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Lainiotis Information Filter

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***Abstract*—Kalman filter and Lainiotis filter are well known algorithms that solve the filtering problem, producing the state estimation as well as the corresponding estimation error covariance matrix. Using the Information estimation error covariance matrix, which is the inverse of the estimation error covariance matrix, the Information filter has been derived from Kalman filter. In this paper, using the Information estimation error covariance matrix, the Lainiotis Information filter is introduced. The Lainiotis filter and the Lainiotis Information filter are equivalent with respect to their behavior, since they produce the same estimations. The computational requirements of the Lainiotis filter and the Lainiotis Information filter are determined and a method is proposed to a-priori (before the filters' implementation) decide which filter is the faster one. In the time invariant systems case, the Lainiotis Information filter provides a faster method than the classical one to solve the Riccati equation emanating from Lainiotis filter.**

***Keywords*—Lainiotis filter, Kalman filter, Information filter, Riccati equation**

I. Introduction

Estimation plays an important role in many fields of science: applications to aerospace industry, chemical process, communication systems design, control, civil engineering, filtering noise from 2-dimensional images, pollution prediction and power systems are mentioned in [1]. Linear estimation is associated with discrete time systems described by the following state space equations:

$$x(k+1) = F(k+1)\, x(k) + w(k) \tag{1}$$

$$z(k) = H(k)\, x(k) + v(k) \tag{2}$$

where x(k) is the n-dimensional state vector, z(k) is the m-dimensional measurement vector, F(k+1) is the nxn transition matrix, H(k) is the mxn output matrix, w(k) is the state noise v(k) is the measurement noise, at time $k\geq 0$. The statistical model expresses the nature of the state and the measurements. The basic assumption is that the state noise and the measurement noise are Gaussian white noises with zero means and known covariances Q(k) of dimension nxn and R(k) of dimension mxm, respectively.

The discrete time Kalman filter [1] and Lainiotis filter [2] are well known algorithms that solve the filtering problem, producing the state estimation x(k/k) as well as the corresponding estimation error covariance matrix P(k/k). Kalman filter produces the state estimation through the state prediction x(k+1/k) as well as the corresponding prediction error covariance matrix P(k+1/k). Kalman filter and Lainiotis filter are equivalent with respect to their behavior, since they produce the same estimations [2].

For time varying systems, the Time Varying Lainiotis filter is derived:

Time Varying Lainiotis Filter (TVLF)

$$x(k+1/k+1) = K_n(k+1)z(k+1) + F_n(k+1)[\mathrm{I}+P(k/k)O_n(k+1)]^{-1}[P(k/k)K_m(k+1)z(k+1)+x(k/k)] \tag{3}$$

$$P(k+1/k+1) = P_n(k+1) + F_n(k+1)[\mathrm{I}+ P(k/k)O_n(k+1)]^{-1} P(k/k)F_n^T(k+1) \tag{4}$$

for k=0,1,… with initial conditions $x(0/0)=x_0$, $P(0/0)=P_0$ where

$$A(k+1) = [H(k+1)Q(k)H^T(k+1)+R(k+1)]^{-1}$$
$$K_n(k+1) = Q(k)H^T(k+1)A(k+1)$$
$$K_m(k+1) = F^T(k+1)H^T(k+1)A(k+1)$$
$$P_n(k+1) = [\mathrm{I}-K_n(k+1)H(k+1)]Q(k)$$
$$F_n(k+1) = [\mathrm{I}-K_n(k+1)H(k+1)]F(k+1)$$
$$O_n(k+1) = F^T(k+1)H^T(k+1)A(k+1)H(k+1)F(k+1)$$

The notation M^T is used for the transpose matrix of matrix M.

The notation I is used for the identity matrix.

For time invariant systems, where the transition matrix F(k+1)=F, the output matrix H(k)=H, as well as the plant and measurement noise covariance matrices Q=Q(k) and R=R(k) are constant matrices, the **Time Invariant Lainiotis Filter (TILF)** is derived. In this case, the constant matrices

$$A = [HQH^T+R]^{-1}$$
$$K_n = QH^TA$$
$$K_m = F^TH^TA$$
$$P_n = [\mathrm{I}-K_nH]Q$$
$$F_n = [\mathrm{I}-K_nH]F$$
$$O_n = F^TH^TAHF$$

are computed off-line.

