Blow-up and Bounds of Solutions for a Class of Semi-Linear Pseudo-Parabolic Equations with $p(.)$-Laplacian Viscoelastic Term

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Abstract: - In a bounded domain subject to Dirichlet boundary conditions, this paper discusses the phenomenon of finite time blow-up of solutions for a particular class of evolution equations that affects the pseudo - Laplacian viscoelastic term. We give the equation by:

$$u_t - \Delta u - \int_0^t g(t-s)\Delta_{p(x)} u(x,s)\mathrm{d}s = |u|^{q(x)-2}u.$$

Our findings show that, regardless of the initial energy and sizable initial values, the classical solutions of this equation blow-up in finite time in two cases. Subject to certain conditions on p, q, g, and the initial given data, we have established a new criterion for blow-up and provided lower and upper bounds on the solutions if blow-up occurs.

Key-Words: - Pseudo-parabolic equation, $p(.)$-Laplacian viscoelastic term, memory term, blow-up time, bounds of the blow-up time, critical exponents, variable nonlinearity.

Received: November 28, 2022. Revised: October 12, 2023. Accepted: November 8, 2023. Published: December 12, 2023.

1 Introduction

The pseudo-parabolic equation in the form of

$$u_t(t) - k\Delta u_t(t) - \Delta u(t) = f(u), \quad x \in \Omega, \quad t \geq 0,$$

is commonly used to describe various physical and biological phenomena, such as the propagation of nonlinear dispersive long waves, [1], population aggregation, [2], heat conduction with two temperatures, [3], and nonstationary processes in semiconductors, [4], fluid dynamics, electrorheological fluids, quantum mechanics theory, [5], [6], [7], [8]. It originated from the study of beams and heats. In reference, [9], the authors provide a comprehensive overview of the system:

$$\begin{cases} u_t - \Delta u - \Delta u_t = u^p, & \text{in } \Omega \times (0,T) \\ u(x,t) = 0, & \text{on } \partial\Omega \times (0,T) \\ u(x,0) = u_0(x), & \text{in } \Omega, \end{cases} \quad (1)$$

where $1 < p < \infty$ if $n = 1,2$; $1 < p \leq \frac{n+2}{n-2}$ if $n \geq 3$.

By exploiting the potential well method and the comparison principle, they obtained global existence and finite-time blow-up results for the solutions with initial data at a high energy level. In recent years, a great deal of attention has been given to the study of mathematical nonlinear models with variable-exponent nonlinearity. For instance, modeling physical phenomena such as flows of electrorheological fluids or fluids with temperature-dependent viscosity, nonlinear viscoelasticity, filtration processes through porous media and image processing. More details on these problems can be found in, [10], [11], [12]. Regarding parabolic problems with nonlinearities of variable-exponent type, many works have appeared. Let us mention some of them. For instance, in the, [13], the author studied the following problem:

$$\begin{cases} u_t(x,t) - \Delta u(x,t) = f(u(x,t)), & \text{in } \Omega \times 0, T) \\ u(x,t) = 0, & \text{on } \partial\Omega \times 0, T) \\ u(x,0) = u_0(x), & \text{in } \Omega, \end{cases} \tag{2}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain with a smooth boundary $\partial\Omega$, and the source term is of the form:

$$f(u(x,t)) = a(x)u^{p(x)}(x,t), \qquad x \in \Omega, \quad t \geq 0$$
$$\text{or } f(u) = a(x)\int_\Omega u^{q(y)}(y,t)dy, \tag{3}$$

with $p(x)$, $q(x): \Omega \to (1,\infty)$ and the continuous function $a(x): \Omega \to \mathbb{R}$ are given functions satisfying specific conditions. He established the local existence of positive solutions and proved that solutions with sufficiently large initial data blow up in finite time. Parabolic problems with sources of the form (3) appear in several branches of applied mathematics and have been used to model chemical reactions, heat transfer or population dynamics. The nonlinear parabolic problems of the diffusion equation with nonstandard $p(.)$-growth conditions in the form:

$$u_t(x,t) - div\varphi(u(x,t)) = f(u(x,t)), \; x \in \Omega, \; t \geq 0, \tag{4}$$

for diverse choices of point functions $\varphi(.)$, $f(.)$ such as one might reasonably expect, this equation arises naturally as the equation of motion in all sorts of physical situations such as heat transfer, flows in porous media, propagation of magnetic fields in media with finite conductivities, and in chemical kinetics or biochemical kinetics, to name just a few. In the case where $\varphi(u) = div(|\nabla u|^{p(x)-2}\nabla u)$, for the choices of the function $p(.)$, problem (4) occurs in many mathematical models in fluid mechanics, elasticity theory recently in image processing, [14], [15], porous medium, [16], [17], the unidirectional propagation of nonlinear, dispersive, long waves and the aggregation of population, [18], and the references therein. A series of papers related to problems in the so-called rheological and electrorheological fluids, which lead to spaces with variable exponents, have appeared recently in, [18]. These topics are novel and attractive. It appears from nonlinear elasticity theory, electrorheological fluids, etc. These fluids possess the impressive property that their viscosity depends on the electric field in the fluids. For a general statement of the underlying physics, [19], and for the mathematical presents, [20]. The results detailed in those papers were collected in the books, [21], [22]. Let Ω be a bounded domain in $\mathbb{R}^n (n \geq 1)$ with a smooth boundary $\partial\Omega = \Gamma$. A class of pseudo parabolic equations with $p(.)$-Laplacian viscoelastic terms subject to homogenous Dirichlet boundary conditions are written in the form of partial integro differential equations by:

