{\sqrt{5}}, \quad k \geq 0, \tag{2}$$

where ψ is the conjugate of φ, i.e.

$$\psi = \frac{1-\sqrt{5}}{2} \approx -0.618033988,$$

so for an arbitrary integer $k \geq 0$ we have

$$F_k = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^k - \left(\frac{1-\sqrt{5}}{2}\right)^k\right]$$

or

$$F_k = \frac{1}{2^k\sqrt{5}}\left[\left(1+\sqrt{5}\right)^k - \left(1-\sqrt{5}\right)^k\right].$$

The process of deriving formula (2) using generating functions can be found, for example, in the article [15]. Since $\dfrac{1-\sqrt{5}}{2} = 1 - \dfrac{1+\sqrt{5}}{2}$, we get that $\psi = 1 - \varphi$.

By repeatedly using of the equality (1), i.e. by means of the formulas

$$F_{k+2} = F_{k+1} + F_k$$

and

$$F_{k+1} = F_{k+2} - F_k\,,$$

where $k \geq 0$ is an arbitrary integer, we obtain the following expressions for F_{k+4}, F_{k+6} and F_{k+8} in terms of F_{k+2} and F_k, which we will use in Sections 4, 5 and 6:

$$\begin{aligned} F_{k+4} &= F_{k+3} + F_{k+2} = (F_{k+2} + F_{k+1}) + F_{k+2} \\ &= 2F_{k+2} + F_{k+1} = 2F_{k+2} + (F_{k+2} - F_k) \\ &= 3F_{k+2} - F_k = F_4F_{k+2} - F_2F_k\,, \end{aligned} \tag{3}$$

$$\begin{aligned} F_{k+6} &= F_{k+5} + F_{k+4} = (F_{k+4} + F_{k+3}) + F_{k+4} \\ &= 2F_{k+4} + F_{k+3} = 2(F_{k+3} + F_{k+2}) + F_{k+3} \\ &= 3F_{k+3} + 2F_{k+2} = 3(F_{k+2} + F_{k+1}) + 2F_{k+2} \\ &= 5F_{k+2} + 3F_{k+1} = 5F_{k+2} + 3(F_{k+2} - F_k) \\ &= 8F_{k+2} - 3F_k = F_6F_{k+2} - F_4F_k\,, \end{aligned} \tag{4}$$

$$\begin{aligned} F_{k+8} &= F_{k+7} + F_{k+6} = (F_{k+6} + F_{k+5}) + F_{k+6} \\ &= 2F_{k+6} + F_{k+5} = 2(F_{k+5} + F_{k+4}) + F_{k+5} \\ &= 3F_{k+5} + 2F_{k+4} = 3(F_{k+4} + F_{k+3}) + 2F_{k+4} \\ &= 5F_{k+4} + 3F_{k+3} = 5(F_{k+3} + F_{k+2}) + 3F_{k+3} \\ &= 8F_{k+3} + 5F_{k+2} = 8(F_{k+2} + F_{k+1}) + 5F_{k+2} \\ &= 13F_{k+2} + 8F_{k+1} = 13F_{k+2} + 8(F_{k+2} - F_k) \\ &= 21F_{k+2} - 8F_k = F_8F_{k+2} - F_6F_k\,. \end{aligned} \tag{5}$$

There are plenty of other relationships for the Fibonacci numbers, but here we present only those that we will continue to use in the paper.

3 The series of the reciprocals of the product of two Fibonacci numbers whose indices differ by two

Let us consider the series

$$\begin{aligned} \sum_{k=1}^{\infty} \frac{1}{F_kF_{k+2}} &= \frac{1}{F_1F_3} + \frac{1}{F_2F_4} + \frac{1}{F_3F_5} + \cdots \\ &= \frac{1}{1\cdot 2} + \frac{1}{1\cdot 3} + \frac{1}{2\cdot 5} + \cdots\,. \end{aligned} \tag{6}$$