For both Kalman and Lainiotis filters, we assume that:

- the state noise covariance matrices Q(k) are positive definite, denoted by Q(k)>0
- the measurement noise covariance matrices R(k) are positive definite, denoted by R(k)>0
- the estimation error covariance matrices P(k/k) are positive definite, denoted by P(k/k)>0

Note that the existence of the inverse of the matrices in the time varying filters is ensured assuming that every covariance matrix R(k) is positive definite; this has the significance that no measurement is exact.

The Information filter [1], [3]-[4] has been derived from the Kalman filter equations using the inverse of the prediction error covariance matrix. Kalman filter and the Information filter are equivalent with respect to their behavior, since they produce the same estimations. Information filter has been implemented for several applications [5]-[8]. Information filter equations are an alternative to Kalman filter equations and on occasions may be more efficient [1]. Finally, a method is proposed in [3] to a-priori decide which filter among the Kalman filter and the Information filter is the faster one.

In this paper, the Lainiotis Information filter is introduced in section II. It is established that the Lainiotis filter and the Lainiotis Information filter are equivalent with respect to their behavior, since they produce the same estimations. The computational requirements of the Lainiotis filter and the Lainiotis Information filter are determined in section III. A method is proposed to decide which filter is the faster one. In section IV it is shown that in the time invariant systems case, the Lainiotis Information filter provides a faster method than the classical one to solve the Riccati equation emanating from Lainiotis filter. Finally, Section VI summarizes the conclusions.

II. Lainiotis Information Filter Derivation

We are going to introduce the Lainiotis Information filter, which – according to the ideas in [1], [3] – uses the Information estimation error covariance matrix S(k/k), which is the inverse of the estimation error covariance matrix P(k/k):

$$S(k/k) = P^{-1}(k/k) \tag{5}$$

and the Information estimation vector y(k/k), which is connected to the estimation vector x(k/k), through the Information estimation error covariance matrix:

$$y(k/k) = S(k/k)\, x(k/k) \tag{6}$$

The Lainiotis Information filter can be derived through the Lainiotis filter equations using the prediction error covariance matrix P(k+1/k) and the Information prediction error covariance matrix

$$S(k+1/k)=P^{-1}(k+1/k) \tag{7}$$

as well as the prediction vector x(k+1/k) and the Information prediction vector

$$y(k+1/k) = \mathrm{S}(k+1/k)\, x(k+1/k) \tag{8}$$

For the derivation of the Lainiotis Information filter, we assume that the inverse of the estimation error covariance matrix P(k/k) exists; then we are able to write:

$$[\mathrm{I}+P(k/k)O_n(k+1)]^{-1}P(k/k) = [P^{-1}(k/k)+O_n(k+1)]^{-1} \tag{9}$$

Also, we use the Matrix Inversion Lemma[1].

In particular, concerning the estimation error covariance matrix P(k/k) from the Lainiotis filter equation in (4) and (9) arises:

$$P(k+1/k+1) = P_n(k+1) + F_n(k+1)[P^{-1}(k/k)+O_n(k+1)]^{-1}F_n^{\mathrm{T}}(k+1) \tag{10}$$

where the Matrix Inversion Lemma yields:

$$P^{-1}(k+1/k+1)=P_n^{-1}(k+1) - P_n^{-1}(k+1)\, F_n(k+1) [P^{-1}(k/k)+O_n(k+1)+F_n^{\mathrm{T}}(k+1)P_n^{-1}(k+1)F_n(k+1)]^{-1} F_n^{\mathrm{T}}(k+1)P_n^{-1}(k+1) \tag{11}$$

Remark 1:

Note that the existence of the inverse matrix $P_n^{-1}(k+1)$ that appears in (11) is guaranteed in the case where the plant and measurements noise covariances Q(k) and R(k) are positive definite, since then the Matrix Inversion Lemma allow us write:

$$P_n^{-1}(k+1) = [Q^{-1}(k)+H^{\mathrm{T}}(k+1)R^{-1}(k+1)\mathrm{H}(k+1)]^{-1} \tag{12}$$