$$\begin{cases} u_t - \Delta u - \int_0^t g(t-s)\Delta_{p(x)}u(x,s)\mathrm{d}s \\ \quad = |u|^{q(x)-2}u, \; x \in \Omega, \; t \geq 0, \\ \quad u(x,t) = 0, \quad x \in \partial\Omega, t \geq 0, \\ \quad u(x,0) = u_0(x), \quad x \in \Omega, \end{cases} \tag{5}$$

where $\Delta_{p(x)}u = -div(|\nabla u|^{p(x)-2}\nabla u)$, $p(.)$ and $q(.)$ are two measurable functions, $\Omega \subset \mathbb{R}^n (n \geq 1)$ is a bounded domain, $\Gamma = \partial\Omega$ is Lipschitz continuous, $u_0 \geq 0$, with $u_0 \in W_0^{1,p(x)}(\Omega)$, and $g: \mathbb{R}^+ \to \mathbb{R}^+$ is a bounded C^1 function. The function $q(.)$ is a continuous function on $\overline{\Omega}$. This particular model involves parabolic equations that are nonlinear concerning the gradient of the solution and have varying degrees of nonlinearity. The most common case is the evolution p-Laplace equation, where the exponent p is dependent on the external electromagnetic field. For further information, please refer to sources such as, [23], as well as their respective references. The viscoelastic model has become increasingly popular for analyzing the dynamics of viscoelastic structures in recent years. There is a common issue known as problem (5), which appears in various mathematical models used in engineering and physics. Over the last few decades, equations containing viscoelastic terms have received significant attention, and numerous findings have been made regarding the existence, uniqueness, and regularity of weak or classical solutions. For more detail on this topic, we recommend referring to source, [24], [25]. In a recent investigation of a homogeneous Dirichlet boundary value problem, the study, [25], found that when q is a constant, and g and p satisfy certain conditions, a weak solution for (5) with positive initial energy will blow-up in finite time. However, the conditions on g and q are quite rigid. When $p(.) = 2$, the conventional Fourier law of heat flux is typically substituted with the following equation

$$q = -d\nabla u - \int_{-\infty}^t \nabla k(x,t)u(x,\tau)]\mathrm{d}\tau, \tag{6}$$

where u is the temperature, d is the diffusion coefficient, and the integral term represents the memory effect in the material. Looking at it mathematically, we expect the primary term in the equation to have the most significant impact on the integral term, allowing us to use the theory of parabolic equation to solve problem (5). The property of finite time blow-up is crucial for many evolutionary equations. Exploring the blow-up of solutions can be done through various methods. Kaplan introduced the first eigenvalue method in 1963, Levine introduced the concavity method during the 1970s, and the comparison method is based on the comparison principle. Recently, for $q(.) = q =$ constant and $p = 2$, the problem (5) reduces to the following equation:

$$\begin{cases} u_t - \Delta u - \int_0^t g(t-s)\Delta u(x,s)\mathrm{d}s \\ \quad = |u|^{q-2}u, \ x \in \Omega, \ t \geq 0, \\ \quad u(x,t) = 0, \quad x \in \partial\Omega, t \geq 0, \\ \quad u(x,0) = u_0(x), \quad x \in \Omega, \end{cases} \tag{7}$$

when g is not equal to zero ($g \neq 0$), in, [26], the author studied the blow-up results of (7), and found a lower bound for the solution's blow-up time if it occurs. Additionally, he created a new blow-up criterion and provided an upper bound for the solution's blow-up time based on certain conditions involving p, g, and u_0. Based on previous research, we have found that the solution of problem (5) blows up when given arbitrary positive initial energy and appropriate large initial values, as long as $q(.) \geq 2$ and $N \geq 1$. Additionally, we have proven that the nonnegative solutions must blow-up in a finite amount when given negative initial energy.

2 Preliminaries

Let $p: \Omega \to 1, \infty]$ be a measurable function. $L^{p(.)}(\Omega)$ denotes the set of the real measurable functions u on Ω such that:

$$\int_\Omega |\lambda u(x)|^{p(x)}\mathrm{d}x < \infty \quad \text{for some } \lambda > 0.$$

The variable-exponent space $L^{p(.)}(\Omega)$ equipped with the Luxemburg-type norm:

$$\|u\|_{p(.)} = \inf\left\{\lambda > 0, \qquad \int_\Omega \left|\frac{u(x)}{\lambda}\right|^{p(x)} \mathrm{d}x \leq 1\right\},$$

is a Banach space. Throughout the paper, we use $\|.\|_q$ to indicate the L^q-norm for $1 \leq q \leq +\infty$. Next, we will define the variable-exponent Sobolev space $W^{1,p(.)}(\Omega)$ in the following manner

$$W^{1,p(.)}(\Omega) = \left\{ \begin{matrix} u \in L^{p(.)}(\Omega): \nabla u \text{ exists and} \\ |\nabla u| \in L^{p(.)}(\Omega) \end{matrix} \right\}.$$

This space is a Banach space, which is defined by its norm:

$$\|u\|_{W_0^{1,p(.)}(\Omega)} = \|u\|_{p(.),\Omega} + \|\nabla u\|_{p(.),\Omega}.$$

