Stick motions and grazing flows in a double belt friction oscillator

GE CHEN
Shandong Normal University
School of Mathematical Sciences
Ji'nan, 250014
PR CHINA

JINJUN FAN*
Shandong Normal University
School of Mathematical Sciences
Ji'nan, 250014
PR CHINA

Abstract: This paper is concerned with the dynamics of a double belt friction oscillator which is subjected to periodic excitation, linear spring-loading, damping force and two friction forces using the flow switchability theory of the discontinuous dynamical systems. Different domains and boundaries for such system are defined according to the friction discontinuity, which exhibits multiple discontinuous boundaries in the phase space. Based on the above domains and boundaries, the analytical conditions of the stick motions and grazing motions are obtained mathematically. There are more theories about such friction oscillators to be discussed in future.

Key–Words: double belt friction oscillator; discontinuous dynamical system; switchability; stick motion; grazing motion

1 Introduction

Discontinuous dynamical systems exist widely in the real word, especially in mechanical engineering. In mechanical engineering, most of the dynamical systems are discontinuous. This is because the dynamical systems in mechanical engineering are constrained by engineering requirements and limitations. The traditional theory of continuous dynamical systems can not be applied to discontinuous dynamical systems and only makes it more complicated and difficult to be solved. Therefore, a theory applicable to discontinuous dynamical systems should be built.

The early study of discontinuous dynamical systems goes back to Den Hartog [1] in 1931. Den Hartog considered a forced oscillator with Coulomb and viscous damping. In 1960, Levitan [2] investigated a friction oscillator with the periodically driven base, and also discussed the stability of the periodic motion. In 1966, Masri and Caughey [3] discussed a discontinuous impact damper, and obtained the stability of the symmetrical period-1 motion of the impact damper. More detailed discussions on the general motion of impact dampers were also developed in Masri [4]. In 1976, Utkin [5] first controlled dynamical system through the discontinuity, this method is called sliding mode control. Utkin [6] applied the sliding mode control in variable structure systems, and more detailed theory of this method was also developed in [7] by Utkin. In 1986, Shaw [8] investigated the non-stick periodic motion of a dry-friction oscillator, and discussed the stability of this motion through the Poincare mapping. In 1988, Filippov [9] investigated the dynamic behaviors of a Coulomb friction oscillator and developed differential equations with discontinuous right-hand sides. The analytical conditions of sliding motion along the discontinuous boundary were developed through differential inclusion, and the existence and uniqueness of the solution were also discussed. Leine etal. [10] investigated the stick-slip vibration induced by an alternate friction models through the shooting method in 1998. In 1999, Galvanetto and Bishop [11] discussed dynamics of a simple dynamical system subjected to an elastic restoring force, viscous damping and dry friction forces and studied the non-standard bifurcations with analytical and numerical tools. Pilipchuk and Tan [12] studied the friction induced vibration of a two-degree-of-freedom friction oscillator in 2004. In 2005, Casini and Vestroni [13] investigated dynamics of two double-belt friction oscillators by means of analytical and numerical tools.

However, the dynamical behaviors of discontinuous dynamical system is stilled difficult to investigate. In 2005, Luo [14] developed a general theory to study discontinuous dynamical systems on connectable domains. Luo [15] introduced the imaginary, sink and source flows, and also developed the sufficient and necessary conditions of sink and source flows. More detailed definitions and theorems can be referred to Luo [16]. In 2008, Luo [17] defined G-functions and developed a theory to determine the flow switchability to the discontinuous boundary through G-functions. The detailed discussion can be referred to Luo [18]. Based on this theory, lots of discontinuous models can be investigated easily, for example $[19-25]$.

In this paper, analytical conditions for stick, non-stick and grazing motions of the double-belt friction oscillator will be developed using the flow switchability theory of the discontinuous dynamical systems. Different domains and boundaries for such system are defined according to the friction discontinuity, which exhibits multiple discontinuous boundaries in the phase space. Based on the above domains and boundaries, the analytical conditions of the stick motions and grazing motions are obtained mathematically. The switching plans and basic mappings will be defined to study grazing motions.

2 Physical Model

Consider a periodically forced oscillator, attached to a fixed wall, as shown in Fig. 1. This friction-induced oscillator includes a mass *m*, a spring of s-tiffness *k* and a damper of viscous damping coefficient *c*. In this configuration, the mass *m* is continuously in contact with both belts which are pushed onto the mass with a constant forced F_N and possess the same friction characteristics. The periodic driving force $A_0 + B_0 \cos \Omega t$ exerts on the mass, where A_0 , B_0 and Ω are the constant force, excitation strength and frequency ratio, respectively.

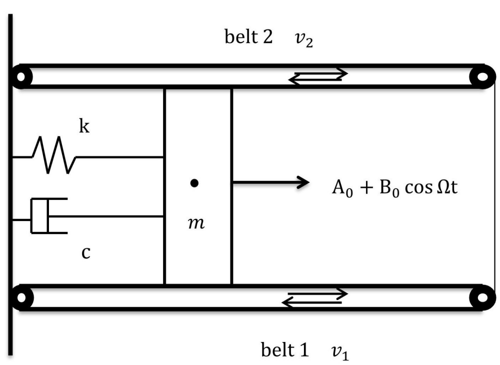

Figure 1: Physical model

Since the mass contacts the moving belts with friction, the mass can move along or rest on the belt 1 or belt 2 surface. Further, a kinetic friction force shown in Fig. 2 is described as

$$F_f(\dot{x}) \begin{cases} = (\mu_1 + \mu_2)F_N, & \dot{x} \in [v_2, +\infty), \\ \in [(\mu_1 - \mu_2)F_N, (\mu_1 + \mu_2)F_N], & \dot{x} = v_2, \\ = (\mu_1 - \mu_2)F_N, & \dot{x} \in [v_1, v_2], \\ \in [-(\mu_1 + \mu_2)F_N, (\mu_1 - \mu_2)F_N], & \dot{x} = v_1, \\ = -(\mu_1 + \mu_2)F_N, & \dot{x} \in (-\infty, v_1], \end{cases} \tag{1}$$

where $\dot{x} := dx/dt$,F_N and $\mu_k (k = 1, 2)$ are a normal force to the contact surface and friction coefficients between the mass *m* and the belt k $(k = 1, 2)$, respectively. Here we assume that $v_2 > v_1$ and $\mu_1 \geq \mu_2$.

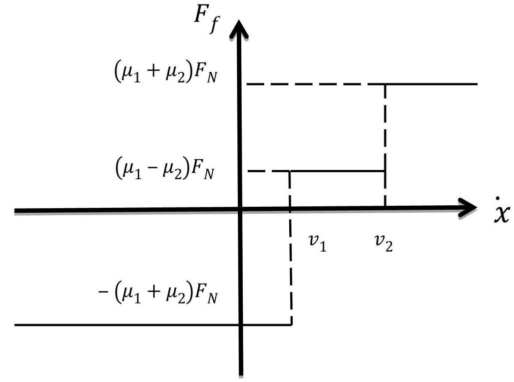

Figure 2: Friction force

The motions of the mass in a double-belt friction oscillator can be divided into two cases. If the mass moves along belt 1 and belt 2, the corresponding motion is called the non-stick motion. If the mass moves together with belt 1 or belt 2, the corresponding motion is called the stick motion.