Denote the sum of the series (6) as $S(2)$ and state, that taking a sum is similar to taking an integral. If we integrate the reciprocal of the product of two functions, then we often use the method of partial fraction decomposition. Inspired by integration we use the decomposition of a general kth term of the series (6) in the form

$$\frac{1}{F_kF_{k+2}} = \frac{A}{F_k} + \frac{B}{F_{k+2}}\,.$$

By multiplying this equation by the product F_kF_{k+2} we get

$$1 = AF_{k+2} + BF_k\,.$$

According to (1) we have $F_{k+2} = F_{k+1} + F_k$, so we obtain an equation

$$1 = A(F_{k+1} + F_k) + BF_k\,,$$

from which we get

$$1 = AF_{k+1} + (A+B)F_k\,.$$

To obtain an equation with an unknown coefficient B, we substitute $A = \dfrac{1}{F_{k+1}}$. Then we obtain an equation

$$1 = 1 + \left(\frac{1}{F_{k+1}} + B\right)F_k\,,$$

i.e.

$$0 = \left(\frac{1}{F_{k+1}} + B\right)F_k\,,$$

from which we get

$$B = -\frac{1}{F_{k+1}}\,.$$

So we have received the coefficients

$$A = \frac{1}{F_{k+1}} \qquad \text{and} \qquad B = -\frac{1}{F_{k+1}}\,.$$

That gives us the partial fraction decomposition in the form

$$\frac{1}{F_kF_{k+2}} = \frac{1}{F_kF_{k+1}} - \frac{1}{F_{k+1}F_{k+2}}\,.$$

Now we can write the series (6) in the form

$$\sum_{k=1}^{\infty}\frac{1}{F_kF_{k+2}}=\sum_{k=1}^{\infty}\left(\frac{1}{F_kF_{k+1}}-\frac{1}{F_{k+1}F_{k+2}}\right).$$

The sum $S(2)$ we determine as a limit of the sequence of partial sums $\{S(2)_n\}$. We get

$$\begin{aligned}S(2)&=\lim_{n\to\infty}S(2)_n=\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{F_kF_{k+2}}\\&=\lim_{n\to\infty}\sum_{k=1}^{n}\left(\frac{1}{F_kF_{k+1}}-\frac{1}{F_{k+1}F_{k+2}}\right).\end{aligned}$$

By writing out the terms of the nth partial sum, we obtain

$$\begin{aligned}S(2)=\lim_{n\to\infty}\Bigg[&\left(\frac{1}{F_1F_2}-\frac{1}{F_2F_3}\right)\\&+\left(\frac{1}{F_2F_3}-\frac{1}{F_3F_4}\right)+\left(\frac{1}{F_3F_4}-\frac{1}{F_4F_5}\right)\\&+\cdots+\left(\frac{1}{F_{n-1}F_n}-\frac{1}{F_nF_{n+1}}\right)\\&+\left(\frac{1}{F_nF_{n+1}}-\frac{1}{F_{n+1}F_{n+2}}\right)\Bigg].\end{aligned}$$

Due to the telescopic properties of this nth partial sum, all summands will be canceled, with the exception of the first and last summands. Because $\lim_{n\to\infty}F_n=\infty$, we have $\lim_{n\to\infty}\frac{1}{F_n}=0$ and also $\lim_{n\to\infty}\frac{1}{F_{n+1}F_{n+2}}=0$. So we get

$$\begin{aligned}S(2)&=\sum_{k=1}^{\infty}\frac{1}{F_kF_{k+2}}\\&=\lim_{n\to\infty}\left(\frac{1}{F_1F_2}-\frac{1}{F_{n+1}F_{n+2}}\right)\\&=\frac{1}{F_1F_2}=\frac{1}{1\cdot 1}=1.\end{aligned}\tag{7}$$

4 The series of the reciprocals of the product of two Fibonacci numbers whose indices differ by four

Let us consider the series

$$\begin{aligned}\sum_{k=1}^{\infty}\frac{1}{F_kF_{k+4}}&=\frac{1}{F_1F_5}+\frac{1}{F_2F_6}+\frac{1}{F_3F_7}+\cdots\\&=\frac{1}{1\cdot 5}+\frac{1}{1\cdot 8}+\frac{1}{2\cdot 13}+\cdots.\end{aligned}\tag{8}$$