In addition, we have established that $W_0^{1,p(.)}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in $W^{1,p(.)}(\Omega)$. It is known that for the elements of $W_0^{1,p(.)}(\Omega)$ the Poincaré inequality holds,

$$\|u\|_{p(.),\Omega} \leq C(n,\Omega)\|\nabla u\|_{p(.),\Omega}, \tag{8}$$

and an equivalent norm of $W_0^{1,p(.)}(\Omega)$ can be defined by:

$$\|u\|_{W_0^{1,p(.)}(\Omega)} = \|\nabla u\|_{p(.),\Omega}.$$

To state and prove our main result, we need to establish the following hypotheses.
The measurable exponent functions $p(.)$ and $q(.)$ provided meet the requirements.

$$2 < q_1 \leq q(x) \leq q_2 < p_1 \leq p(x) \leq p_2 < \frac{2n}{n-2} \text{ for } n \geq 3,$$

where for a given measurable function φ on $\overline{\Omega}$;

$$\varphi_2 = \underset{x\in\Omega}{esssup}\varphi(x), \varphi_1 = \underset{x\in\Omega}{essinf}\varphi(x),$$

assuming except that $p(.)$, and $q(.)$ everifies the log-Hölder continuity condition:

$$|\varphi(x) - \varphi(y)| \leq M(|x-y|), \tag{9}$$

where $M(r)$ satisfies

$$\underset{r\to 0^+}{\text{limsup}} M(r)\ln\left(\frac{1}{r}\right) = c < \infty.$$

The memory kernel $g: [0, +\infty) \to 0, +\infty)$ is a C^1 function satisfying:

$$g(t) \geq 0, \ g'(t) \leq 0, 1 - \int_0^\infty g(s)\mathrm{d}s = \kappa > 0, \tag{10}$$

$$1 - \int_0^\infty g(s)\mathrm{d}s = \kappa \in \left[\tfrac{1}{(q_1-1)^2}, 1\right]. \tag{11}$$

3 Blow-up in Finite Time and Bounds of Blow-Up Time

In this section, we will prove that the blow-up of solutions to problem (5) with arbitrary positive energy and suitable initial data, besides, we get a new bounds for the blow-up time if the variable exponents and the initial data satisfy some conditions.

As it is well known that degenerate equations do not have classical solutions, we give a precise definition of the weak solution.

Definition 1 *A function $u(x,t) \in L^{\infty}(\Omega \times (0,T)) \cap L^{p(.)}\left(0,T; W_0^{1,p(.)}(\Omega)\right)$, $u_t \in L^2(0,T;L^2(\Omega))$ is called weak solution of problem (5), if and if only if the equality*

$$\int_\Omega \int_0^T u_t \varphi \mathrm{d}t\mathrm{d}x + \int_\Omega \int_0^T \nabla\varphi.\left(\nabla u - \int_0^t g(t-s)|\nabla u(s)|^{p(.)-2}\nabla u(s)\mathrm{d}s\right)\mathrm{d}t\mathrm{d}x = \int_\Omega \int_0^T |u|^{q(.)-2}u\varphi \mathrm{d}t\mathrm{d}x,$$

Holds for all

$$\varphi \in L^2(Q) \cap L^{p(.)}\left(0,T; W_0^{1,p(.)}(\Omega)\right).$$

The proof of the first main result relies heavily on the significance of these two lemmas:

Lemma 2 *Suppose that a positive, twice-differentiable function $\theta(t)$ satisfies the inequality*

$$\theta''(t)\theta(t) - (1+\beta)\theta'(t)^2 \geq 0, \quad t > 0,$$

where $\beta > 0$ is some constant. If $\theta(0) > 0$ and $\theta'(0) > 0$, then there exists $0 < T_1 < \frac{\theta(0)}{\beta\theta'(0)}$ such that $\theta(t)$ tends to infinity as $t \to T_1$.

In the following, we prepare some lemmas needed in the proof of the main results.

Lemma 3 (Sobolev-Poincarà inequality) *If $q(.)$ satisfy $(H1)$ For all $u \in H_0^1(\Omega)$, then the following embedding*

$$H_0^1(\Omega) \hookrightarrow L^{q_2}(\Omega) \hookrightarrow L^{q(.)}(\Omega) \hookrightarrow L^{q_1}(\Omega) \hookrightarrow L^2(\Omega),$$

are continuous, and we get:

$$\| u \|_{q(.)} \leq B \| \nabla u \|_2,$$

where the optimal constant of the Sobolev embedding is denoted by B, and the norm of $L^{q(.)}(\Omega)$ is represented by $\|.\|_{q(.)}$. The following property is associated:

$$\min\left(\|u\|_{q(.)}^{q_1}, \|u\|_{q(.)}^{q_2}\right) \leq \varrho(u) = \int_\Omega |u(x)|^{q(x)}\mathrm{d}x \leq \max\left(\|u\|_{q(.)}^{q_1}, \|u\|_{q(.)}^{q_2}\right), \quad (12)$$

for any $u \in L^{q(.)}(\Omega)$.

Our main result is presented here.

We assert the local existence of a solution for (5), even without proof. This can be gained through the Faedo-Galerkin methods, in combination with the fixed point theorem in Banach spaces.

Theorem 1 *Assuming that both $(H1)$ and $(H2)$ are valid. Problem (5) has a local solution, denoted as u, that satisfies $u(x,t) \in L^{\infty}(\Omega \times (0,T_0)) \cap L^{p(.)}\left(0,T_0; W_0^{1,p(.)}(\Omega)\right)$ and $u_t \in L^2(0,T_0;L^2(\Omega))$ for $T_0 > 0$.*

3.1 First blow-up Result

One of the primary techniques for proving the blow-up of solutions involves calculating the energy function and using the concavity argument.