For the mass moving with the same speed of the belt 1 surface, the force acting on the mass in the x-direction is defined as

$$F_{s1} = A_0 + B_0 \cos \Omega t - kx - c\dot{x} + \mu_2 F_N \text{ for } \dot{x} = v_1. \tag{2}$$

If this force cannot overcome the friction force $\mu_1 F_N$ (i.e., $|F_{s1}| \leq \mu_1 F_N$), the mass does not have any relative motion to the belt 1. The equation of the motion for the mass in such state is described as

$$\dot{x} = v_1, \qquad \ddot{x} = 0. \tag{3}$$

For the mass moving with the same speed of the belt 2 surface, we can also obtain the equation for the mass as follows

$$\dot{x} = v_2, \qquad \ddot{x} = 0. \tag{4}$$

For the non-stick motions of the friction-induced oscillator, we can obtain the equations of the motions as follows

$$\begin{cases} m\ddot{x} = A_0 + B_0 \cos \Omega t - kx - c\dot{x} + (\mu_1 + \mu_2)F_N \\ \qquad\qquad\qquad\qquad \text{for } \dot{x} < v_1, \\ m\ddot{x} = A_0 + B_0 \cos \Omega t - kx - c\dot{x} - (\mu_1 - \mu_2)F_N \\ \qquad\qquad\qquad\qquad \text{for } v_1 < \dot{x} < v_2, \\ m\ddot{x} = A_0 + B_0 \cos \Omega t - kx - c\dot{x} - (\mu_1 + \mu_2)F_N \\ \qquad\qquad\qquad\qquad \text{for } \dot{x} > v_2. \end{cases} \tag{5}$$

3 Domains and Boundaries

From the previous discussion, there are five motion states including three non-stick motions in the three regions and two stick motions on the boundaries. The phase plane can be partitioned into three domains and two boundaries, as shown in Fig. 3. In each domain, the motion can be described through a continuous dynamical system.

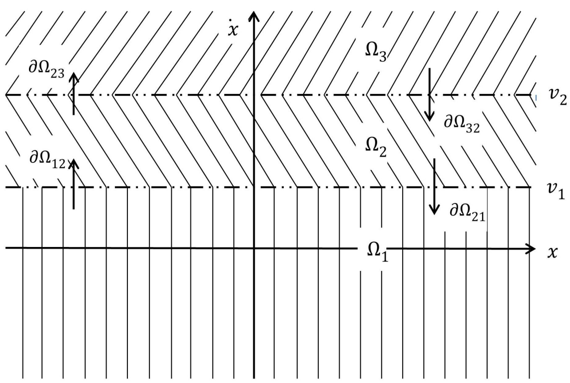

Figure 3: Domains and boundaries

The three domains are expressed by $\Omega_\alpha(\alpha = 1, 2, 3)$:

$$\begin{aligned}
\Omega_1 &= \Big\{(x,\dot{x})\,|\,x \in (-\infty,+\infty),\ \dot{x} \in (-\infty, v_1)\Big\},\\
\Omega_2 &= \Big\{(x,\dot{x})\,|\,x \in (-\infty,+\infty),\ \dot{x} \in (v_1, v_2)\Big\},\\
\Omega_3 &= \Big\{(x,\dot{x})\,|\,x \in (-\infty,+\infty),\ \dot{x} \in (v_2, +\infty)\Big\}.
\end{aligned} \tag{6}$$

The corresponding boundaries are defined as:

$$\begin{aligned}
\partial\Omega_{12} = \partial\Omega_{21} &= \Big\{(x,\dot{x})\,|\,x \in (-\infty,+\infty),\ \dot{x} = v_1\Big\},\\
\partial\Omega_{23} = \partial\Omega_{32} &= \Big\{(x,\dot{x})\,|\,x \in (-\infty,+\infty),\ \dot{x} = v_2\Big\}.
\end{aligned} \tag{7}$$

Based on the above domains and boundaries, the vectors for motions of the mass in the domains can be introduced as follows

$$\mathbf{x}_{(\lambda)} = (x_{(\lambda)}, \dot{x}_{(\lambda)})^{\mathrm{T}}, \qquad \mathbf{F}_{(\lambda)} = (\dot{x}_{(\lambda)}, F_{(\lambda)})^{\mathrm{T}}, \tag{8}$$

where $\lambda = 1, 2, 3$ and

$$\begin{aligned}
F_{(1)}(\mathbf{x}_{(1)}, t) &= -\frac{c}{m}\dot{x}_{(1)} - \frac{k}{m}x_{(1)} + \frac{B_0}{m}\cos\Omega t\\
&\quad + \frac{1}{m}[A_0 + (\mu_1 + \mu_2)F_N],\\
F_{(2)}(\mathbf{x}_{(2)}, t) &= -\frac{c}{m}\dot{x}_{(2)} - \frac{k}{m}x_{(2)} + \frac{B_0}{m}\cos\Omega t\\
&\quad + \frac{1}{m}[A_0 - (\mu_1 - \mu_2)F_N],\\
F_{(3)}(\mathbf{x}_{(3)}, t) &= -\frac{c}{m}\dot{x}_{(3)} - \frac{k}{m}x_{(3)} + \frac{B_0}{m}\cos\Omega t\\
&\quad + \frac{1}{m}[A_0 - (\mu_1 + \mu_2)F_N].
\end{aligned} \tag{9}$$

From Eq. (5), the equations of the non-stick motions for the mass are rewritten in the vector form of

$$\dot{\mathbf{x}}_{(\lambda)} = \mathbf{F}_{(\lambda)}(\mathbf{x}_{(\lambda)}, t) \quad \text{for} \quad \lambda \in \{1, 2, 3\}. \tag{10}$$

For the stick motion, the equations of the motion for the mass are rewritten in the vector form of

$$\dot{\mathbf{x}}^{(0)}_{(\lambda)} = \mathbf{F}^{(0)}_{(\lambda)}(\mathbf{x}_{(\lambda)}, t) \quad \text{for} \quad \lambda \in \{1, 2\} \tag{11}$$

and

$$F^{(0)}_{(\lambda)}(\mathbf{x}^{(0)}_{(\lambda)}, t) = 0, \tag{12}$$

where

$$\mathbf{x}^{(0)}_{(\lambda)} = (x^{(0)}_{(\lambda)}, \dot{x}^{(0)}_{(\lambda)})^{\mathrm{T}}, \qquad \mathbf{F}^{(0)}_{(\lambda)} = (v_\lambda, F^{(0)}_{(\lambda)})^{\mathrm{T}}.$$

4 Analytical Conditions

By the theory of the flow switchability to a specific boundary in discontinuous dynamical system in [17], the switching conditions of the passability, stick motions and grazing flows of the double-belt friction oscillator will be developed in this section.

For convenience, we first introduce some concepts and several lemmas in flow switching theory.