After some algebra we get the **Time Varying Lainiotis Information Filter (TVLIF)** for time varying systems:

$$P(k/k) = S^{-1}(k/k) \tag{13}$$
$$x(k/k) = S^{-1}(k/k)\, y(k/k) \tag{14}$$
$$S(k+1/k)=Q^{-1}(k)- \Gamma(k+1)[S(k/k)+\mathrm{E}(k+1)]^{-1}\Gamma^{\mathrm{T}}(k+1) \tag{15}$$
$$y(k+1/k)=S(k+1/k)F(k+1)S^{-1}(k/k)y(k/k) \tag{16}$$
$$S(k+1/k+1)=\mathrm{B}(k+1)-\Gamma(k+1)[S(k/k)+\mathrm{E}(k+1)]^{-1}\Gamma^{\mathrm{T}}(k+1) \tag{17}$$
$$y(k+1/k+1)=y(k+1/k)+H^{\mathrm{T}}(k+1)R^{-1}(k+1)z(k+1) \tag{18}$$

for k=0,1,… with initial conditions $y(0/0)=y_0=S_0x_0$, $S(0/0)=S_0=P_0^{-1}$
where

$$\mathrm{B}(k+1)=P_n^{-1}(k+1)=Q^{-1}(k)+H^{\mathrm{T}}(k+1)R^{-1}(k+1)H(k+1) \tag{19}$$
$$\Gamma(k+1)=P_n^{-1}(k+1)F_n(k+1)=Q^{-1}(k)F(k+1) \tag{20}$$
$$\Delta(k+1)=F_n(k+1)P_n^{-1}(k+1)F_n(k+1) =F^T(k+1)[Q^{-1}(k)-H^{\mathrm{T}}(k+1)A(k+1)H(k+1]F(k+1) \tag{21}$$
$$\mathrm{E}(k+1)= O_n(k+1)\Delta(k+1)=F^{\mathrm{T}}(k+1)Q^{-1}(k)F(k+1) \tag{22}$$

Remark 2

The existence of the inverse matrices that appear in the Lainiotis filter and the Lainiotis Information filter equations is guaranteed in the case where

- the state noise covariance matrices Q(k) are positive definite
- the measurements noise covariance matrices R(k) are positive definite; this happens in the case where no measurement is exact
- the estimation error covariance matrices P(k/k) are positive definite

Remark 3

If the initial condition $P(0/0)=P_0$ is singular, then we compute $P(1/1)=P_1$ and $x(1/1)=x_1$ from (3) and (4) and we implement Lainiotis Information filter for k=1,2,… with initial conditions $y(1/1)=y_1=S_1x_1$, $S(1/1)=S_1=P_1^{-1}$

For time invariant systems, the **Time Invariant Lainiotis Information Filter (TILIF)** is derived. In this case, the constant matrices

$$\mathrm{B} = P_n^{-1} = Q^{-1}+H^{\mathrm{T}}R^{-1}H$$
$$\Gamma = P_n^{-1}F_n = Q^{-1}F$$
$$\Delta = F_nP_n^{-1}F_n = F^T[Q^{-1}-H^{\mathrm{T}}AH]F$$
$$\mathrm{E} = O_n\Delta = F^{\mathrm{T}}Q^{-1}F$$

are computed off-line.

[1] Let the matrices A,B,C,D with A,C be nonsingular. Then the following equation holds:

$$[A+BCD]^{-1} = A^{-1} - A^{-1}B[C^{-1}+DA^{-1}B]^{-1}DA^{-1}$$

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III. COMPUTATIONAL REQUIREMENTS

It is then established that the Lainiotis Information filter equations are derived by the Lainiotis filter equations. Thus the two filters are equivalent with respect to their behavior, since they calculate theoretically the same estimates. The two filters are iterative algorithms; then, it is reasonable to assume that both the Lainiotis filter and the Information filter compute the estimation x(k/k) and the estimation error covariance P(k/k) executing the same number of iterations. Thus, in order to compare the algorithms with respect to their computational time, we have to compare their per step (iteration) calculation burden (CB) required for the on-line calculations; the calculation burden of the off-line calculations (initialization process for time invariant filters) is not taken into account.

Scalar operations are involved in matrix manipulation operations, which are needed for the implementation of the filtering algorithms. Table I summarizes the calculation burden of needed matrix operations. Note that the identity matrix is denoted by I and a symmetric matrix by S. The details are given in [2].

The per iteration calculation burdens of Lainiotis filter and Lainiotis Information filter are calculated using Table I and summarized in Table II.