Let

$$M_1 = \left(1 - \frac{2}{q_1}\frac{\left(1+\tau-\int_0^t g(s)\mathrm{d}s\right)}{1-\int_0^t g(s)\mathrm{d}s}\right), \quad M_2 = \frac{\left(1+\tau-\int_0^t g(s)\mathrm{d}s\right)}{1-\int_0^t g(s)\mathrm{d}s}, \quad (13)$$

for any positive τ such that

$$\tau \in \left[\frac{1}{2}\left(\sqrt{1-\int_0^t g(s)\mathrm{d}s} + \int_0^t g(s)\mathrm{d}s - 1\right), \frac{1}{2}\left(1-\int_0^t g(s)\mathrm{d}s\right)(q_1-2)\right].$$

Theorem 2 *Let us consider the assumptions of Theorem 1 . If $E(u_0) > 0$ for any given u_0 such that*

$$\int_\Omega |u_0|^2\mathrm{d}x \geq \frac{1}{M_1}(2M_2\mathrm{E}(0) + M_1|\Omega|),$$

with

$$0 < E(0) < \frac{|\Omega|}{4(p_1+1)}. \quad (14)$$

If $p(.)$ and $q(.)$ satisfy (9) and (H1) $-$ (H2) hold, then the solution $u(x,t)$ can exist for a finite amount of time. However, if there exists a $T_1 \leq T_{\max}$ such that $\lim_{t\to T_1}\int_0^t \|u(s)\|_2^2\mathrm{d}s = +\infty$, this means that the solution u blows up in finite time in $L^2(\Omega)$-norm. M_1 and M_2 are given in (13).

Lemma 6 *Under the assumptions of Theorem 5, the corresponding energy to problem (5)* $E: W_0^{1,p(.)}(\Omega) \cap L^{q(.)}(\Omega) \to \mathbb{R}$, *is considered by*

$$E(t) := E\big(u(t)\big)$$
$$= \frac{1}{2}\left(1 - \int_0^t g(s)ds\right) \| \nabla u(t) \|_2^2 - \int_\Omega \frac{1}{q(x)} |u|^{q(x)} \mathrm{d}x + \frac{1}{2}(g \circ \nabla u)(t), \qquad (15)$$

$E(t)$ decreasing, that is

$$E'(t) = -\frac{1}{2} g(t) \int_\Omega |\nabla u(t)|^2 \mathrm{d}x + \frac{1}{2}(g' \circ \nabla u)(t) - \int_\Omega |u_t(t)|^2 \mathrm{d}x$$
$$\leq -\int_\Omega |u_t(t)|^2 \mathrm{d}x \leq 0. \qquad (16)$$

where

$$(g \circ \nabla u)(t) = \int_0^t g(t-s) \left\| \nabla u - |\nabla u(s)|^{p(x)-2} \nabla u(s) \right\|_2^2 \mathrm{d}s.$$

Proof. For a solution u to problem (5), multiplying Equation (5) $_{(2)}$ by u_t, integrating the result over Ω, using the Green's formula, we find:

$$\frac{1}{2}\frac{d}{dt}\left(\int_\Omega |\nabla u(t)|^2 \mathrm{d}x - \int_\Omega \frac{1}{q(x)} |u|^{q(x)} \mathrm{d}x\right)$$
$$- \int_0^t g(t-s) |\nabla u|^{p(x)-2} \nabla u(x,s) \nabla u_t(t) \mathrm{d}s = -\int_\Omega |u_t(t)|^2 \mathrm{d}x. \qquad (17)$$

A direct calculation of the last term on the left side of (17) can views as follows:

$$-\int_0^t g(t-s) |\nabla u|^{p(x)-2} \nabla u(x,s) \nabla u_t(t) \mathrm{d}s = \frac{1}{2} g(t) \int_\Omega |\nabla u|^2 \mathrm{d}x - \frac{1}{2}(g' \circ \nabla u)(t)$$
$$+ \frac{1}{2}\frac{d}{dt}(g \circ \nabla u)(t) - \frac{1}{2}\frac{d}{dt}\left(\int_0^t g(s)ds \int_\Omega |\nabla u|^2 \mathrm{d}x\right) \qquad (18)$$

Putting (18) in (17), we get:

$$\frac{d}{dt}\begin{pmatrix} \frac{1}{2}\int_\Omega \left|\nabla u(t)\right|^2 \mathrm{d}x - \frac{1}{2}\int_0^t g(s)ds \int_\Omega \left|\nabla u(t)\right|^2 \mathrm{d}x \\ -\int_\Omega \frac{1}{q(x)} |u(t)|^{q(x)} \mathrm{d}x + \frac{1}{2}(g \circ \nabla u)(t) \end{pmatrix}$$
$$-\frac{1}{2} g(t) \int_\Omega |\nabla u(t)|^2 \mathrm{d}x + \frac{1}{2}(g' \circ \nabla u)(t) = -\int_\Omega |u_t(t)|^2 \mathrm{d}x$$

Integrating the above identify over $(0, t)$, we obtain

$$E(t) - E(0) \leq -\frac{1}{2}\int_0^t g(s) \int_\Omega |\nabla u(s)|^2 \mathrm{d}x ds + \frac{1}{2}\int_0^t (g' \circ \nabla u)(s) ds - \int_0^t \int_\Omega |u_t|^2 \mathrm{d}x ds \leq 0. \qquad (19)$$

Proving Theorem 2 relies on the significance of the following lemma 7.