Consider a discontinuous dynamical system

$$\dot{\mathbf{x}}^{(\alpha)} \equiv \mathbf{F}^{(\alpha)}(\mathbf{x}^{(\alpha)}, t, \mathbf{P}_\alpha) \in R^n \tag{13}$$

in domain $\Omega_\alpha(\alpha = i, j)$ which has a flow $\mathbf{x}^{(\alpha)}_t = \mathbf{\Phi}(t_0, \mathbf{x}^{(\alpha)}_0, \mathbf{P}_\alpha, t)$ with an initial condition $(t_0, \mathbf{x}^{(\alpha)}_0)$, and on the boundary

$$\begin{aligned}
\partial\Omega_{ij} = \Big\{ &\mathbf{x} \mid \varphi_{ij}(\mathbf{x}, t, \lambda) = 0,\\
&\varphi_{ij} \text{ is } C^r-\text{continuous } (r \geq 1)\Big\} \subset R^{n-1},
\end{aligned} \tag{14}$$

there is a flow $\mathbf{x}^{(0)}_t = \mathbf{\Phi}(t_0, \mathbf{x}^{(0)}_0, \lambda, t)$ with an initial condition $(t_0, \mathbf{x}^{(0)}_0)$. The 0-order G-functions of the flow $\mathbf{x}^{(\alpha)}_t$ to the flow $\mathbf{x}^{(0)}_t$ on the boundary in the normal direction of the boundary $\partial\Omega_{ij}$ are defined as

$$G^{(\alpha)}_{\partial\Omega_{ij}}(\mathbf{x}^{(0)}_t, t_\pm, \mathbf{x}^{(\alpha)}_{t_\pm}, \mathbf{P}_\alpha, \lambda)$$

$$\begin{aligned}&\equiv G^{(0,\alpha)}_{\partial\Omega_{ij}}(\mathbf{x}^{(0)}_t, t_\pm, \mathbf{x}^{(\alpha)}_{t_\pm}, \mathbf{P}_\alpha, \lambda)\\ &= D_{\mathbf{x}^{(0)}_t}{}^t\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}}\cdot(\mathbf{x}^{(\alpha)}_{t_\pm}-\mathbf{x}^{(0)}_t)\\ &+{}^t\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}}\cdot(\dot{\mathbf{x}}^{(\alpha)}_{t_\pm}-\dot{\mathbf{x}}^{(0)}_t).\end{aligned} \tag{15}$$

The 1-order G-functions for a flow $\mathbf{x}^{(\alpha)}_t$ to a boundary flow $\mathbf{x}^{(0)}_t$ in the normal direction of the boundary $\partial\Omega_{ij}$ are defined as

$$\begin{aligned}&G^{(1,\alpha)}_{\partial\Omega_{ij}}(\mathbf{x}^{(0)}_t, t^{(\alpha)}_\pm, \mathbf{x}^{(\alpha)}_{t_\pm}, \mathbf{P}_\alpha, \lambda)\\ &= D^2_{\mathbf{x}^{(0)}_t}{}^t\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}}\cdot(\mathbf{x}^{(\alpha)}_{t_\pm}-\mathbf{x}^{(0)}_t)\\ &+2D_{\mathbf{x}^{(0)}_t}{}^t\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}}\cdot(\dot{\mathbf{x}}^{(\alpha)}_{t_\pm}-\dot{\mathbf{x}}^{(0)}_t)\\ &+{}^t\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}}\cdot(\ddot{\mathbf{x}}^{(\alpha)}_{t_\pm}-\ddot{\mathbf{x}}^{(0)}_t),\end{aligned} \tag{16}$$

where the total derivative

$$D_{\mathbf{x}^{(0)}_t}(\cdot) := \frac{\partial(\cdot)}{\partial\mathbf{x}^{(0)}_t}\cdot\dot{\mathbf{x}}^{(0)}_t + \frac{\partial(\cdot)}{\partial t},$$

the normal vector of the boundary surface $\partial\Omega_{ij}$ at point $\mathbf{x}^{(0)}(t)$ is given by

$$\begin{aligned}&{}^t\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}}(\mathbf{x}^{(0)}, t, \lambda) = \bigtriangledown\varphi_{ij}(\mathbf{x}^{(0)}, t, \lambda)\\ &= (\tfrac{\partial\varphi_{ij}}{\partial x^{(0)}_1}, \tfrac{\partial\varphi_{ij}}{\partial x^{(0)}_2}, \cdots, \tfrac{\partial\varphi_{ij}}{\partial x^{(0)}_n})^{\mathrm{T}}_{(t,\,\mathbf{x}^{(0)})},\end{aligned} \tag{17}$$

and $t_\pm = t \pm 0$.

If the flow $\mathbf{x}^{(\alpha)}_t$ contacts with the boundary at the time t_m, that is $\mathbf{x}^{(\alpha)}_{t_m} = \mathbf{x}_m = \mathbf{x}^{(0)}_{t_m}$, and the boundary $\partial\Omega_{ij}$ is linear, independent of time t, we have

$$\begin{aligned}&G^{(0,\alpha)}_{\partial\Omega_{ij}}(\mathbf{x}_m, t_m, \mathbf{P}_\alpha, \lambda)\\ &:= G^{(0,\alpha)}_{\partial\Omega_{ij}}(\mathbf{x}^{(0)}_{t_m}, t_{m\pm}, \mathbf{x}^{(\alpha)}_{t_{m\pm}}, \mathbf{P}_\alpha, \lambda)\\ &= {}^t\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}}\cdot\dot{\mathbf{x}}^{(\alpha)}_t\Big|_{(\mathbf{x}_m, t_{m\pm})},\end{aligned} \tag{18}$$

$$\begin{aligned}&G^{(1,\alpha)}_{\partial\Omega_{ij}}(\mathbf{x}_m, t_m, \mathbf{P}_\alpha, \lambda)\\ &:= G^{(1,\alpha)}_{\partial\Omega_{ij}}(\mathbf{x}^{(0)}_{t_m}, t_{m\pm}, \mathbf{x}^{(\alpha)}_{t_{m\pm}}, \mathbf{P}_\alpha, \lambda)\\ &= {}^t\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}}\cdot\ddot{\mathbf{x}}^{(\alpha)}_t\Big|_{(\mathbf{x}_m, t_{m\pm})}.\end{aligned} \tag{19}$$

Here t_{m+} and t_{m-} are the time before approaching and after departing the corresponding boundary, respectively.

Lemma 1 [17] *For a discontinuous dynamical system $\dot{\boldsymbol{x}}^{(\alpha)} = \boldsymbol{F}^{(\alpha)}(\boldsymbol{x}^{(\alpha)}, t, \boldsymbol{P}_\alpha) \in R^n$, $\boldsymbol{x}(t_m) = \boldsymbol{x}_m \in \partial\Omega_{ij}$ at time t_m. For an arbitrarily small $\varepsilon > 0$, there is a time interval $[\,t_{m-\varepsilon}, t_m\,)$. Suppose $\boldsymbol{x}^{(i)}(t_{m-}) = \boldsymbol{x}_m = \boldsymbol{x}^{(j)}(t_{m-})$. Both flows $\boldsymbol{x}^{(i)}(t)$ and $\boldsymbol{x}^{(j)}(t)$ are $C^r_{[t_{m-\varepsilon}, t_m)}$-continuous $(r \geq 1)$ for time t, and $\|d^{r+1}\boldsymbol{x}^{(\alpha)}/dt^{r+1}\| < \infty$ $(\alpha \in \{i,j\})$. The necessary and sufficient conditions for a sliding motion on $\partial\Omega_{\alpha\beta}$ are*

$$\left.\begin{aligned}&G^{(0,\alpha)}_{\partial\Omega_{ij}}(\boldsymbol{x}_m, t_{m-}, \boldsymbol{P}_\alpha, \lambda) < 0\\ &G^{(0,\beta)}_{\partial\Omega_{ij}}(\boldsymbol{x}_m, t_{m-}, \boldsymbol{P}_\beta, \lambda) > 0\end{aligned}\right\}\ for \quad \boldsymbol{n}_{\partial\Omega_{\alpha\beta}} \to \Omega_\alpha, \tag{20}$$

where $\alpha, \beta \in \{i,j\}$ and $\alpha \neq \beta$.