Denote the sum of the series (8) as $S(4)$ and as in the previous section, we determine the sum $S(4)$ by using the decomposition of a general kth term of the series (8) into the sum of partial fractions. So, we suppose that we can write

$$\frac{1}{F_kF_{k+4}}=\frac{A}{F_k}+\frac{B}{F_{k+4}}.$$

By multiplying by the product F_kF_{k+4} we get

$$1=AF_{k+4}+BF_k.$$

According to (3) we obtain an equation

$$1=A(3F_{k+2}-F_k)+BF_k,$$

from which we get

$$1=3AF_{k+2}+(B-A)F_k.$$

To obtain an equation with an unknown coefficient B, we substitute $A=\frac{1}{3F_{k+2}}$. Then we receive an equation

$$1=1+\left(B-\frac{1}{3F_{k+2}}\right)F_k,$$

i.e.

$$0=\left(B-\frac{1}{3F_{k+2}}\right)F_k,$$

whence we get

$$B=\frac{1}{3F_{k+2}}.$$

So we have received the coefficients

$$A=\frac{1}{3F_{k+2}}\qquad\text{and also}\qquad B=\frac{1}{3F_{k+2}}.$$

That gives us the partial fraction decomposition in the form

$$\frac{1}{F_kF_{k+4}}=\frac{1}{3F_kF_{k+2}}+\frac{1}{3F_{k+2}F_{k+4}}.$$

Therefore we can write the series (6) in the form

$$\sum_{k=1}^{\infty}\frac{1}{F_kF_{k+4}}=\frac{1}{3}\sum_{k=1}^{\infty}\left(\frac{1}{F_kF_{k+2}}+\frac{1}{F_{k+2}F_{k+4}}\right).$$

In Section 3 we show that the series $\sum_{k=1}^{\infty}\frac{1}{F_kF_{k+2}}$ converges to 1. The series $\sum_{k=1}^{\infty}\frac{1}{F_{k+2}F_{k+4}}$ is clearly of the same type of convergent series that can be obtained by suitable re-indexing. So now we can write

$$\sum_{k=1}^{\infty}\frac{1}{F_kF_{k+4}}=\frac{1}{3}\left(\sum_{k=1}^{\infty}\frac{1}{F_kF_{k+2}}+\sum_{k=1}^{\infty}\frac{1}{F_{k+2}F_{k+4}}\right)$$
$$=\frac{1}{3}\left(\sum_{k=1}^{\infty}\frac{1}{F_kF_{k+2}}+\sum_{k=3}^{\infty}\frac{1}{F_kF_{k+2}}\right).$$

Because we can write the second series using the first one in the form

$$\sum_{k=3}^{\infty}\frac{1}{F_kF_{k+2}}=\sum_{k=1}^{\infty}\frac{1}{F_kF_{k+2}}-\frac{1}{F_1F_3}-\frac{1}{F_2F_4},$$

we have

$$\sum_{k=1}^{\infty}\frac{1}{F_kF_{k+4}}$$
$$=\frac{1}{3}\left(\sum_{k=1}^{\infty}\frac{1}{F_kF_{k+2}}+\sum_{k=1}^{\infty}\frac{1}{F_kF_{k+2}}-\frac{1}{F_1F_3}-\frac{1}{F_2F_4}\right)$$
$$=\frac{1}{3}\left(2\sum_{k=1}^{\infty}\frac{1}{F_kF_{k+2}}-\frac{1}{1\cdot 2}-\frac{1}{1\cdot 3}\right).$$

According to (7) we obtain

$$\begin{aligned}S(4)&=\sum_{k=1}^{\infty}\frac{1}{F_kF_{k+4}}\\&=\frac{1}{3}\left(\frac{2}{F_1F_2}-\frac{1}{F_1F_3}-\frac{1}{F_2F_4}\right)\\&=\frac{1}{3}\left(\frac{2}{1}-\frac{1}{2}-\frac{1}{3}\right)=\frac{1}{3}\cdot\frac{7}{6}=\frac{7}{18}.\end{aligned}\tag{9}$$