Lemma 7 *Under the assumptions of Theorem 2, the solution of problem (5) satisfies the following inequalities*

$$|u(t)|^2 \mathrm{d}x \geq \mathrm{e}^{2M_1 t}\left[G(0) - 2\frac{M_2}{M_1}\mathrm{E}(0) - |\Omega|\right] \int_\Omega + 2\frac{M_2}{M_1}\mathrm{E}(0) + |\Omega|$$
$$= \mathrm{e}^{2M_1 t} G(0) + \left(2\frac{M_2}{M_1}\mathrm{E}(0) + |\Omega|\right)(1 - \mathrm{e}^{2M_1 t}), \qquad (20)$$

and

$$\int_\Omega u u_t \mathrm{d}x \geq M_1 \mathrm{e}^{2M_1 t}\left[\|u_0\|_2^2 - 2\frac{M_2}{M_1}\mathrm{E}(0) - |\Omega|\right] + M_2 \int_0^t \|u_t(.,s)\|_2^2 \mathrm{d}s, \quad > 0, \qquad (21)$$

where M_1 and M_2 as in (13).

Proof. Set

$$G(t) = \int_\Omega |u(t)|^2 \mathrm{d}x.$$

Integrating by parts, and using Eq. (5), we obtain

$$G'(t) = -2 \| \nabla u(t) \|_2^2 + 2\int_0^t g(t-s) \int_\Omega \nabla u(|\nabla u(s)|^{p(x)-2} \nabla u(s) - \nabla u(t)) \mathrm{d}x \mathrm{d}s$$
$$+2\int_0^t g(s) \mathrm{d}s \| \nabla u(t) \|_2^2 + 2\int_\Omega |u|^{q(x)} \mathrm{d}x. \qquad (22)$$

Applying Young and Hölder inequalities, the second term in the right-hand side of (22) can be estimated as follows

$$\left| \int_0^t g(t-s) \int_\Omega \nabla u(t)(|\nabla u(s)|^{p(x)-2} \nabla u(s) - \nabla u(t)) \mathrm{d}x \mathrm{d}s \right|$$

$$=\left|\begin{array}{c}\int_\Omega \nabla u(t)\int_0^t g(t-s)\\ (|\nabla u(s)|^{p(x)-2}\nabla u(s)-\nabla u(t))\mathrm{d}s\mathrm{d}x\end{array}\right|$$
$$\leq \tau\int_\Omega |\nabla u(t)|^2\mathrm{d}x$$
$$+\frac{1}{4\tau}\int_\Omega\left(\begin{array}{c}\int_0^t g(t-s)\\ \left(|\nabla u(s)|^{p(x)-2}\nabla u(s)-\nabla u(t)\right)\mathrm{d}s\end{array}\right)^2\mathrm{d}x$$
$$\leq \tau\parallel\nabla u(t)\parallel_2^2$$
$$+\tfrac{1}{4\tau}\int_\Omega\left(\int_0^t g(t-s)\mathrm{d}s\right)$$
$$\left(\int_0^t g(t-s)\left||\nabla u(s)|^{p(x)-2}\nabla u(s)\right.\right.$$
$$\left.\left.-\nabla u(t)\right|^2\mathrm{d}s\right)dx$$
$$=\tau\parallel\nabla u(t)\parallel_2^2$$
$$+\frac{1}{4\tau}(g\circ\nabla u)(t)\int_0^t g(s)\mathrm{d}s, \tag{23}$$

for any $\tau>0$. Using (22) and (23), we conclude

$$G'(t)\geq 2\left(-1-\tau+\int_0^t g(s)\mathrm{d}s\right)\parallel\nabla u(t)\parallel_2^2-\frac{1}{2\tau}\int_0^t g(s)\mathrm{d}s(g\circ\nabla u)(t)+2\int_\Omega |u|^{q(x)}\mathrm{d}x. \tag{24}$$

by $q_1>2$, it is clear to check that

$$\int_\Omega |u|^{q(x)}\mathrm{d}x=\int_{\{x\in\Omega:|u|\leq 1\}}|u|^{q(x)}\mathrm{d}x+\int_{\{x\in\Omega:|u|\geq 1\}}|u|^{q(x)}\mathrm{d}x$$
$$\geq\int_{\{x\in\Omega:|u|\geq 1\}}|u|^2\mathrm{d}x$$
$$\geq\int_\Omega |u|^2\mathrm{d}x-\int_{\{x\in\Omega:|u|\leq 1\}}|u|^2\mathrm{d}x$$
$$\geq\int_\Omega |u|^2\mathrm{d}x-|\Omega|, \tag{25}$$