Lemma 2 [17] *For a discontinuous dynamical system $\dot{\boldsymbol{x}}^{(\alpha)} = \boldsymbol{F}^{(\alpha)}(\boldsymbol{x}^{(\alpha)}, t, \boldsymbol{P}_\alpha) \in R^n$, $\boldsymbol{x}(t_m) = \boldsymbol{x}_m \in \partial\Omega_{ij}$ at time t_m. For an arbitrarily small $\varepsilon > 0$, there are two time intervals $[\,t_{m-\varepsilon}, t_m\,)$ and $(\,t_m, t_{m+\varepsilon}\,]$. Suppose $\boldsymbol{x}^{(i)}(t_{m-}) = \boldsymbol{x}_m = \boldsymbol{x}^{(j)}(t_{m+})$. Both flows $\boldsymbol{x}^{(i)}(t)$ and $\boldsymbol{x}^{(j)}(t)$ are $C^r_{[t_{m-\varepsilon}, t_m)}$ and $C^r_{(t_m, t_{m+\varepsilon}]}$-continuous $(r \geq 1)$ for time t, respectively, and $\|d^{r+1}\boldsymbol{x}^{(\alpha)}/dt^{r+1}\| < \infty$ $(\alpha \in \{i,j\}$. The flow $\boldsymbol{x}^{(i)}(t)$ and $\boldsymbol{x}^{(j)}(t)$ to the boundary $\partial\Omega_{ij}$ is semi-passable from domain Ω_i to Ω_j iff*

$$either\ \left.\begin{aligned}&G^{(0,i)}_{\partial\Omega_{ij}}(\boldsymbol{x}_m, t_{m-}, \boldsymbol{P}_i, \lambda) > 0\\ &G^{(0,j)}_{\partial\Omega_{ij}}(\boldsymbol{x}_m, t_{m+}, \boldsymbol{P}_j, \lambda) > 0\end{aligned}\right\} for\ \boldsymbol{n}_{\partial\Omega_{\alpha\beta}} \to \Omega_j, \tag{21}$$

$$or\ \left.\begin{aligned}&G^{(0,i)}_{\partial\Omega_{ij}}(\boldsymbol{x}_m, t_{m-}, \boldsymbol{P}_i, \lambda) < 0\\ &G^{(0,j)}_{\partial\Omega_{ij}}(\boldsymbol{x}_m, t_{m+}, \boldsymbol{P}_j, \lambda) < 0\end{aligned}\right\} for\ \boldsymbol{n}_{\partial\Omega_{\alpha\beta}} \to \Omega_i. \tag{22}$$

Lemma 3 [17] *For a discontinuous dynamical system $\dot{\boldsymbol{x}}^{(\alpha)} = \boldsymbol{F}^{(\alpha)}(\boldsymbol{x}^{(\alpha)}, t, \boldsymbol{P}_\alpha) \in R^n$, $\boldsymbol{x}(t_m) = \boldsymbol{x}_m \in \partial\Omega_{ij}$ at time t_m. For an arbitrarily small $\varepsilon > 0$, there is a time interval $[\,t_{m-\varepsilon}, t_{m+\varepsilon}\,]$. Suppose $\boldsymbol{x}^{(\alpha)}(t_{m\pm}) = \boldsymbol{x}_m$. The flow $\boldsymbol{x}^{(\alpha)}(t)$ is $C^r_{[t_{m-\varepsilon}, t_{m+\varepsilon}]}$-continuous $(r_\alpha \geq 2)$ for time t, and $\|d^{r+1}\boldsymbol{x}^{(\alpha)}/dt^{r+1}\| < \infty$ $(\alpha \in \{i,j\})$. A flow $\boldsymbol{x}^{(\alpha)}(t)$ in Ω_α is tangential to the boundary $\partial\Omega_{ij}$ iff*

$$G^{(0,\alpha)}_{\partial\Omega_{ij}}(\boldsymbol{x}_m, t_m, \boldsymbol{P}_\alpha, \lambda) = 0 \quad for \quad \alpha \in \{i,j\}; \tag{23}$$

$$\left.\begin{aligned}&either\ G^{(1,\alpha)}_{\partial\Omega_{ij}}(\boldsymbol{x}_m, t_m, \boldsymbol{P}_\alpha, \lambda) < 0\ for\ \boldsymbol{n}_{\partial\Omega_{\alpha\beta}} \to \Omega_\beta,\\ &or\qquad G^{(1,\alpha)}_{\partial\Omega_{ij}}(\boldsymbol{x}_m, t_m, \boldsymbol{P}_\alpha, \lambda) > 0\ for\ \boldsymbol{n}_{\partial\Omega_{\alpha\beta}} \to \Omega_\alpha.\end{aligned}\right\} \tag{24}$$

More detailed theory on the flow switchability such as high-order G-functions, the definitions or theorems about various flow passability in discontinuous dynamical systems can be referred to [17] and [18].

From the aforementioned definitions and lemmas, we give the analytical conditions for the flow switching in the double-belt friction oscillator.

For the double-belt friction oscillator in Section 2, the normal vectors of the boundaries $\partial\Omega_{12}$ and $\partial\Omega_{23}$ are given as

$$\mathbf{n}_{\partial\Omega_{12}} = \mathbf{n}_{\partial\Omega_{21}} = (0,1)^{\mathrm{T}},\ \mathbf{n}_{\partial\Omega_{23}} = \mathbf{n}_{\partial\Omega_{32}} = (0,1)^{\mathrm{T}}. \tag{25}$$

The G-functions for such friction oscillator are simplified as $G^{(0,\alpha)}_{\partial\Omega_{ij}}(\mathbf{x}_{(\alpha)}, t_{m\pm})$ or $G^{(1,\alpha)}_{\partial\Omega_{ij}}(\mathbf{x}_{(\alpha)}, t_{m\pm})$.