5 The series of the reciprocals of the product of two Fibonacci numbers whose indices differ by six

Let us consider the series

$$\begin{aligned}\sum_{k=1}^{\infty}\frac{1}{F_kF_{k+6}}&=\frac{1}{F_1F_7}+\frac{1}{F_2F_8}+\frac{1}{F_3F_9}+\cdots\\&=\frac{1}{1\cdot 13}+\frac{1}{1\cdot 21}+\frac{1}{2\cdot 34}+\cdots\end{aligned}\tag{10}$$

and denote the sum of this series as $S(6)$. We will proceed similarly to the previous section, so we will limit ourselves to only a brief comment on individual steps.

We again assume that we can write

$$\frac{1}{F_kF_{k+6}}=\frac{A}{F_k}+\frac{B}{F_{k+6}},$$

i.e.

$$1=AF_{k+6}+BF_k.$$

According to (4) we obtain an equation

$$1=A(8F_{k+2}-3F_k)+BF_k,$$

i.e.

$$1=8AF_{k+2}+(B-3A)F_k.$$

When we substitute $A=\dfrac{1}{8F_{k+2}}$, we receive an equation

$$1=1+\left(B-\frac{3}{8F_{k+2}}\right)F_k,$$

whence we get

$$B=\frac{3}{8F_{k+2}}.$$

These values of the coefficients A and B lead to decomposition into partial fractions of the form

$$\frac{1}{F_kF_{k+6}}=\frac{1}{8F_kF_{k+2}}+\frac{3}{8F_{k+2}F_{k+6}}.$$

Therefore we can write the series (10) in the form

$$\sum_{k=1}^{\infty}\frac{1}{F_kF_{k+6}}=\frac{1}{8}\sum_{k=1}^{\infty}\left(\frac{1}{F_kF_{k+2}}+\frac{3}{F_{k+2}F_{k+6}}\right).$$

In Section 4 we show that the series $\displaystyle\sum_{k=1}^{\infty}\frac{1}{F_kF_{k+4}}$ converges to $\dfrac{7}{18}$. The series $\displaystyle\sum_{k=1}^{\infty}\frac{3}{F_{k+2}F_{k+6}}$ is of the same type of convergent series as the series $\displaystyle\sum_{k=1}^{\infty}\frac{1}{F_kF_{k+4}}$. So we can write

$$\sum_{k=1}^{\infty}\frac{1}{F_kF_{k+6}}=\frac{1}{8}\left(\sum_{k=1}^{\infty}\frac{1}{F_kF_{k+2}}+\sum_{k=1}^{\infty}\frac{3}{F_{k+2}F_{k+6}}\right)$$
$$=\frac{1}{8}\left(\sum_{k=1}^{\infty}\frac{1}{F_kF_{k+2}}+3\sum_{k=3}^{\infty}\frac{1}{F_kF_{k+4}}\right).$$

Because

$$\sum_{k=3}^{\infty}\frac{1}{F_kF_{k+4}}=\sum_{k=1}^{\infty}\frac{1}{F_kF_{k+4}}-\frac{1}{F_1F_5}-\frac{1}{F_2F_6},$$

we have

$$\sum_{k=1}^{\infty}\frac{1}{F_kF_{k+6}}=\frac{1}{8}\left[\sum_{k=1}^{\infty}\frac{1}{F_kF_{k+2}}\right.$$
$$\left.+3\left(\sum_{k=1}^{\infty}\frac{1}{F_kF_{k+4}}-\frac{1}{F_1F_5}-\frac{1}{F_2F_6}\right)\right]$$
$$=\frac{1}{8}\left[\sum_{k=1}^{\infty}\frac{1}{F_kF_{k+2}}+3\left(\sum_{k=1}^{\infty}\frac{1}{F_kF_{k+4}}-\frac{1}{1\cdot 5}-\frac{1}{1\cdot 8}\right)\right].$$

