Application of Bivariate Conditional Erlang Distribution to Sequential Medical Procedures

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Abstract: - This investigation applied Bivariate Conditional Erlang Distribution (BCED) to model two key sequential medical procedures: Time to Blood Glucose Monitoring (T1) and Time to Follow-Up Treatment (T2). The findings estimated T1 to have a shape parameter of 24.14 and a scale parameter of 0.642, while T2 had a shape parameter of 25.87 and a scale parameter of 0.847. The Goodness-of-fit tests using the Kolmogorov-Smirnov method yielded p-values of 0.9129 for T1 and 0.9462 for T2, confirming that the Erlang distribution adequately describes both processes. A strong positive correlation of 0.983 between T1 and T2, with a highly significant p-value of $3.9 \times 10^{-223.9}$, indicated a close relationship between the two procedures. These outcomes confirmed that, when the time for blood glucose monitoring increases, the time for follow-up treatment also rises proportionately. Visualizations using bivariate Gaussian kernel density estimates further reinforced these findings by demonstrating the concentration of data points around the joint mode of the two variables. This investigation illuminates the appropriateness of the BCED for modelling real-world dependent stochastic processes, particularly in healthcare where interrelated tasks like T1 and T2 can be adequately trapped. The analysis offers a solid foundation for using the Erlang distribution in healthcare procedure modelling and offers insights for improving time efficiency in sequential medical operations.

Key-Words: - Bivariate, Kolmogorov-Smirnov, Random Variable, Density Function, Distribution.

Received: March 19, 2024. Revised: October 14, 2024. Accepted: November 3, 2024. Published: December 20, 2024.

1 Introduction

Univariate distribution relationships, [8] delve into the intricate connections between Erlang distribution and other probability distributions, [5], [12]. Lawrence's work encompasses an investigation of Erlang's placement within the family of exponential distributions and its ties to the gamma distribution. Moreover, Saralees noted in 2005 that there have been limited proposals for bivariate gamma distributions within statistical literature [13].

The Erlang distribution, pioneered by Erlang, originally aimed to model the number of telephone calls received simultaneously by an operator at a switching station, [8]. This distribution holds significant relevance in telecommunications and queueing theory, [14]. Halliday in 1990 established the complementary relationship between the Erlang and Poisson distributions [6]. While the Poisson distribution counts events occurring within a fixed time frame, the Erlang distribution quantifies the time until a fixed number of events occur, [3]. Consequently, the Poisson, Exponential, Erlang, and Gamma distributions are closely intertwined, [11]. The gamma distribution widely employed in reliability analysis, queueing theory, and finance, shares mathematical properties with the Erlang distribution, [14], [1]. Specifically, the bivariate gamma distribution is notably linked to the Erlang distribution due to their shared underlying mathematical characteristics, [10]. As we embark on this study, our focus turns to the bivariate conditional Erlang distribution. Building upon the foundational research of Gongskin and Saporu [5] and Saralees and Arjun [13], we aim to unravel the complexities and implications of this bivariate conditional extension.

2 Problem Formulation

The method of extending family of distributions for added flexibility and potentiality is a familiar technique in the literature. In random phenomena, modeling and analyzing lifetime data are very essential in the fields of sciences and applied sciences such medicine, engineering, finance, economics, biomedical sciences, public health, among others. Several lifetime distributions have been used to analyze such kinds of data in practice, but the quality of the procedure used in statistical study depends on the assumed probability model [8].

2.1 Bivariate conditional Erlang distribution model

The Erlang distribution is a continuous probability distribution that is widely used in various fields such as queueing theory, telecommunications, and reliability engineering [2] [14] [7]. It is a special case of the Gamma distribution, specifically designed for modeling the sum of several exponential variables, where the shape parameter α is an integer ($\alpha = k$) and the rate parameter β is equal to λ.

The Gamma distribution's PDF is given by:

$$f(x;\alpha,\beta) = \frac{\beta^{\alpha} x^{\alpha-1} e^{-\beta x}}{\Gamma(\alpha)} \tag{1}$$

where $\Gamma(\alpha)$ is the Gamma function, which generalizes the factorial function to real and complex numbers.