Theorem 4 *For the double-belt friction oscillator described in Section 2, we have the following results:*

(i) *The stick motion on $\boldsymbol{x}_m \in \partial\Omega_{12}$ at time t_m appears iff the following conditions can be obtained:*

$$F_{(1)}(\boldsymbol{x}_m, t_{m-}) > 0 \ \ and \ \ F_{(2)}(\boldsymbol{x}_m, t_{m-}) < 0. \tag{26}$$

(ii) *The stick motion on $\boldsymbol{x}_m \in \partial\Omega_{23}$ at time t_m appears iff the following conditions can be obtained:*

$$F_{(2)}(\boldsymbol{x}_m, t_{m-}) > 0 \ \ and \ \ F_{(3)}(\boldsymbol{x}_m, t_{m-}) < 0. \tag{27}$$

Proof: From the aforementioned definitions, the 0-order G-functions for the stick boundaries $\partial\Omega_{12}$ and $\partial\Omega_{23}$ in the double-belt friction oscillator are

$$\begin{aligned} G^{(0,1)}_{\partial\Omega_{12}}(\mathbf{x}_m, t_{m\pm}) &= \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{12}} \cdot \mathbf{F}_{(1)}(\mathbf{x}_m, t_{m\pm}), \\ G^{(0,2)}_{\partial\Omega_{12}}(\mathbf{x}_m, t_{m\pm}) &= \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{12}} \cdot \mathbf{F}_{(2)}(\mathbf{x}_m, t_{m\pm}), \end{aligned} \tag{28}$$

and

$$\begin{aligned} G^{(0,2)}_{\partial\Omega_{23}}(\mathbf{x}_m, t_{m\pm}) &= \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{23}} \cdot \mathbf{F}_{(2)}(\mathbf{x}_m, t_{m\pm}), \\ G^{(0,3)}_{\partial\Omega_{23}}(\mathbf{x}_m, t_{m\pm}) &= \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{23}} \cdot \mathbf{F}_{(3)}(\mathbf{x}_m, t_{m\pm}). \end{aligned} \tag{29}$$

From (25), the Eqs. (28) and (29) can be computed as:

$$\begin{aligned} G^{(0,1)}_{\partial\Omega_{12}}(\mathbf{x}_m, t_{m-}) &= F_{(1)}(\mathbf{x}_m, t_{m-}), \\ G^{(0,2)}_{\partial\Omega_{12}}(\mathbf{x}_m, t_{m-}) &= F_{(2)}(\mathbf{x}_m, t_{m-}), \end{aligned} \tag{30}$$

and

$$\begin{aligned} G^{(0,2)}_{\partial\Omega_{23}}(\mathbf{x}_m, t_{m-}) &= F_{(2)}(\mathbf{x}_m, t_{m-}), \\ G^{(0,3)}_{\partial\Omega_{23}}(\mathbf{x}_m, t_{m-}) &= F_{(3)}(\mathbf{x}_m, t_{m-}). \end{aligned} \tag{31}$$

By Lemma 1, the stick motion on $\mathbf{x}_m \in \partial\Omega_{12}$ at time t_m appears iff

$$G^{(0,1)}_{\partial\Omega_{12}}(\mathbf{x}_{(m)}, t_{m-}) > 0 \ \ and \ \ G^{(0,2)}_{\partial\Omega_{12}}(\mathbf{x}_{(m)}, t_{m-}) < 0, \tag{32}$$

i.e.

$$F_{(1)}(\mathbf{x}_m, t_{m-}) > 0 \ \ and \ \ F_{(2)}(\mathbf{x}_m, t_{m-}) < 0. \tag{33}$$

Therefore, (i) holds. Similarly, (ii) holds. □

Theorem 5 *For the double-belt friction oscillator described in Section 2, we have the following results:*

(i) *The non-stick motion (or called passable motion to boundary) on $\boldsymbol{x}_m \in \partial\Omega_{12}$ at time t_m appears iff the following condition can be obtained:*

$$F_{(1)}(\boldsymbol{x}_m, t_{m\pm}) \times F_{(2)}(\boldsymbol{x}_m, t_{m\mp}) > 0. \tag{34}$$

(ii) *The non-stick motion on $\boldsymbol{x}_m \in \partial\Omega_{23}$ at time t_m appears iff the following condition can be obtained:*

$$F_{(2)}(\boldsymbol{x}_m, t_{m\pm}) \times F_{(3)}(\boldsymbol{x}_m, t_{m\mp}) > 0. \tag{35}$$

Proof: By Lemma 2, passable motion on the boundary $\mathbf{x}_m \in \partial\Omega_{12}$ at time t_m appears iff

$$G^{(0,1)}_{\partial\Omega_{12}}(\mathbf{x}_m, t_{m\pm}) \times G^{(0,2)}_{\partial\Omega_{12}}(\mathbf{x}_m, t_{m\mp}) > 0. \tag{36}$$

By (25), we obtain

$$\begin{aligned} G^{(0,1)}_{\partial\Omega_{12}}(\mathbf{x}_m, t_{m\pm}) &= F_{(1)}(\mathbf{x}_m, t_{m\pm}), \\ G^{(0,2)}_{\partial\Omega_{12}}(\mathbf{x}_m, t_{m\mp}) &= F_{(2)}(\mathbf{x}_m, t_{m\mp}). \end{aligned} \tag{37}$$

The Eqs. (36) and (37) implies that (i) holds. The proof for (ii) is similar. □

Theorem 6 *For the double-belt friction oscillator described in Section 2, we have the following results:*

(i) *The grazing motion on $\boldsymbol{x}_m \in \partial\Omega_{12}$ at time t_m appears iff the following conditions can be obtained:*

$$F_{(\alpha)}(\boldsymbol{x}_m, t_{m\pm}) = 0 \quad \text{for} \quad \alpha \in \{1, 2\}, \tag{38}$$

$$\nabla F_{(1)}(\boldsymbol{x}_m, t_{m\pm}) \cdot \boldsymbol{F}_{(1)}(\boldsymbol{x}_m, t_{m\pm}) + \frac{\partial F_{(1)}(\boldsymbol{x}_m, t_{m\pm})}{\partial t_m} < 0, \tag{39}$$

$$\nabla F_{(2)}(\boldsymbol{x}_m, t_{m\pm}) \cdot \boldsymbol{F}_{(2)}(\boldsymbol{x}_m, t_{m\pm}) + \frac{\partial F_{(2)}(\boldsymbol{x}_m, t_{m\pm})}{\partial t_m} > 0. \tag{40}$$

(ii) *The grazing motion on $\boldsymbol{x}_m \in \partial\Omega_{23}$ at time t_m appears iff the following conditions can be obtained:*

$$F_{(\alpha)}(\boldsymbol{x}_m, t_{m\pm}) = 0 \quad \text{for} \quad \alpha \in \{2, 3\}, \tag{41}$$

$$\nabla F_{(2)}(\boldsymbol{x}_m, t_{m\pm})\cdot\boldsymbol{F}_{(2)}(\boldsymbol{x}_m, t_{m\pm})+\frac{\partial F_{(2)}(\boldsymbol{x}_m, t_{m\pm})}{\partial t_m}<0, \tag{42}$$

$$\nabla F_{(3)}(\boldsymbol{x}_m, t_{m\pm})\cdot\boldsymbol{F}_{(3)}(\boldsymbol{x}_m, t_{m\pm})+\frac{\partial F_{(3)}(\boldsymbol{x}_m, t_{m\pm})}{\partial t_m}>0. \tag{43}$$