According to (7) and (9) we obtain

$$\sum_{k=1}^{\infty}\frac{1}{F_kF_{k+6}}=\frac{1}{8}\left[1+3\left(\frac{7}{18}-\frac{1}{5}-\frac{1}{8}\right)\right]$$
$$=\frac{1}{8}\left(1+3\cdot\frac{140-72-45}{360}\right)=\frac{1}{8}\cdot\frac{360+69}{360},$$

so

$$S(6)=\sum_{k=1}^{\infty}\frac{1}{F_kF_{k+6}}=\frac{1}{8}\left\{\frac{1}{F_1F_2}\right.$$
$$\left.+3\left[\frac{1}{3}\left(\frac{2}{F_1F_2}-\frac{1}{F_1F_3}-\frac{1}{F_2F_4}\right)-\frac{1}{F_1F_5}-\frac{1}{F_2F_6}\right]\right\}$$
$$=\frac{1}{8}\left(\frac{3}{F_1F_2}-\frac{1}{F_1F_3}-\frac{1}{F_2F_4}-\frac{3}{F_1F_5}-\frac{3}{F_2F_6}\right)$$
$$=\frac{1}{8}\cdot\frac{143}{120}=\frac{143}{960}. \tag{11}$$

6 The series of the reciprocals of the product of two Fibonacci numbers whose indices differ by eight

Let us consider the series

$$\sum_{k=1}^{\infty}\frac{1}{F_kF_{k+8}}=\frac{1}{F_1F_9}+\frac{1}{F_2F_{10}}+\frac{1}{F_3F_{11}}$$
$$=\frac{1}{1\cdot 34}+\frac{1}{1\cdot 55}+\frac{1}{2\cdot 89}+\cdots \tag{12}$$

and denote the sum of this series as $S(8)$. We will proceed briefly and similarly to the previous sections, We again write

$$\frac{1}{F_kF_{k+8}}=\frac{A}{F_k}+\frac{B}{F_{k+8}},$$

i.e.

$$1=AF_{k+8}+BF_k.$$

According to (5) we get an equation

$$1=A(21F_{k+2}-8F_k)+BF_k,$$

i.e.

$$1=21AF_{k+2}+(B-8A)F_k.$$

When we substitute $A=\dfrac{1}{21F_{k+2}}$, we receive an equation

$$1=1+\left(B-\frac{8}{21F_{k+2}}\right)F_k,$$

whence we get

$$B=\frac{8}{21F_{k+2}}.$$

These values of the coefficients A and B lead to decomposition into partial fractions of the form

$$\frac{1}{F_kF_{k+8}}=\frac{1}{21F_kF_{k+2}}+\frac{8}{21F_{k+2}F_{k+8}}.$$

Therefore we can write the series (12) in the form

$$\sum_{k=1}^{\infty}\frac{1}{F_kF_{k+8}}=\frac{1}{21}\sum_{k=1}^{\infty}\left(\frac{1}{F_kF_{k+2}}+\frac{8}{F_{k+2}F_{k+8}}\right).$$

In Section 5 we show that the series $\displaystyle\sum_{k=1}^{\infty}\frac{1}{F_kF_{k+6}}$ converges to $\dfrac{143}{960}$ and the series $\displaystyle\sum_{k=1}^{\infty}\frac{8}{F_{k+2}F_{k+8}}$ is of the same type of convergent series as the series $\displaystyle\sum_{k=1}^{\infty}\frac{1}{F_kF_{k+6}}$. So we can write

$$\sum_{k=1}^{\infty}\frac{1}{F_kF_{k+8}}=\frac{1}{21}\left(\sum_{k=1}^{\infty}\frac{1}{F_kF_{k+2}}+\sum_{k=1}^{\infty}\frac{8}{F_{k+2}F_{k+8}}\right)$$
$$=\frac{1}{21}\left(\sum_{k=1}^{\infty}\frac{1}{F_kF_{k+2}}+8\sum_{k=3}^{\infty}\frac{1}{F_kF_{k+6}}\right).$$