In this article, we will explore the Erlang distribution and its connection to the exponential family of distributions. The Erlang distribution is parameterized by two values: k, a positive integer, and λ, a positive real number. The probability density function (PDF) for $x \geq 0$, $k \in Z^{+}$ (positive integers), and $\lambda > 0$ of an Erlang distributed random variable X is given by:

$$f(x;k,\lambda) = \frac{\lambda^{k} x^{k-1} e^{-\lambda x}}{(k-1)!} \tag{2}$$

Given a random variable Y that depends on another random variable X, where the distribution of Y given $X = x$ follows an Erlang distribution with scale parameter x and shape parameter λ, we can express the probability density function (PDF) of $Y|X = x$.

Recall that the PDF of an Erlang distributed random variable with shape parameter k (which is an integer) and rate parameter λ is given by:

$$f(y;k,\lambda) = \frac{\lambda^{k} y^{k-1} e^{-\lambda y}}{(k-1)!} \tag{3}$$

Note that x must be a positive integer for this distribution to be an Erlang distribution, as the shape parameter of the Erlang distribution is defined as a positive integer. In this scenario, since $Y|X = x$ follows an Erlang distribution with shape parameter x and rate parameter λ. Therefore; for $y \geq 0$, the probability density function of Y given $X = x$ is

$$f_{Y|X}(y \mid x) = \frac{\lambda^{x} y^{x-1} e^{-\lambda y}}{(x-1)!} \tag{4}$$

This PDF characterizes the conditional distribution of Y given the value of X, showing that Y follows an Erlang distribution whose shape parameter is determined by the realization of X and whose scale parameter is λ. To derive the joint probability density function known as the bivariate conditional Erlang distribution (BCED) for X and Y in terms of the marginal and conditional density functions, we can utilize the concept of conditional probability. Given that Y follows an Erlang distribution conditional on the realization of X, we can express the joint PDF of X and Y using the conditional PDF of Y given X and the marginal PDF of X.

Let $f_X(x)$ denote the marginal PDF of X, and $f_{Y|X}(y|x)$ denote the conditional PDF of Y given $X = x$. Then, the joint PDF of X and Y, denoted as $f_{XY}(x,y)$ can be derived using the conditional PDF and the marginal PDF as given in Equation (5) where,

$$f_{XY}(x,y) = f_{Y|X}(y|x) \cdot f_X(x),$$

and the BCED for $x > 0$, $y > 0$, and $\lambda > 0$ is then given by:

$$\int_0^\infty \int_0^\infty \frac{\lambda^{k+x} x^{k-1} y^{x-1} e^{-\lambda(x+y)}}{(k-1)! \cdot (x-1)!} \, dy \, dx \tag{5}$$

$$f_{X,Y}(x,y) = \frac{\lambda^{k+x} x^{k-1} y^{x-1} e^{-\lambda(x+y)}}{(k-1)! \cdot (x-1)!} \tag{6}$$

Theorem 1:
If $f(x,y)$ is a true bivariate probability density function for $x, y \in (0, \infty)$ then,

$$f_{X,Y}(x,y) = \frac{\lambda^{k+x} x^{k-1} y^{x-1} e^{-\lambda(x+y)}}{(k-1)! \cdot (x-1)!} \tag{7}$$

Proof:
Let

$$M = \int_0^\infty \int_0^\infty f(x,y) \, dy \, dx$$

$$M = \int_0^\infty \int_0^\infty \frac{(k-1)! \cdot (x-1)! \lambda^{k+x} x^{k-1} y^{x-1} e^{-\lambda(x+y)}}{\lambda^{k+x} x^{k-1} y^{x-1} e^{-\lambda(x+y)}} \, dy \, dx$$

$$= \int_0^\infty \int_0^\infty \frac{(k-1)! \cdot (x-1)!}{\lambda^{k+x} x^{k-1} y^{x-1}} e^{-\lambda x} e^{-\lambda y} \, dy \, dx$$