Proof: By Lemma 3, the sufficient and necessary conditions for the grazing flows on the boundary $\partial\Omega_{12}$ are

$$G^{(0,\alpha)}_{\partial\Omega_{12}}(\mathbf{x}_m, t_{m\pm}) = 0 \quad \text{for} \quad \alpha = 1, 2, \tag{44}$$

$$G^{(1,1)}_{\partial\Omega_{12}}(\mathbf{x}_m, t_{m\pm}) < 0,\ G^{(1,2)}_{\partial\Omega_{12}}(\mathbf{x}_m, t_{m\pm}) > 0. \tag{45}$$

From (25), (28) and (29), we have

$$\begin{aligned} G^{(0,\alpha)}_{\partial\Omega_{12}}(\mathbf{x}_m, t_{m\pm}) &= \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{12}}\cdot\mathbf{F}_{(\alpha)}(\mathbf{x}_m, t_{m\pm}) \\ &= F_{(\alpha)}(\mathbf{x}_m, t_{m\pm}) \quad \text{for } \alpha = 1, 2. \end{aligned} \tag{46}$$

From (19), we obtain

$$\begin{aligned} &G^{(1,1)}_{\partial\Omega_{12}}(\mathbf{x}_m, t_{m\pm}) = \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{12}}\cdot D_{\mathbf{x}^{(0)}_{t_m}}\mathbf{F}_{(1)}(\mathbf{x}_m, t_{m\pm}) \\ &= (0,1)\cdot D_{\mathbf{x}^{(0)}_{t_m}}\Big(\dot{\mathbf{x}}_{(1)}, F_{(1)}(\mathbf{x}_m, t)\Big)^{\mathrm{T}}\Big|_{(\mathbf{x}_m, t_{m\pm})} \\ &= \nabla F_{(1)}(\mathbf{x}_m, t_{m\pm})\cdot\mathbf{F}_{(1)}(\mathbf{x}_m, t_{m\pm})+\frac{\partial F_{(1)}(\mathbf{x}_m, t_{m\pm})}{\partial t_m}. \end{aligned} \tag{47}$$

Similarly,

$$\begin{aligned} &G^{(1,2)}_{\partial\Omega_{12}}(\mathbf{x}_m, t_{m\pm}) \\ &= \nabla F_{(2)}(\mathbf{x}_m, t_{m\pm})\cdot\mathbf{F}_{(2)}(\mathbf{x}_m, t_{m\pm})+\frac{\partial F_{(2)}(\mathbf{x}_m, t_{m\pm})}{\partial t_m}. \end{aligned} \tag{48}$$

From (46),(47) and (48), (i) holds. In a similar manner, (ii) holds. □

5 Switching Plan and Mappings

The switching plans are introduced as $(\lambda = 1, 2)$:

$$\begin{aligned} \Sigma^0_{(\lambda)} &= \{(x_i, \dot{x}_i, \Omega t_i)|\dot{x}_i = v_\lambda\}, \\ \Sigma^1_{(\lambda)} &= \{(x_i, \dot{x}_i, \Omega t_i)|\dot{x}_i = v^-_\lambda\}, \\ \Sigma^2_{(\lambda)} &= \{(x_i, \dot{x}_i, \Omega t_i)|\dot{x}_i = v^+_\lambda\}, \end{aligned} \tag{49}$$

where $v^-_\lambda = \lim_{\delta\to 0}(v_\lambda - \delta)$ and $v^+_\lambda = \lim_{\delta\to 0}(v_\lambda + \delta)$ for arbitrary small $\delta > 0$. Therefore, eight basic mappings will be defined as:

$$\begin{aligned} &P_1 : \Sigma^0_{(1)} \to \Sigma^0_{(1)}, \quad P_2 : \Sigma^1_{(1)} \to \Sigma^1_{(1)}, \\ &P_3 : \Sigma^2_{(1)} \to \Sigma^2_{(1)}, \quad P_4 : \Sigma^0_{(2)} \to \Sigma^0_{(2)}, \\ &P_5 : \Sigma^1_{(2)} \to \Sigma^1_{(2)}, \quad P_6 : \Sigma^2_{(2)} \to \Sigma^2_{(2)}, \\ &P_7 : \Sigma^1_{(2)} \to \Sigma^2_{(1)}, \quad P_8 : \Sigma^2_{(1)} \to \Sigma^1_{(2)}. \end{aligned} \tag{50}$$

From foregoing (49) and (50), we obtain

$$\begin{aligned} P_1 :&\quad (x_i, v_1, \Omega t_i) \to (x_{i+1}, v_1, \Omega t_{i+1}), \\ P_2 :&\quad (x_i, v^-_1, \Omega t_i) \to (x_{i+1}, v^-_1, \Omega t_{i+1}), \\ P_3 :&\quad (x_i, v^+_1, \Omega t_i) \to (x_{i+1}, v^+_1, \Omega t_{i+1}), \\ P_4 :&\quad (x_i, v_2, \Omega t_i) \to (x_{i+1}, v_2, \Omega t_{i+1}), \\ P_5 :&\quad (x_i, v^-_2, \Omega t_i) \to (x_{i+1}, v^-_2, \Omega t_{i+1}), \\ P_6 :&\quad (x_i, v^+_2, \Omega t_i) \to (x_{i+1}, v^+_2, \Omega t_{i+1}), \\ P_7 :&\quad (x_i, v^-_2, \Omega t_i) \to (x_{i+1}, v^+_1, \Omega t_{i+1}), \\ P_8 :&\quad (x_i, v^+_1, \Omega t_i) \to (x_{i+1}, v^-_2, \Omega t_{i+1}). \end{aligned} \tag{51}$$

With (11) and (12), the governing equations for $P_\lambda(\lambda = 1, 4)$ can be described as

$$\begin{cases} x_{i+1} = v_1(t_{i+1} - t_i) + x_i, \\ A_0 + B_0\cos\Omega t_{i+1} - kx_{i+1} - cv_1 + \mu_2 F_N = \mu_1 F_N, \end{cases} \tag{52}$$

$$\begin{cases} x_{i+1} = v_2(t_{i+1} - t_i) + x_i, \\ A_0 + B_0\cos\Omega t_{i+1} - kx_{i+1} - cv_2 - \mu_1 F_N = \mu_2 F_N, \end{cases} \tag{53}$$

respectively.

For the double-belt friction oscillator, the domains Ω_α $(\alpha \in \{1, 2, 3\})$ are unboubded. From the basic theorems of discontinuous dynamical system, only three possible bounded motions exist in the three domains, from which the governing equations of mapping P_λ $(\lambda \in \{1, 2, \cdots, 8\})$ are obtained. With (51), the governing equations of each mapping P_λ $(\lambda \in \{1, 2, \cdots, 8\})$ can be expressed as

$$\begin{aligned} f^{(\lambda)}_1(x_i, \Omega t_i, x_{i+1}, \Omega t_{i+1}) = 0, \\ f^{(\lambda)}_2(x_i, \Omega t_i, x_{i+1}, \Omega t_{i+1}) = 0. \end{aligned} \tag{54}$$

The grazing motion occurs when a flow in a domain is tangential to the boundary and then returns back to this domain. The analytical conditions for the grazing motion in the double-belt friction oscillator were described as Lemma 3 and Theorem 6. If the grazing motion occurs at $(\mathbf{x}_m, t_m) \in \partial\Omega_{\alpha\beta}$ $(\alpha, \beta \in \{1,2,3\})$, more detailed theorem on the grazing motions will be developed.