Because

$$\sum_{k=3}^{\infty}\frac{1}{F_kF_{k+6}}=\sum_{k=1}^{\infty}\frac{1}{F_kF_{k+6}}-\frac{1}{F_1F_7}-\frac{1}{F_2F_8},$$

we have

$$\sum_{k=1}^{\infty}\frac{1}{F_kF_{k+8}}=\frac{1}{21}\left[\sum_{k=1}^{\infty}\frac{1}{F_kF_{k+2}}\right.$$
$$\left.+8\left(\sum_{k=1}^{\infty}\frac{1}{F_kF_{k+6}}-\frac{1}{F_1F_7}-\frac{1}{F_2F_8}\right)\right]$$
$$=\frac{1}{21}\left(\sum_{k=1}^{\infty}\frac{1}{F_kF_{k+2}}\right.$$
$$\left.+8\left(\sum_{k=1}^{\infty}\frac{1}{F_kF_{k+6}}-\frac{1}{1\cdot 13}-\frac{1}{1\cdot 21}\right)\right].$$

According to (7) and (11) we obtain

$$\sum_{k=1}^{\infty}\frac{1}{F_kF_{k+8}}=\frac{1}{21}\left[1+8\left(\frac{143}{960}-\frac{1}{13}-\frac{1}{21}\right)\right]=$$
$$=\frac{1}{21}\left(1+8\cdot\frac{13013-6720-4160}{87360}\right)$$
$$=\frac{1}{21}\cdot\frac{87360+17064}{87360},$$

so

$$
\begin{aligned}
S(8) &= \sum_{k=1}^{\infty} \frac{1}{F_k F_{k+8}} = \frac{1}{21}\left\{ \frac{1}{F_1F_2} \right.\\
&+ 8\left[\frac{1}{8}\left(\frac{3}{F_1F_2} - \frac{1}{F_1F_3} - \frac{1}{F_2F_4}\right.\right.\\
&\left.\left.\left. - \frac{3}{F_1F_5} - \frac{3}{F_2F_6}\right) - \frac{1}{F_1F_7} - \frac{1}{F_2F_8}\right]\right\} \\
&= \frac{1}{21}\left(\frac{4}{F_1F_2} - \frac{1}{F_1F_3} - \frac{1}{F_2F_4} - \frac{3}{F_1F_5}\right.\\
&\left. - \frac{3}{F_2F_6} - \frac{8}{F_1F_7} - \frac{8}{F_2F_8}\right)\\
&= \frac{1}{21}\cdot\frac{104424}{87360} = \frac{4351}{76440}.
\end{aligned} \tag{13}
$$

7 The series of the reciprocals of the product of two Fibonacci numbers whose indices differ in the even number

Let us consider the series

$$
\begin{aligned}
\sum_{k=1}^{\infty} \frac{1}{F_k F_{k+2n}} &= \frac{1}{F_1F_{k+2n}} + \frac{1}{F_2F_{2+2n}}\\
&+ \frac{1}{F_3F_{3+2n}} + \frac{1}{F_4F_{4+2n}} + \cdots ,
\end{aligned} \tag{14}
$$

where n is an arbitrary positive integer, and denote the sum of this series as $S(2n)$. From the above formulas (7), (9), (11), (13), which can be written in the forms

$$S(2) = \sum_{k=1}^{\infty} \frac{1}{F_kF_{k+2}} = \frac{1}{F_1F_2},$$

$$S(4) = \sum_{k=1}^{\infty} \frac{1}{F_kF_{k+4}} = \frac{1}{3}\left(\frac{2}{F_1F_2} - \frac{1}{F_1F_3} - \frac{1}{F_2F_4}\right),$$

$$
\begin{aligned}
S(6) &= \sum_{k=1}^{\infty} \frac{1}{F_kF_{k+6}} = \frac{1}{8}\left(\frac{3}{F_1F_2} - \frac{1}{F_1F_3} - \frac{1}{F_2F_4}\right.\\
&\left. - \frac{3}{F_1F_5} - \frac{3}{F_2F_6}\right),
\end{aligned}
$$