Since $e^{-\lambda(x+y)} = e^{-\lambda x} e^{-\lambda y}$, the integrand can be separated:

$$= \int_0^\infty \int_0^\infty \frac{(k-1)! \cdot (x-1)!}{\lambda^{k+x} x^{k-1} y^{x-1}} e^{-\lambda x} e^{-\lambda y} \, dy \, dx$$

$$= \int_0^\infty \left(\int_0^\infty \frac{(k-1)! \cdot (x-1)!}{\lambda^{k+x} x^{k-1} y^{x-1}} e^{-\lambda x} e^{-\lambda y} \, dy \right) dx$$

Evaluating the inner integral with respect to y, we have

$$= \int_0^\infty \frac{(k-1)! \cdot (x-1)!}{\lambda^{k+x} x^{k-1}} e^{-\lambda x} \left(\int_0^\infty y^{x-1} e^{-\lambda y} \, dy \right) dx$$

The inner integral is the Gamma function, $\Gamma(x)$ for $y^{x-1} e^{-\lambda y}$

$$\int_0^\infty y^{x-1} e^{-\lambda y} \, dy = \frac{\Gamma(x)}{\lambda^x}$$

Substitute the result back into the integral

$$\int_0^\infty \frac{\lambda^{k+x} x^{k-1} e^{-\lambda x}}{(k-1)! \cdot (x-1)!} \cdot \frac{\Gamma(x)}{\lambda^x} \, dx = \int_0^\infty \frac{\lambda^k \lambda^x x^{k-1} e^{-\lambda x} \Gamma(x)}{(k-1)! \cdot (x-1)! \lambda^x} dx$$

Since $\Gamma(x) = (x-1)!$, then

$$\int_0^\infty \frac{\lambda^k x^{k-1} e^{-\lambda x} (x-1)!}{(k-1)! \cdot (x-1)!} dx = \frac{\lambda^k}{(k-1)!} \int_0^\infty x^{k-1} e^{-\lambda x} \, dx$$

Again,

$$\int_0^\infty x^{k-1} e^{-\lambda x} \, dx = \frac{\Gamma(k)}{\lambda^k}$$

Therefore,

$$\frac{\lambda^k}{(k-1)!} \cdot \frac{\Gamma(k)}{\lambda^k} = \frac{\Gamma(k)}{(k-1)!}$$

Since $\Gamma(k) = (k-1)!$, the expression simplifies to

$$\frac{(k-1)!}{(k-1)!} = 1$$

Thus, the given integral evaluates to 1, which verifies the normalization condition for the bivariate conditional Erlang distribution is

$$M = \int_0^\infty \int_0^\infty \frac{\lambda^{k+x} x^{k-1} y^{x-1} e^{-\lambda(x+y)}}{(k-1)! \cdot (x-1)!} \, dy \, dx = 1$$

Since $M = 1$, it indicates that Equation (7) is a true bivariate probability density function and represented in Fig. 1 for different values of λ and k.

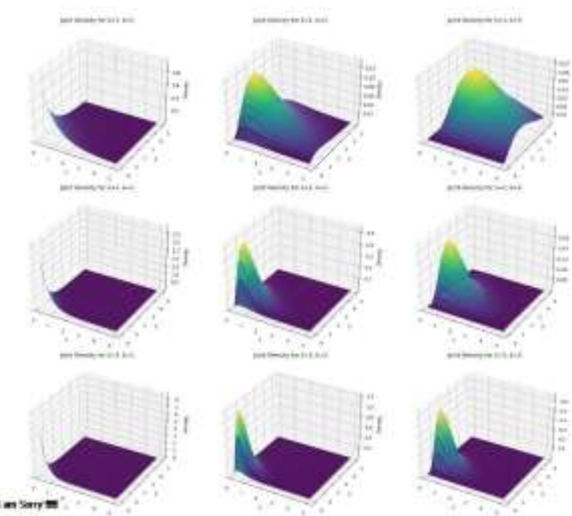

Fig 1: Density curves of the bivariate conditional Erlang distribution for various parameter values