For the double belt friction oscillator described in Section 2, there are four cases of grazing motions on the boundaries: the flow in domain Ω_1 tangential to the boundary $\partial\Omega_{12}$, the flow in domain Ω_2 tangential to the boundary $\partial\Omega_{21}$, the flow in domain Ω_2 tangential to the boundary $\partial\Omega_{23}$, and the flow in domain Ω_3 tangential to the boundary $\partial\Omega_{32}$, corresponding to the mapping P_2, P_3, P_5 and P_6, respectively. With (54), we can obtain the following theorem.

Theorem 7 *For the double-belt friction oscillator described in Section 2, there are four kinds of grazing motions:*

(i) *Suppose the flow in domain Ω_1 reaches $\boldsymbol{x}_m \in \partial\Omega_{12}$ at time t_m, the grazing motion on the boundary $\partial\Omega_{12}$ appears (i.e. the mapping P_2 is tangential to the boundary $\partial\Omega_{12}$) iff*

$$\left.\begin{array}{l}
\mathrm{mod}(\Omega t_m, 2\pi) \in [\,0, \pi + |\Theta_2^{\mathrm{cr}}|\,) \cup (\,2\pi - |\Theta_2^{\mathrm{cr}}|, 2\pi\,] \\
\hfill \textit{for } 0 < \gamma_2 < \frac{B_0}{m}\Omega; \\
\mathrm{mod}(\Omega t_m, 2\pi) \in [\,0, \frac{3}{2}\pi\,) \cup (\,\frac{3}{2}\pi, 2\pi\,] \\
\hfill \textit{for } 0 < \gamma_2 = \frac{B_0}{m}\Omega; \\
\mathrm{mod}(\Omega t_m, 2\pi) \in [\,0, 2\pi\,] \\
\hfill \textit{for } 0 < \frac{B_0}{m}\Omega < \gamma_2; \\
\mathrm{mod}(\Omega t_m, 2\pi) \in (\,0, \pi\,) \\
\hfill \textit{for } \gamma_2 = 0; \\
\mathrm{mod}(\Omega t_m, 2\pi) \in (\,\Theta_2^{\mathrm{cr}}, \pi - \Theta_2^{\mathrm{cr}}) \subset (\,0, \pi\,) \\
\hfill \textit{for } \gamma_2 < 0 \textit{ and } \frac{B_0}{m}\Omega > |\gamma_2|; \\
\mathrm{mod}(\Omega t_m, 2\pi) \in \{\varnothing\} \\
\hfill \textit{for } \gamma_2 < 0 \textit{ and } \frac{B_0}{m}\Omega < |\gamma_2|,
\end{array}\right\} \tag{55}$$

where

$$\Theta_2^{\mathrm{cr}} = \arcsin(-\frac{\gamma_2 m}{B_0 \Omega}),$$

and

$$\gamma_2 = \frac{c}{m}\ddot{x}_{(1)}(t_m) + \frac{k}{m}\dot{x}_{(1)}(t_m).$$

(ii) *Suppose the flow in domain Ω_2 reaches $\boldsymbol{x}_m \in \partial\Omega_{21}$ at time t_m, the grazing motion on the boundary $\partial\Omega_{21}$ appears (i.e. the mapping P_3 is tangential to the boundary $\partial\Omega_{21}$) iff*

$$\left.\begin{array}{l}
\mathrm{mod}(\Omega t_m, 2\pi) \in (\,\pi + |\Theta_3^{\mathrm{cr}}|, 2\pi - |\Theta_3^{\mathrm{cr}}|\,) \subset (\,\pi, 2\pi) \\
\hfill \text{for } 0 < \gamma_3 < \frac{B_0}{m}\Omega; \\
\mathrm{mod}(\Omega t_m, 2\pi) \in \{\varnothing\} \\
\hfill \text{for } 0 < \frac{B_0}{m}\Omega \le \gamma_3; \\
\mathrm{mod}(\Omega t_m, 2\pi) \in (\,\pi, 2\pi\,) \\
\hfill \text{for } \gamma_3 = 0; \\
\mathrm{mod}(\Omega t_m, 2\pi) \in [\,0, \Theta_3^{\mathrm{cr}}\,) \cup (\,\pi - \Theta_3^{\mathrm{cr}}, 2\pi\,] \\
\hfill \text{for } \gamma_3 < 0 \text{ and } \frac{B_0}{m}\Omega > |\gamma_3|; \\
\mathrm{mod}(\Omega t_m, 2\pi) \in [\,0, \frac{\pi}{2}\,) \cup (\,\frac{\pi}{2}, 2\pi\,] \\
\hfill \text{for } \gamma_3 < 0 \text{ and } \frac{B_0}{m}\Omega = |\gamma_3|; \\
\mathrm{mod}(\Omega t_m, 2\pi) \in [\,0, 2\pi\,] \\
\hfill \text{for } \gamma_3 < 0 \text{ and } \frac{B_0}{m}\Omega < |\gamma_3|,
\end{array}\right\} \tag{56}$$

where

$$\Theta_3^{\mathrm{cr}} = \arcsin(-\frac{\gamma_3 m}{B_0 \Omega}),$$

and

$$\gamma_3 = \frac{c}{m}\ddot{x}_{(2)}(t_m) + \frac{k}{m}\dot{x}_{(2)}(t_m).$$

(iii) *Suppose the flow in domain Ω_2 reaches $\boldsymbol{x}_m \in \partial\Omega_{23}$ at time t_m, the grazing motion on the boundary $\partial\Omega_{23}$ appears (i.e. the mapping P_5 is tangential to the boundary $\partial\Omega_{23}$) iff*

$$\left.\begin{array}{l}
\mathrm{mod}(\Omega t_m, 2\pi) \in [\,0, \pi + |\Theta_5^{\mathrm{cr}}|\,) \cup (\,2\pi - |\Theta_5^{\mathrm{cr}}|, 2\pi\,] \\
\hfill \textit{for } 0 < \gamma_5 < \frac{B_0}{m}\Omega; \\
\mathrm{mod}(\Omega t_m, 2\pi) \in [\,0, \frac{3}{2}\pi\,) \cup (\,\frac{3}{2}\pi, 2\pi\,] \\
\hfill \textit{for } 0 < \gamma_5 = \frac{B_0}{m}\Omega; \\
\mathrm{mod}(\Omega t_m, 2\pi) \in [\,0, 2\pi\,] \\
\hfill \textit{for } 0 < \frac{B_0}{m}\Omega < \gamma_5; \\
\mathrm{mod}(\Omega t_m, 2\pi) \in (\,0, \pi\,) \\
\hfill \textit{for } \gamma_5 = 0; \\
\mathrm{mod}(\Omega t_m, 2\pi) \in (\,\Theta_5^{\mathrm{cr}}, \pi - \Theta_5^{\mathrm{cr}}) \subset (\,0, \pi\,) \\
\hfill \textit{for } \gamma_5 < 0 \textit{ and } \frac{B_0}{m}\Omega > |\gamma_5|; \\
\mathrm{mod}(\Omega t_m, 2\pi) \in \{\varnothing\} \\
\hfill \textit{for } \gamma_5 < 0 \textit{ and } \frac{B_0}{m}\Omega < |\gamma_5|,
\end{array}\right\} \tag{57}$$