which connect with (24) give

$$\frac{\mathrm{d}}{\mathrm{d}t}G(t)\geq$$
$$2\left(-1-\tau+\int_0^t g(s)\mathrm{d}s\right)\times$$
$$\frac{2\mathrm{E}(t)-(g\circ\nabla u)(t)+\frac{2}{q_2}\int_\Omega |u|^{q(x)}\mathrm{d}x}{1-\int_0^t g(s)\mathrm{d}s}$$
$$+2\int_\Omega |u|^{q(x)}\mathrm{d}x-\frac{1}{2\tau}\int_0^t g(s)\mathrm{d}s(g\circ\nabla u)(t)$$
$$\geq -4\frac{\left(1+\tau-\int_0^t g(s)\mathrm{d}s\right)}{1-\int_0^t g(s)\mathrm{d}s}\mathrm{E}(t)$$
$$+2\left[\frac{\left(1+\tau-\int_0^t g(s)\mathrm{d}s\right)}{1-\int_0^t g(s)\mathrm{d}s}-\frac{1}{4\tau}\int_0^t g(s)\mathrm{d}s\right]\times$$
$$(g\circ\nabla u)(t)$$
$$+2\left(1-\frac{2}{q_1}\frac{\left(1+\tau-\int_0^t g(s)\mathrm{d}s\right)}{1-\int_0^t g(s)\mathrm{d}s}\right)\int_\Omega |u|^2\mathrm{d}x$$
$$+2\left(\frac{2}{q_1}\frac{\left(1+\tau-\int_0^t g(s)\mathrm{d}s\right)}{1-\int_0^t g(s)\mathrm{d}s}-1\right)|\Omega| \tag{26}$$

Inequality (26) confirms that

$$\frac{\mathrm{d}}{\mathrm{d}t}G(t)\geq\quad 2M_1\left(G(t)-2\frac{M_2}{M_1}\mathrm{E}(t)-|\Omega|\right)$$
$$\geq 2M_1\left(G(t)-2\tfrac{M_2}{M_1}\mathrm{E}(0)-|\Omega|\right)+2M_2\int_0^t\|u_t(.,s)\|_2^2\mathrm{d}s \tag{27}$$

because $\int_0^t\|u_t(.,s)\|_2^2\mathrm{d}s$ is positive, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}G(t)\geq 2M_1\left(G(t)-2\frac{M_2}{M_1}\mathrm{E}(0)-|\Omega|\right)$$

By solving nonhomogeneous ordinary differential equation, we can obtain

$$G(t)\geq\mathrm{e}^{2M_1t}\left[G(0)-2\frac{M_2}{M_1}\mathrm{E}(0)-|\Omega|\right]$$
$$+2\tfrac{M_2}{M_1}\mathrm{E}(0)+|\Omega|=\mathrm{e}^{2M_1t}G(0)$$
$$+\left(2\tfrac{M_2}{M_1}\mathrm{E}(0)+|\Omega|\right)(1-\mathrm{e}^{2M_1t}). \tag{28}$$

Substituting (28) into (27), it follows that

$$\frac{\mathrm{d}}{\mathrm{d}t}G(t)\geq 2M_1\mathrm{e}^{2M_1t}\left[G(0)-2\frac{M_2}{M_1}\mathrm{E}(0)-|\Omega|\right]$$
$$+2M_2\int_0^t\|u_t(.,s)\|_2^2\mathrm{d}s.$$

Here is the proof for the first main result:
We point out that the the main method employed in this proof is based on the concavity technique, taking into account the idea used in, [26], Theorem 2.2.

Proof of Theorem 2. We first assume that u exists in the classical sense on $\Omega\times 0,\infty)$ i.e., $T_{\max}=+\infty$ (The interval of existence of u is unbounded, or u is defined in the whole interval $(0,+\infty)$), and then show that this leads to a contradiction. We select an $\varphi(t)$ of the following form for $0<t<\infty$,

$$\varphi(t)=\int_0^t\parallel u(\tau)\parallel_2^2\mathrm{d}\tau$$

Then $\varphi'(t)=\|u\|_2^2$, (29)

we distinguish two cases:

1. **Case.1** $\mathrm{E}(u(t)) \geq 0$, for all $t > 0$. Through (14) we can choose β as such

$$1 < \beta < \frac{|\Omega|}{4\mathrm{E}(0)(p_1+1)}. \quad (30)$$

By adding $4(p_1+1)\beta\mathrm{E}(t) - 4(p_1+1)\beta\mathrm{E}(t)$, and making us (21), (29), and (25) it yields

$$\begin{aligned}\varphi'(t) \geq\ & e^{2M_1 t}\left[\varphi'(0) - 2\frac{M_2}{M_1}\mathrm{E}(0) - |\Omega|\right] + \left(2\frac{M_2}{M_1} + 4(p_1+1)\beta\right)\mathrm{E}(t) \\ & - 4(p_1+1)\beta\mathrm{E}(t) + |\Omega| \\ \geq\ & e^{2M_1 t}\left[\varphi'(0) - 2\frac{M_2}{M_1}\mathrm{E}(0) - |\Omega|\right] \\ & - 4(p_1+1)\beta\mathrm{E}(0) + |\Omega| + 4(p_1+1)\beta\int_0^t \|u_t(.,s)\|_2^2 \mathrm{d}s, \quad (31)\end{aligned}$$

Let ψ be an auxiliary function defined as

$$\psi(t) = \varphi^2(t) + \varepsilon^{-1}\varphi'(0)\varphi(t) + \gamma,$$

where $\varepsilon > 0$, is taken small enough such that

$$0 < \varepsilon \leq \frac{\left[\varphi'(0) - 2\frac{M_2}{M_1}\mathrm{E}(0) - |\Omega|\right] + |\Omega| - 4(p_1+1)\beta\mathrm{E}(0)}{(p_1+1)\beta\varphi(0)},$$

and $\gamma > 0$ large enough (if needed), so that

$$4\varepsilon^2\gamma > \varphi^2(0). \quad (32)$$

Therefore,

$$\psi'(t) = \left(2\varphi(t) + \varepsilon^{-1}\varphi'(0)\right)\varphi'(t); \quad (33)$$

$$\psi''(t) = \left(2\varphi(t) + \varepsilon^{-1}\varphi'(0)\right)\varphi''(t) + 2(\varphi'(t))^2. \quad (34)$$