2.2 Generalization

Let Erlang density defined as in Equation (3) and the conditional Erlang density function of $Y_i|X = x$, for $x \geq 0$ be given by

$$f_{Y_i|X=x}(x; y, \beta) = \frac{\beta^y x^{y-1} e^{-\beta x}}{(y-1)!}$$

Consider $p+1$ random variables $X, Y_1, \ldots, Y_p$ such that $Y_i|X = x$ and $Y_j|X = x$, $i, j = 1, 2, \ldots, p$,

$i \neq j$, are independent. It is easy to show that the joint density of $X, Y_1, \dots, Y_p$ is given by

$$f\left(x, y_1, \dots, y_p\right) = f\left(x\right)\prod_{i=1}^{p} f\left(y_i \middle| X = x\right) \quad (8)$$

2.3 Properties of BCED (Marginal distribution of Y)

Theorem 2:

The random variable Y in the joint density (6) has a marginal distribution, given by:

$$f_Y(y) = \int_0^{\infty} f_{Y|X}(y|x) f_X(x)\, dx$$

where X follows a Poisson distribution with parameter λ, i.e.,

$$f_X\left(x\right) = \frac{\lambda^x e^{-\lambda}}{x!}, \quad x = 0,1,2,\dots$$

$$f_Y\left(y\right) = \int_0^{\infty} f\left(x, y\right) dx$$

Proof:

$$\int_0^{\infty} \frac{\lambda^x y^{x-1} e^{-\lambda y}}{\left(x-1\right)!} \square \frac{\lambda^x e^{-\lambda}}{x!} dx = \sum_{x=1}^{\infty} \frac{\lambda^x y^{x-1} e^{-\lambda y}}{\left(x-1\right)!} \square \frac{\lambda^x e^{-\lambda}}{x!}$$

$$\Rightarrow \qquad f_Y\left(y\right) = \sum_{x=1}^{\infty} \frac{\lambda^x y^{x-1} e^{-\lambda y}}{\left(x-1\right)!} \square \frac{\lambda^x e^{-\lambda}}{x \cdot \left(x-1\right)!}$$

$$f_Y\left(y\right) = \sum_{x=1}^{\infty} \frac{\lambda^{2x} y^{x-1} e^{-\lambda y} e^{-\lambda x}}{\left(x \cdot \left(\left(x-1\right)\right)^2 !\right)}$$

$$= e^{-\lambda(1+y)} \sum_{x=1}^{\infty} \frac{\lambda^{2x} y^{x-1}}{x \cdot \left(\left(x-1\right)!\right)^2} \qquad (9)$$

Given that Y given X follows an Erlang distribution and X follows a Poisson distribution, we can identify that the marginal distribution of Y (which is the sum of X exponential random variables) follows a Gamma distribution $f_Y(y) = \lambda e^{-\lambda y}$, $y \geq 0$, which is the form of an exponential distribution with rate λ.

2.4 Moments of BCED (Joint Moment)

The (r, s)th joint moments for X and Y can be derived as follows:

$$E[X^r Y^s] = \int_0^{\infty}\int_0^{\infty} x^r y^s \frac{\lambda^{k+x} x^{k-1} y^{x-1} e^{-\lambda(x+y)}}{(k-1)! \cdot (x-1)!}\, dy\, dx$$

Simplifying

$$\frac{\lambda^{k+x} x^{k-1} y^{x-1} e^{-\lambda(x+y)}}{(k-1)! \cdot (x-1)!} = \frac{\lambda^k \lambda^x x^{k-1} y^{x-1} e^{-\lambda x} e^{-\lambda y}}{(k-1)!(x-1)!}$$

Combining the exponential terms, we have

$$\frac{\lambda^k \lambda^x x^{k-1} y^{x-1} e^{-\lambda(x+y)} \lambda x}{(k-1)!(x-1)!} = \frac{\lambda^k \lambda^x x^{k-1} y^{x-1} e^{-\lambda(x+y)} \lambda x}{(k-1)!(x-1)!}$$

Separating the integral gives

$$E[X^r Y^s] = \lambda^k \int_0^{\infty} x^r x^{k-1} e^{-\lambda x} \frac{\lambda^x}{(k-1)!(x-1)!} \left(\int_0^{\infty} y^s y^{x-1} e^{-\lambda y} dy\right) dx$$