where

$$\Theta_5^{\mathrm{cr}} = \arcsin(-\frac{\gamma_5 m}{B_0 \Omega}),$$

and

$$\gamma_5 = \frac{c}{m}\ddot{x}_{(2)}(t_m) + \frac{k}{m}\dot{x}_{(2)}(t_m).$$

(iv) *Suppose the flow in domain Ω_3 reaches $\boldsymbol{x}_m \in \partial\Omega_{32}$ at time t_m, the grazing motion on the boundary $\partial\Omega_{32}$ appears (i.e. the mapping P_6 is tangential to the* *boundary $\partial\Omega_{32}$) iff*

$$\left.\begin{array}{l}\mathrm{mod}(\Omega t_m,2\pi)\in(\,\pi+|\Theta_6^{\mathrm{cr}}|,2\pi-|\Theta_6^{\mathrm{cr}}|\,)\subset(\,\pi,2\pi)\\ \hfill\text{for } 0<\gamma_6<\frac{B_0}{m}\Omega;\\ \mathrm{mod}(\Omega t_m,2\pi)\in\{\varnothing\}\\ \hfill\text{for } 0<\frac{B_0}{m}\Omega\le\gamma_6;\\ \mathrm{mod}(\Omega t_m,2\pi)\in(\,\pi,2\pi\,)\\ \hfill\text{for } \gamma_6=0;\\ \mathrm{mod}(\Omega t_m,2\pi)\in[\,0,\Theta_6^{\mathrm{cr}}\,)\cup(\,\pi-\Theta_6^{\mathrm{cr}},2\pi\,]\\ \hfill\text{for } \gamma_6<0 \text{ and } \frac{B_0}{m}\Omega>|\gamma_6|;\\ \mathrm{mod}(\Omega t_m,2\pi)\in[\,0,\frac{\pi}{2}\,)\cup(\,\frac{\pi}{2},2\pi\,]\\ \hfill\text{for } \gamma_6<0 \text{ and } \frac{B_0}{m}\Omega=|\gamma_6|;\\ \mathrm{mod}(\Omega t_m,2\pi)\in[\,0,2\pi\,]\\ \hfill\text{for } \gamma_6<0 \text{ and } \frac{B_0}{m}\Omega<|\gamma_6|,\end{array}\right\} \tag{58}$$

where

$$\Theta_6^{\mathrm{cr}}=\arcsin(-\frac{\gamma_6 m}{B_0\Omega}),$$

and

$$\gamma_6=\frac{c}{m}\ddot{x}_{(3)}(t_m)+\frac{k}{m}\dot{x}_{(3)}(t_m).$$

Proof: For the double-belt friction oscillator described in Section 2, by Theorem 6, the grazing motion conditions for the flow $\mathbf{x}_{(1)}(t)$ in domain Ω_1 on the boundary $\partial\Omega_{12}$ at time t_m are given as

$$F_{(1)}(\mathbf{x}_m,t_{m\pm})=0, \tag{59}$$

$$\nabla F_{(1)}(\mathbf{x}_m,t_{m\pm})\cdot\mathbf{F}_{(1)}(\mathbf{x}_m,t_{m\pm})+\frac{\partial F_{(1)}(\mathbf{x}_m,t_{m\pm})}{\partial t_m}<0. \tag{60}$$

With (9), the Eqs. (59) and (60) can be computed as

$$-\frac{c}{m}\dot{x}_{(1)}(t_m)-\frac{k}{m}x_{(1)}(t_m)+\frac{B_0}{m}\cos\Omega t_m +\frac{1}{m}[A_0+(\mu_1+\mu_2)F_N]=0, \tag{61}$$

$$-\frac{c}{m}\ddot{x}_{(1)}(t_m)-\frac{k}{m}\dot{x}_{(1)}(t_m)-\frac{B_0\Omega}{m}\sin\Omega t_m<0. \tag{62}$$

The grazing conditions are computed through (54), (61) and (62). Three equations and an inequality have four unknowns, then one unknown must be given.

From (62), the critical value for $\mathrm{mod}(\Omega t_m,2\pi)$ is introduced through

$$\Theta_2^{\mathrm{cr}}=\arcsin(-\frac{\gamma_2 m}{B_0\Omega}),$$

where $\gamma_2=\frac{c}{m}\ddot{x}_{(1)}(t_m)+\frac{k}{m}\dot{x}_{(1)}(t_m)$, and the superscript "cr" represents a critical value relative to grazing.

If $0<\gamma_2<\frac{B_0}{m}\Omega$, then $-1<-\frac{\gamma_2 m}{B_0\Omega}<0$, we have

$$\mathrm{mod}(\Omega t_m,2\pi)\in[\,0,\pi+|\Theta_2^{\mathrm{cr}}|\,)\cup(\,2\pi-|\Theta_2^{\mathrm{cr}}|,2\pi\,].$$

If $0<\gamma_2=\frac{B_0}{m}\Omega$, then $-\frac{\gamma_2 m}{B_0\Omega}=-1$, we have

$$\mathrm{mod}(\Omega t_m,2\pi)\in[\,0,\frac{3}{2}\pi\,)\cup(\,\frac{3}{2}\pi,2\pi\,].$$

If $0<\frac{B_0}{m}\Omega<\gamma_2$, then $-\frac{\gamma_2 m}{B_0\Omega}<-1$, we have

$$\mathrm{mod}(\Omega t_m,2\pi)\in[\,0,2\pi\,].$$

If $\gamma_2=0$, then $-\frac{\gamma_2 m}{B_0\Omega}=0$, we have

$$\mathrm{mod}(\Omega t_m,2\pi)\in(\,0,\pi\,).$$

If $\gamma_2<0$ and $\frac{B_0}{m}\Omega>|\gamma_2|$, then $0<-\frac{\gamma_2 m}{B_0\Omega}<1$, we have

$$\mathrm{mod}(\Omega t_m,2\pi)\in(\,\Theta_2^{\mathrm{cr}},\pi-\Theta_2^{\mathrm{cr}})\subset(\,0,\pi\,).$$

If $\gamma_2<0$ and $\frac{B_0}{m}\Omega<|\gamma_2|$, then $-\frac{\gamma_2 m}{B_0\Omega}>1$, we have

$$\mathrm{mod}(\Omega t_m,2\pi)\in\{\varnothing\}.$$

Therefore (i) holds. Similarly we can prove that (ii), (iii) and (iv) hold. □

6 Conclusion

In this paper, analytical results of complex motions of a double belt friction oscillator which was subjected to periodic excitation, linear spring-loading, damping force and two friction forces were investigated using the flow switchability theory of the discontinuous dynamical systems. Different domains and boundaries for such system were defined according to the friction discontinuity, which exhibited multiple discontinuous boundaries in the phase space. Analytical conditions of the stick motions and grazing motions of such system were obtained in the form of theorem mathematically. More theories about the double belt friction oscillator need to be investigated in the next.

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⋆ Corresponding author: Jinjun Fan.

E-mail: fjj18@126.com(J.Fan).

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