From (33), we obtain

$$\begin{aligned}\left(\psi'(t)\right)^2 &= \left(2\varphi(t) + \varepsilon^{-1}\varphi'(0)\right)^2(\varphi'(t))^2 \\ &= \left(4\varphi^2(t) + \varepsilon^{-2}\left(\varphi'(0)\right)^2 + 4\varepsilon^{-1}\varphi(t)\varphi'(0)\right) \times (\varphi'(t))^2 \\ &= (4\varphi^2(t) + 4\varepsilon^{-1}\varphi(t)\varphi'(0) + 4\gamma - \delta)\left(\varphi'(t)\right)^2 \\ &= (4\psi(t) - \delta)(\varphi'(t))^2, \quad (35)\end{aligned}$$

where $\delta = 4\gamma - \varepsilon^{-2}\left(\varphi'(0)\right)^2 > 0$, then

$$\left(\psi'(t)\right)^2 + \delta(\varphi'(t))^2 = 4\psi(t)(\varphi'(t))^2. \quad (36)$$

Noting that

$$\begin{aligned}\int_0^t (u_t(.,s), u)\mathrm{d}s &= \frac{1}{2}\int_0^t \left(\frac{\mathrm{d}}{\mathrm{d}s}\|u\|_2^2\right)\mathrm{d}s \\ &= \frac{1}{2}\|u(t)\|_2^2 - \frac{1}{2}\|u_0\|_2^2.\end{aligned}$$

Therefore,

$$\|u(t)\|_2^2 = \|u_0\|_2^2 + 2\int_0^t\int_\Omega u_t(.,s)u(s)\mathrm{d}x\mathrm{d}s.$$

Using Holder and Young's inequalities gives

$$\begin{aligned}(\varphi'(t))^2 = \|u(t)\|_2^4 &= \left(\|u_0\|_2^2 + 2\int_0^t\int_\Omega u_t(.,s)u(s)\mathrm{d}x\mathrm{d}s\right)^2 \\ &\leq \left(\|u_0\|_2^2 + 2\left(\int_0^t \|u\|_2^2\mathrm{d}s\right)^{\frac{1}{2}}\left(\int_0^t \|u_t(.,s)\|_2^2\mathrm{d}s\right)^{\frac{1}{2}}\right)^2 \\ &\leq \|u_0\|_2^4 + 2\|u_0\|_2^2\left(\int_0^t \|u\|_2^2\mathrm{d}s + \int_0^t \|u_t(.,s)\|_2^2\mathrm{d}s\right) \\ &\quad + 4\left(\int_0^t \|u\|_2^2\mathrm{d}s\right)\left(\int_0^t \|u_t(.,s)\|_2^2\mathrm{d}s\right) \\ &= \|u_0\|_2^4 + 2\varepsilon^{-1}\|u_0\|_2^2\int_0^t \|u_t(.)\|_2^2\mathrm{d}s \\ &\quad + 4\varphi(t)\int_0^t \|u_t(.,s)\|_2^2\mathrm{d}s + 2\varepsilon\|u_0\|_2^2\varphi(t). \quad (37)\end{aligned}$$

From (34) and (36), we get

$$\begin{aligned}2\psi''(t)\psi(t) &= 2\left(2\phi(t) + \varepsilon^{-1}\varphi(0)\right)\varphi''(t)\psi(t) + 4(\varphi'(t))^2\psi(t) \\ &= 2\left(2\varphi(t) + \varepsilon^{-1}\varphi(0)\right)\varphi''(t)\psi(t) + \left(\psi'(t)\right)^2 + \delta(\varphi'(t))^2. \quad (38)\end{aligned}$$

Now, from (38), (35), (31) and (37), the following estimates ensured:

$$\begin{aligned}& 2\psi''(t)\psi(t) - (1+\beta)\left(\psi'(t)\right)^2 \\ &= 2\left(2\varphi(t) + \varepsilon^{-1}\varphi(0)\right)\varphi''(t)\psi(t) + \delta(\varphi'(t))^2 - \beta\left(\psi'(t)\right)^2 \\ &= 2\left(2\varphi(t) + \varepsilon^{-1}\varphi(0)\right)\varphi''(t)\psi(t) + \delta(\varphi'(t))^2 - \beta(4\psi(t) - \delta)(\varphi'(t))^2 \\ &= 2\left(2\varphi(t) + \varepsilon^{-1}\varphi(0)\right)\varphi''(t)\psi(t) - 4\beta\psi(t)(\varphi'(t))^2\end{aligned}$$

$$+\delta(1+\beta)(\varphi'(t))^2$$
$$\geq 2\psi(t)\big(2\varphi(t)+\varepsilon^{-1}\varphi(0)\big)\times$$
$$\left(\begin{array}{l} 2M_1 e^{2M_1 t}\left[\|u_0\|_2^2 - 2\frac{M_2}{M_1}\mathrm{E}(0) - |\Omega|\right] \\ +2M_2\int_0^t \|u_t(.,s)\|_2^2 ds \end{array}\right)$$
$$-4\beta\psi(t)\times$$
$$\left(\begin{array}{l} \|u_0\|_2^4 + 2\varepsilon^{-1}\|u_0\|_2^2\int_0^t \|u_t(.,s)\|_{2,\Omega_2}^2 ds \\ +2\varepsilon\|u_0\|_2^2\varphi(t) + 4\varphi(t)\int_0^t \|u_t(.,s)\|_2^2 ds \end{array}\right)$$