Therefore,

$$\int_0^{\infty} y^{s+x-1} e^{-\lambda y} dy = \frac{\Gamma(s+x)}{\lambda^{s+x}}$$

Substitute this result becomes

$$E[X^r Y^s] = \lambda^k \int_0^{\infty} x^{r+k-1} e^{-\lambda x} \frac{\lambda^x}{(k-1)!(x-1)!} \frac{\Gamma(s+x)}{\lambda^{s+x}} dx$$

Simplifying further

$$E[X^r Y^s] = \lambda^k \int_0^{\infty} x^{r+k-1} e^{-\lambda x} \frac{\lambda^x \Gamma(s+x)}{(k-1)!(x-1)!\lambda^{s+x}}\, dx$$

$$= \frac{\lambda^k}{(k-1)!} \int_0^{\infty} x^{r+k-1} e^{-\lambda x} \frac{\Gamma(s+x)}{\lambda^s (x-1)!}\, dx$$

$$E[X^r Y^s] = \frac{\lambda^{k-s}}{(k-1)!} \qquad (10)$$

2.5 Marginal moments and covariance matrix

The marginal moments of X and Y can be derived from the joint moment in Equation (10) by appropriate substitution. When $s = 0$, we obtain the rth marginal moment of X given by

$$E[X^r] = \frac{\lambda^k \Gamma(r+k)}{(k-1)!} \qquad (11)$$

Similarly, when $r = 0$, we obtain the sth marginal moment of Y as

$$E[Y^s] = \frac{(k-1)!}{\lambda^{k-s}} \int_0^{\infty} (\lambda + t)^{s+x-1}\, dt \qquad (12)$$

while, the covariance between X and Y is defined by:

$$cov(X, Y) = E(XY) - E(X)E(Y)$$

From Equation (9),

$$E(XY) = \frac{e^{\lambda^2}}{\lambda^2}$$

Then,

$$cov(X, Y) = \frac{e^{\lambda^2}}{\lambda^2} - \frac{\lambda^k \Gamma(r+k)}{(k-1)!} \cdot \frac{(k-1)!}{\lambda^{k-s}} \int_0^{\infty} (\lambda + t)^{s+x-1}\, dt$$

$$= \frac{e^{\lambda^2}}{\lambda^2} - \frac{\lambda^{k+s}\Gamma(r+k)}{(s+x)(k-1)!}$$

$$\operatorname{var}(X) = E(X^2) - (E(X))^2$$

$$\operatorname{var}(X) = \frac{\lambda^{k+s-1}\Gamma(r+k)}{(k-1)!(s+x-1)} - \left(\frac{\lambda^k\Gamma(r+k)}{(k-1)!}\right)^2$$

$$\operatorname{var}(Y) = E(Y^2) - (E(Y))^2$$

$$\operatorname{var}(Y) = \frac{(k-1)!}{\lambda^{k-s}}\left(e^{\lambda} - 1 - \left(\int_0^{\infty}(\lambda+t)^{s+x-1}\,dt\right)^2\right.$$

Thus, a random vector $X = (X, Y)$ having the bivariate conditional Erlang distribution has the mean vector:

$$\mu = \begin{pmatrix} \dfrac{\lambda_k\Gamma(r+k)}{(k-1)!} \\ \dfrac{(k-1)!}{\lambda^{k-s}}\displaystyle\int_0^{\infty}(\lambda+t)^{s+x-1}\,dt \end{pmatrix} \tag{13}$$