$$
\begin{aligned}
S(8) &= \sum_{k=1}^{\infty} \frac{1}{F_kF_{k+8}} = \frac{1}{21}\left(\frac{4}{F_1F_2} - \frac{1}{F_1F_3} - \frac{1}{F_2F_4}\right.\\
&\left. - \frac{3}{F_1F_5} - \frac{3}{F_2F_6} - \frac{8}{F_1F_7} - \frac{8}{F_2F_8}\right),
\end{aligned}
$$

i.e. the formulas that have the forms

$$S(2) = \sum_{k=1}^{\infty} \frac{1}{F_kF_{k+2}} = \frac{1}{F_2}\cdot\frac{1}{F_1F_2},$$

$$S(4) = \sum_{k=1}^{\infty} \frac{1}{F_kF_{k+4}} = \frac{1}{F_4}\left(\frac{2}{F_1F_2} - \frac{F_2}{F_1F_3} - \frac{F_2}{F_2F_4}\right),$$

$$
\begin{aligned}
S(6) &= \sum_{k=1}^{\infty} \frac{1}{F_kF_{k+6}} = \frac{1}{F_6}\left(\frac{3}{F_1F_2} - \frac{F_2}{F_1F_3} - \frac{F_2}{F_2F_4}\right.\\
&\left. - \frac{F_4}{F_1F_5} - \frac{F_4}{F_2F_6}\right),
\end{aligned}
$$

$$
\begin{aligned}
S(8) &= \sum_{k=1}^{\infty} \frac{1}{F_kF_{k+8}} = \frac{1}{F_8}\left(\frac{4}{F_1F_2} - \frac{F_2}{F_1F_3} - \frac{F_2}{F_2F_4}\right.\\
&\left. - \frac{F_4}{F_1F_5} - \frac{F_4}{F_2F_6} - \frac{F_6}{F_1F_7} - \frac{F_6}{F_2F_8}\right),
\end{aligned}
$$

it can be concluded that the formula for the sum of the reciprocals of two Fibonacci numbers whose indices differ in the even number $2n$ will have the form

$$
\begin{aligned}
S(2n) &= \sum_{k=1}^{\infty} \frac{1}{F_kF_{k+2n}}\\
&= \frac{1}{F_{2n}}\left(\frac{n}{F_1F_2} - F_2\sum_{k=1}^{2}\frac{1}{F_kF_{k+2}}\right.\\
&\left. - F_4\sum_{k=1}^{2}\frac{1}{F_kF_{k+4}} - \cdots - F_{2n-2}\sum_{k=1}^{2}\frac{1}{F_kF_{k+2n-2}}\right).
\end{aligned}
$$

Now we can formulate the main result of this paper:

Theorem 1. *For each positive integer n,*

$$
\begin{aligned}
S(2n) &= \sum_{k=1}^{\infty} \frac{1}{F_kF_{k+2n}}\\
&= \frac{1}{F_{2n}}\left[\frac{n}{F_1F_2} - \sum_{i=1}^{n-1}\left(F_{2i}\sum_{k=1}^{2}\frac{1}{F_kF_{k+2i}}\right)\right].
\end{aligned} \tag{15}
$$

8 Numerical verification

We solve the problem of determining the values of the sum $S(2n)$ for $n = 1, 2, \ldots, 10$, for $n = 20, 30, \ldots, 100$, for $n = 500$ and for $n = 1000$. We use on the one hand an approximate direct evaluation of the sum

$$S(2n, t) = \sum_{k=1}^{t} \frac{1}{F_kF_{k+2n}},$$

where the upper index $t = 10^3$, using the basic programming language of the computer algebra system