From Table II, it is clear that the algorithms' calculation burdens depend on the state vector dimension n and the measurement vector dimension m.

From table II we conclude that the selection of the faster implementation depends on the relationship between n and m.

TABLE I. CALCULATION BURDEN OF MATRIX OPERATIONS

Matrix Operation	**Matrix Dimensions**	***Calculation Burden***
$C = A + B$	$(nxm)+(nxm)$	nm
$S = A + B$	$(nxn)+(nxn)$	$(n^2+n)/2$
$B = I + A$	$(nxn)+(nxn)$	n
$C = A \cdot B$	$(nxm)\cdot(mxl)$	$2nml–nl$
$S = A \cdot B$	$(nxm)\cdot(mxn)$	$n^2m+nm–(n^2+n)/2$
$B = A^{-1}$	nxn	$(16n^3–3n^2–n)/6$

TABLE II. PER ITERATION CALCULATION BURDEN OF LAINIOTIS FILTER (LF) AND LAINIOTIS INFORMATION FILTER (LIF)

System	***Filter***	***Calculation Burden***
Time Varying	LF	$CB_{TVLF} = (41n^3–12n^2+n)/3 +(16m^3–3m^2–m)/6 +15n^2m–2nm+3nm^2$
	LIF	$CB_{TVLIF} = (56n^3–6n^2–3n)/3 +(16m^3–3m^2–m)/3 +13n^2m–3nm+5nm^2$
Time Invariant	LF	$CB_{TILF} = (58n^3–3n^2–n)/6 +2n^2m+2n$
	LIF	$CB_{TILIF} = (74n^3+12n^2–8n)/6 +2nm$

For time varying systems, we have:

$$CB_{TVLIF}–CB_{TVLF} = (15n^3+6n^2–4n)/3+(16m^3–3m^2–m)/6 –(2n^2m+nm–2nm^2)$$

Then, for time varying systems, the areas (depending on the model dimensions) where the Lainiotis Information Filter or the traditional Lainiotis Filter is faster, are shown in Fig. 1. The following Rule of Thumb is derived: Lainiotis Information Filter is faster than Lainiotis Filter, when $m/n<1.12$.

For time invariant systems, we have

$$CB_{TILIF}–CB_{TILF} = (16n^3+15n^2–7n)/6 –2n^2m$$

Then, for time invariant systems, the areas (depending on the model dimensions) where the Lainiotis Information Filter or the traditional Lainiotis Filter is faster, are shown in Fig. 2. The following Rule of Thumb is derived: Lainiotis Information Filter is faster than Lainiotis Filter, when $m/n>1.35$.

Finally, from Table II and Table II in [3] we conclude that Lainiotis Information filter may be faster than (Kalman) Information filter, depending on the state vector dimension n and the measurement vector dimension m.

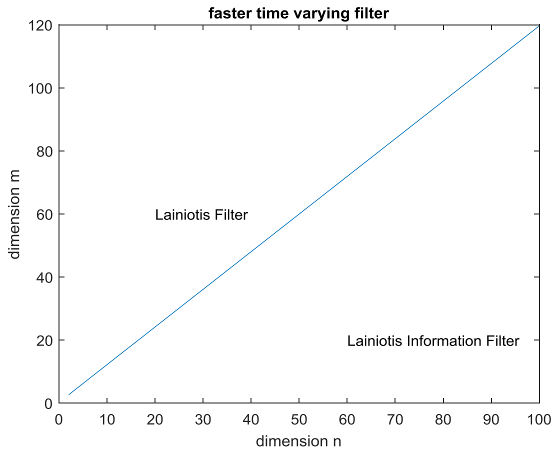

Fig. 1. The faster time varying filter

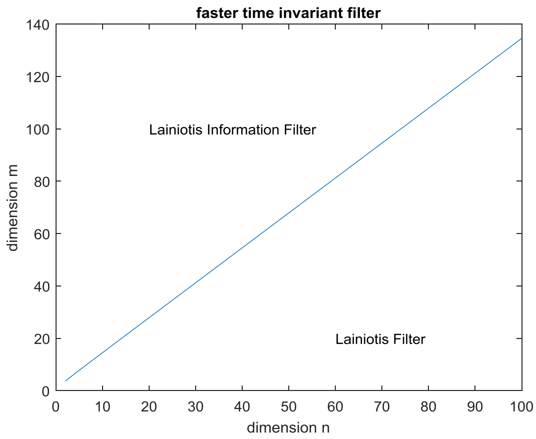

Fig. 2. The faster time invariant filter

IV. Application to Riccati Equation

We consider the **time invariant case**. Concerning the estimation error covariance matrix, the ***algebraic*** **Riccati equation emanating from the Time Invariant Lainiotis Filter** becomes:

$$P = P_n + F_n\,[I+PO_n]^{-1}PF_n^{\mathrm{T}} \qquad (23)$$

The classical algorithm to solve the algebraic Riccati equation emanating from the Time Invariant Lainiotis Filter equation [9]-[10] is to implement iteratively the **Riccati equation emanating from the Time Invariant Lainiotis Filter (RETILF)**:

$$P(k+1/k+1) = P_n + F_n[I+P(k/k)O_n]^{-1}P(k/k)F_n^{\mathrm{T}} \qquad (24)$$

for k=0,1,… with initial condition $P(0/0)=P_0$ until $\|P(k+1/k+1)-P(k/k)\|<\varepsilon$ (convergence criterion).

The ***algebraic*** **Riccati equation emanating from the Time Invariant Lainiotis Information Filter** becomes:

$$S = \mathrm{B} - \Gamma[S+\mathrm{E}]^{-1}\Gamma^{\mathrm{T}} \qquad (25)$$

The algorithm to solve the Riccati equation emanating from the Time Invariant Lainiotis Information Filter equation is to implement iteratively the **Riccati equation emanating from the Time Invariant Information Lainiotis Filter (RETILIF)**:

$$S(k+1/k+1) = \mathrm{B} - \Gamma[S(k/k)+\mathrm{E}]^{-1}\Gamma^{\mathrm{T}} \qquad (26)$$

for k=0,1,… with initial condition $S(0/0)=S_0=P_0^{-1}$ until $\|S(k+1/k+1)-S(k/k)\|<\varepsilon$ (convergence criterion).

The two algorithms are iterative algorithms; then, it is reasonable to assume that the first algorithm computes the steady state estimation error covariance P and the second algorithm computes the inverse $S=P^{-1}$ of the steady state estimation error covariance, executing the same number of iterations. Thus, in order to compare the algorithms with respect to their computational time, we have to compare their per iteration calculation burden (CB) required for the on-line calculations; the calculation burden of the off-line calculations is not taken into account. The per iteration calculation burdens of the algorithms that solve the Riccati equation emanating from Lainiotis filter and Lainiotis Information filter are calculated using Table I and summarized in Table III.

From Table III, it is clear that the algorithms' calculation burdens depend on the state vector dimension n. From table II we conclude that the Lainiotis Information Filter provides a faster method than the classical one to solve the Riccati equation emanating from Lainiotis filter, since

$$CB_{RETILF}-CB_{RETILIF} = (2n^3-3n^2+3n)/6 > 0$$

TABLE III. Per Iteration Calculation Burden Of Riccati Equation Solution Algorithms

Filter	***Calculation Burden***
LF	$CB_{RETILF} = (52n^3-6n^2+2n)/6$
LIF	$CB_{RETILIF} = (50\,n^3-3n^2-n)/6$

V. Conclusions

Kalman filter and Lainiotis filter are well known algorithms that solve the filtering problem, producing the state estimation as well as the corresponding estimation error covariance matrix. Kalman filter and Lainiotis filter are equivalent with respect to their behavior, since they produce the same estimations [2].

Using the Information estimation error covariance matrix, which is the inverse of the estimation error covariance matrix, the (Kalman) Information filter has been derived from Kalman filter. Also, Kalman filter and (Kalman) Information Filter are equivalent [1].

In this paper, using the Information estimation error covariance matrix, the Lainiotis Information filter is introduced. The Lainiotis filter and the Lainiotis Information filter are equivalent with respect to their behavior, since they produce the same estimations.

The computational requirements of the Lainiotis filter and the Lainiotis Information filter are determined and a method is proposed to a-priori (before the filters' implementation) decide which filter is the faster one.

In the time invariant systems case, the Lainiotis Information filter provides a faster method than the classical one to solve the Riccati equation emanating from Lainiotis filter.

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