And the variance–covariance matrix of X is given by:

$$\Sigma = \begin{pmatrix} \dfrac{\lambda^{k+s-1}\Gamma(r+k)}{(k-1)!(s+x-1)} - \left[\dfrac{\lambda^k\Gamma(r+k)}{(k-1)!}\right]^2 & \dfrac{e^{\lambda^2}}{\lambda^2} - \dfrac{\lambda^k\Gamma(r+k)}{(k-1)!}\square\dfrac{(k-1)}{\lambda^{k-s}}\displaystyle\int_0^{\infty}(\lambda+t)^{s+x-1}\,dt \\ \dfrac{e^{\lambda^2}}{\lambda^2} - \dfrac{\lambda^k\Gamma(r+k)}{(k-1)!}\square\dfrac{(k-1)!}{\lambda^{k-s}}\displaystyle\int_0^{\infty}(\lambda+t)^{s+x-1}\,dt & \dfrac{(k-1)!}{\lambda^{k-s}}\left[e^{\lambda} - 1 - \left(\displaystyle\int_0^{\infty}(\lambda+t)^{s+x-1}\,dt\right)^2\right] \end{pmatrix} \tag{14}$$

The correlation between X and Y is defined by

$$\rho_{XY} = \frac{\dfrac{e^{\lambda^2}}{\lambda^2} - \dfrac{\Gamma(r+k)}{s+x}}{\sqrt{\dfrac{\lambda^{k+s-1}\Gamma(r+k)}{(k-1)!(s+x-1)} - \left(\dfrac{\lambda^k\Gamma(r+k)}{(k-1)!}\right)^2}\sqrt{\dfrac{(k-1)!}{\lambda^{k-s}}\left(e^{\lambda} - 1 - \left(\int_0^{\infty}(\lambda+t)^{s+x-1}\,dt\right)^2\right)}} \tag{15}$$

2.5 **Method of Parameter Estimation**

The maximum likelihood method of estimation of the parameter of BCED can be obtained as follow:

$$L(\lambda; x, y) = \sum_{i=1}^{n}\Big((k+x_i)\log(\lambda) + (k-1)\log(x_i) + (x_i-1)\log(y_i) - \lambda(x_i+y_i) - \log((k-1)!) - \log((x_i-1)!)\Big) \tag{16}$$

The derivatives of the log likelihood function (16) with respect to λ is given below

$$\frac{\partial L(\lambda; x, y)}{\partial\lambda} = \sum_{i=1}^{n}\left(\frac{k+x_i}{\lambda} - (x_i+y_i)\right) \tag{17}$$

Set the Derivative to Zero and Solve for λ

$$\sum_{i=1}^{n}\left(\frac{k+x_i}{\lambda} - (x_i+y_i)\right) = 0$$

$$\sum_{i=1}^{n}\frac{k+x_i}{\lambda} = \sum_{i=1}^{n}(x_i+y_i)$$

$$\sum_{i=1}\frac{k+x_i}{\lambda} = \sum_{i=1}(x_i+y_i)$$

$$\frac{n(k+\bar{x})}{\lambda} = n(\bar{x}+\bar{y})$$

where

$$x = \frac{1}{n}\sum_{i=1}^{n}x_i$$

and

$$y - \frac{1}{n}\sum_{i=1}^{n}y_i$$

Thus, the maximum likelihood estimate for λ is:

$$\hat{\lambda} = \frac{x+y}{k+\bar{x}} \tag{18}$$

3 Problem Solution

In this study, the time required for two sequential medical procedures in diabetes management, modeled using a bivariate Erlang distribution was analyzed. The dataset comprises measurement of the time to complete blood glucose monitoring (T1) and the time to finalize follow-up treatment based on monitoring results (T2).

The estimated Erlang distribution parameters for the Time to Blood Glucose Monitoring (T1) were a shape parameter of 24.14 and a scale parameter of 0.642. While, for the Time to Follow-Up Treatment (T2), the parameters were a shape parameter of 25.87 and a scale parameter of 0.847.

The Kolmogorov-Smirnov tests for both T1 and T2 produced high p-values (0.9129 for T1 and 0.9462 for T2). These results suggested that the Erlang model provides a good fit for both procedures, capturing their distributional characteristics effectively.

A strong positive correlation of 0.983 between T1 and T2 with a highly significant p-value of 3.9×10^{-22} also indicates that the times for these two sequential medical procedures are closely related. The plots in Figs. 2, 3, 4 and 5 strengthens the statistical outcomes. The contour lines in Fig. 2 represent the bivariate Gaussian kernel density estimate (KDE).