Maple 2020, and on the other hand the formula (15) for evaluating the sum $S(2n)$ for the same values of the variable n. We compare 21 pairs of these ways obtained sums $S(2n,t)$ and $S(2n)$ to verify the formula (15). We use the following simple procedure `recfiboeven`.

```
> with(combinat,fibonacci):
> recfiboeven:=proc(n)
    local i,k,s,st,s1; st:=0;
    s1:=sum(fibonacci(2*i)*sum(1/(fibonacci(k)
        *fibonacci(k+2*i)),k=1..2),i=1..n-1);
    s:= (1/fibonacci(2*n))*(n/(fibonacci(1)
        *fibonacci(2))-s1);
    print("n=",n,"s=",evalf[12](s));
    for k from 1 to 1000 do
        st:=st+1/(fibonacci(k)*fibonaci(k+2*n));
    end do;
    print("st=",evalf[12](st));
    print("diff=",evalf[12](abs(st-s)));
  end proc:
> for n from 1 to 10 do
    recfiboeven(n);
  end do;
  for n from 20 by 10 to 100 do
    recfiboeven(n);
  end do;
  recfiboeven(500); recfiboeven(1000);
```

The approximate values of the sum $S(2n)$ obtained by this procedure are stated in Table 1. Computation of 21 pairs of the sums $S(2n)$ and $S(2n,10^3)$ took about 7 seconds. The absolute errors, i.e. the differences $|S(2n)-S(2n,10^3)|$, range between about 10^{-418} (for $n=1$) and 10^{-836} (for $n=1000$).

$S(2)$	$S(4)$	$S(6)$
1.000000	0.388889	0.148958
$S(8)$	$S(10)$	$S(12)$
$5.692046\cdot10^{-2}$	$2.174299\cdot10^{-2}$	$8.305156\cdot10^{-3}$
$S(14)$	$S(16)$	$S(18)$
$3.172291\cdot10^{-3}$	$1.211708\cdot10^{-3}$	$4.628312\cdot10^{-4}$
$S(20)$	$S(40)$	$S(60)$
$1.767858\cdot10^{-4}$	$1.168677\cdot10^{-8}$	$7.725769\cdot10^{-13}$
$S(80)$	$S(100)$	$S(120)$
$5.107271\cdot10^{-17}$	$3.376262\cdot10^{-21}$	$2.231944\cdot10^{-25}$
$S(140)$	$S(160)$	$S(180)$
$1.475470\cdot10^{-29}$	$9.753886\cdot10^{-34}$	$6.447998\cdot10^{-38}$
$S(200)$	$S(500)$	$S(1000)$
$4.262575\cdot10^{-42}$	$2.751439\cdot10^{-209}$	$2.830868\cdot10^{-418}$

Table 1: Some approximate values of the sums $S(2n)$ [source: own modeling in Maple 2020]

9 Conclusion

In this paper we dealt with the sum of the series of reciprocals of the product of two Fibonacci numbers whose indices differ by an even number, i.e. with the series

$$\sum_{k=1}^{\infty}\frac{1}{F_kF_{k+2n}},$$

where n is an arbitrary positive integer. We derived that the sum $S(2n)$ of this series is given by the formula

$$S(2n)=\sum_{k=1}^{\infty}\frac{1}{F_kF_{k+2n}} = \frac{1}{F_{2n}}\left[\frac{n}{F_1F_2}-\sum_{i=1}^{n-1}\left(F_{2i}\sum_{k=1}^{2}\frac{1}{F_kF_{k+2i}}\right)\right].$$

We verified this result by computing 20 sums using the computer algebra system Maple 2020. The series above so belong to special types of infinite series, such as geometric and telescoping series, whose sums are given analytically by means of a simple formula in closed form.

Area of Further Development

The results of the paper can be generalized and further extended to similar series of reciprocals of the expressions containing the Fibonacci numbers F_k or *Lucas numbers* L_k, Note that the Lucas numbers are defined by the analogous recurrence relation

$$L_k = L_{k-1}+L_{k-2}\,,\qquad k\ge 2\,,$$

as the Fibonacci numbers, but the first two Lucas numbers are

$$L_0=2\quad\text{and}\quad L_1=1\,.$$

Series and their sums that can be studied further can be, for example, series of the form

$$\sum_{k=1}^{\infty}\frac{1}{L_kL_{k+2n}},$$

where n is an arbitrary positive integer, or

$$\sum_{k=1}^{\infty}\frac{1}{F_kF_{k+2}F_{k+4}}\,.$$

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