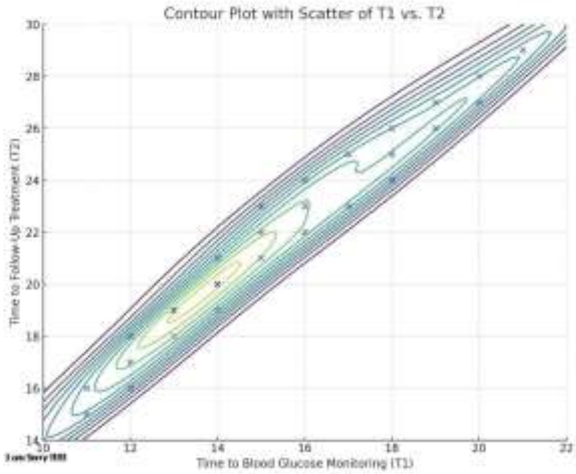

Fig 2: The contour plot overlaid with a scatter plot of T1 (Time to Blood Glucose Monitoring) versus T2 (Time to Follow-Up Treatment).

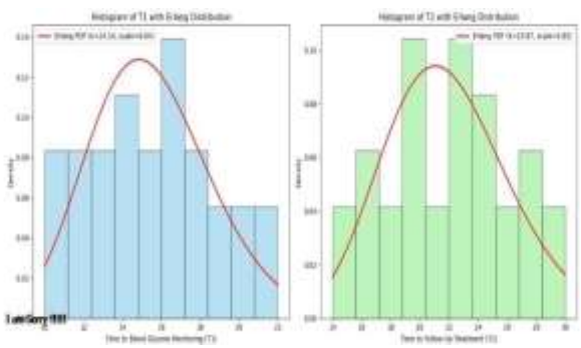

Fig 3: The histograms for T1 (Time to Blood Glucose Monitoring) and T2 (Time to Follow-Up Treatment), each overlaid with the fitted Gamma (Erlang) distribution curve.

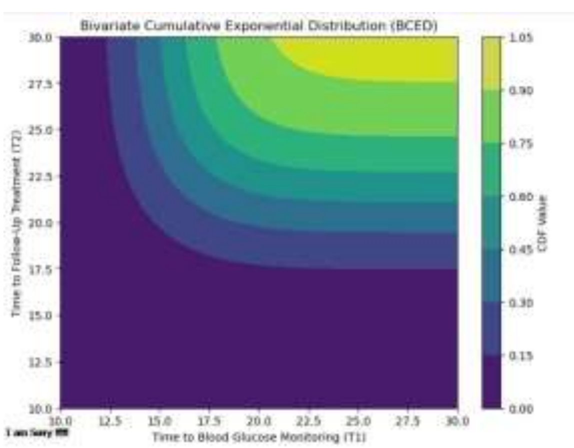

Fig 4: Bivariate Cumulative Exponential Distribution (BCED) Plot for T1 and T2.

The BCED plot of Fig 4, typically show a smooth gradient from lower to higher cumulative probabilities as both T1 and T2 increase.

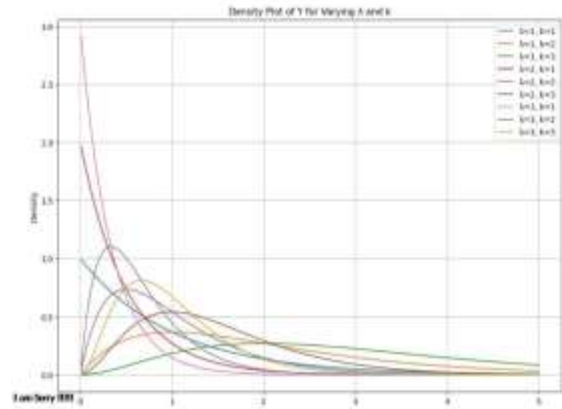

Fig 5: Density plot of Y for varying λ and k.

The results of numerical integration of the area under curve of the marginal distribution y are given as publicized in Table 1 to Table 4.

Table 1. The results of numerical integration of the area under curve of the marginal distribution y for Fig. 1

Parameter value			
a	b	c	Numerical Integration
0.5	0.5	0.5	0.999926060672114
1	1	1	1.000000000002793
1.5	1.5	1.5	1.000000000004236
2.5	2.5	2.5	1.000000000006227
3.5	3.5	3.5	1.000000000179541
4.5	4.5	4.5	1.000000000237848
5.5	5.5	5.5	1.000000000007505
7.5	7.5	7.5	0.999999999989926

Table 2. The results of numerical integration of the area under curve of the marginal distribution y for Fig. 2.

Parameter value			
a	b	c	Numerical Integration
0.5	0.5	0.5	0.999977140718329
1	1	1	1.0000000000007145
1.5	1.5	1.5	1.0000000000000645
2.5	2.5	2.5	1.0000000000008510
3.5	3.5	3.5	1.0000000000015686

4.5	4.5	4.5	1.0000000000045356
5.5	5.5	5.5	0.9999999999967937
7.5	7.5	7.5	0.9999999999878894

Table 3. The results of numerical integration of the area under curve of the marginal distribution y for Fig. 3.

Parameter value			
a	b	c	Numerical Integration
0.5	0.5	0.5	1.0000000000019935
1	1	1	1.0000000000006379
1.5	1.5	1.5	1.0000000000003677
2.5	2.5	2.5	1.0000000000004343
3.5	3.5	3.5	1.0000000000288260
4.5	4.5	4.5	0.9999999999961643
5.5	5.5	5.5	0.9999999999979663
7.5	7.5	7.5	1.0000000000012034

Table 4. The results of numerical integration of the area under curve of the marginal distribution y for Fig. 4.

Parameter value			
a	b	c	Numerical Integration
0.5	0.5	0.5	0.9999999999122686
1	1	1	1.0000000000016196
1.5	1.5	1.5	0.9999999999999252
2.5	2.5	2.5	1.0000000000027870
3.5	3.5	3.5	1.0000000000003789
4.5	4.5	4.5	1.0000000000046242
5.5	5.5	5.5	1.0000000000047694
7.5	7.5	7.5	0.9999999999894141

4 Conclusion

In this paper, Bivariate Conditional Erlang Distribution was introduced with properties such as moments, generating functions, quantile function, random number generation, and other statistics, were extensively considered and analyzed. The analysis of the time to blood glucose monitoring (T1) and the time to follow-up treatment (T2) was conducted using the bivariate Erlang distribution, demonstrating a significant relationship between these two sequential medical procedures. Through parameter estimation and goodness-of-fit tests, the individual distributions of T1 and T2 were confirmed to align well with the Erlang distribution, as evidenced by the Kolmogorov-Smirnov test results. The high Pearson correlation coefficient further supported the positive dependency between T1 and T2, indicating that as the time for one procedure increases, the other is likely to follow suit. Visualizations, including scatter plots and BCED contour plots, provided a clear depiction of this relationship, which showcase the joint cumulative probabilities of the two times. The marginal density plots of T1 and T2 confirmed the consistency of the data with the assumed distribution, justifying the use of the bivariate Erlang model in this context.

Acknowledgement:
It is an optional section where the authors may write a short text on what should be acknowledged regarding their manuscript.

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Contribution of Individual Authors to the Creation of a Scientific Article (Ghostwriting Policy)

Toluwani John Dare carried out the data acquisition, interpretation, simulation, and safeguards the validity, creativity and reliability of the article content.

Olubunmi Temitope Olorunpomi organized the manuscript for publication, making sure the manuscript is logically structured, with coherent arguments, a smooth flow of ideas and following the specific formatting and submission guidelines provided by the journal.

Anselm Onyekachukwu Oyem is the corresponding author and carried out the proofreading of the text and ensured that the editorial policies on authorship and principles of publishing ethic are implemented.

Sources of Funding for Research Presented in a Scientific Article or Scientific Article Itself

No funding was received for conducting this study.

Conflict of Interest

The authors have no conflicts of interest to declare that are relevant to the content of this article.