Recalling the values of β and ε and taking into account that $e^{M_0 t} > 1, p_1+1 > 2, \psi > 0$, it result

$$2\psi''(t)\psi(t) - (1+\beta)\big(\psi'(t)\big)^2$$
$$\geq 4\beta\psi(t)\big(2\varphi(t)+\varepsilon^{-1}\varphi(0)\big)\Big((p_1+1)\int_0^t \|u_t(.,s)\|_2^2 ds + (p_1+1)\varepsilon\varphi(0)\Big) - 4\beta\psi(t)\times$$
$$\left(\begin{array}{l} \|u_0\|_2^4 + 2\varepsilon^{-1}\|u_0\|_2^2\int_0^t \|u_t(.,s)\|_2^2 ds + \\ 2\varepsilon\|u_0\|_2^2\varphi(t) + 4\varphi(t)\int_0^t \|u_t(.,s)\|_2^2 ds \end{array}\right)$$
$$\geq 0$$

Now, in this case we show that T cannot be infinite, and therefore there is no weak solution all the time.

From Lemma 2, it follows that there exists a

$$0 < t_1 < +\infty$$

such that $\psi(t)\to\infty$ as $t\to t_1$, where

$$0 < t_1 < \frac{2\psi(0)}{(\beta-1)\psi'(0)} = \frac{2\gamma\varepsilon}{(\beta-1)\|u_0\|_2^4} < +\infty.$$

Since ψ is continuous with respect to φ, we conclude that there exists a $T_1 \leq t_1$ such that $\lim_{t\to T_1}\int_0^t \|u(s)\|_2^2 ds = +\infty \Rightarrow \limsup_{t\to T_1}\|u(t)\|_2^2 = +\infty$.

Hence, $u(x,t)$ discontinuing at some finite time T_1, that is to means, $u(x,t)$ not exist for all time, i.e. $u(x,t)$ blows up at a time T_1, which will lead to the nonexistence result stated in the theorem, then φblows up at time T_1 in $L^2(\Omega)$-norm, which contradicts. Hence, for the data satisfies (14) any solution possesses finite explosion time.

2. **Case 2.** Assume that there exists $t_0 > 0$ such that $\mathrm{E}(u(t_0)) < 0$, $(u(t_0)\neq 0)$. We define $v(x,t) = u(x,t+t_0)$, so $E(v(0)) = E\big(u(t_0)\big) < 0$. By the fact that $E(t)$ is deceasing in t, we can get:

$$E((v(t)) \leq E(v(0)) \leq 0. \tag{39}$$

Define $G(t) = \int_\Omega v^2(x,t)dx$, then we have as in (26)

$$\frac{d}{dt}G(t) \geq -4\frac{\left(1+\tau-\int_0^t g(s)ds\right)}{1-\int_0^t g(s)ds}\mathrm{E}(t)$$
$$+2\left(1-\frac{2\left(1+\tau-\int_0^t g(s)ds\right)}{q_1\left(1-\int_0^t g(s)ds\right)}\right)\int_\Omega |v|^{q(x)}dx$$
$$\geq 2\left(1-\frac{2\left(1+\tau-\int_0^t g(s)ds\right)}{q_1\left(1-\int_0^t g(s)ds\right)}\right)\int_\Omega |v|^{q(x)}dx.$$

Then, it follows that:

$$G'(t) \geq 2M_1\int_\Omega |v|^{q(x)}dx. \tag{40}$$

For $q(.)$ satisfy (H2), the following embedding

$$L^{q_2}(\Omega)\hookrightarrow L^{q(.)}(\Omega)\hookrightarrow L^{q_1}(\Omega)\hookrightarrow L^2(\Omega),$$

hold. Therefore, from $L^{q(.)}(\Omega)\hookrightarrow L^2(\Omega)$, we obtain that:

$$\| v \|_2 \leq \mathrm{C_e} \| v \|_{q(x)}. \tag{41}$$

Using (12), and (41) from (40), we can get

$$G'(t) \geq 2M_1\min\left\{\begin{array}{l} \left(\frac{1}{\mathrm{C_e}}\right)^{q_1} v \|_2^{q_1} \\ ,\left(\frac{1}{\mathrm{C_e}}\right)^{q_2} \| v \|_2^{q_2} \end{array}\right\}$$
$$\geq C_5\min\left\{G^{\frac{q_1}{2}}(t), G^{\frac{q_2}{2}}(t)\right\}, \tag{42}$$

where $\mathrm{C_e}$ is a best embedding constant and

$$C_5 = 2M_1\min\left\{\left(\frac{1}{\mathrm{C_e}}\right)^{q_1}, \left(\frac{1}{\mathrm{C_e}}\right)^{q_2}\right\}.$$

By $G'(t) > 0$, so $G(t)\geq G(0)$. We can conclude that

$$\left[\frac{G(t)}{G(0)}\right]^{\frac{q_2}{2}} \geq \left[\frac{G(t)}{G(0)}\right]^{\frac{q_1}{2}},$$

that is

$$G(t)]^{\frac{q_2}{2}} \geq G(0)^{\frac{q_2-q_1}{2}}[G(t)]^{\frac{q_1}{2}}. \tag{43}$$

Using (43) and (44), we have

$$G'(t) \geq C_5\min\left\{G^{\frac{q_1}{2}}(t), G(0)^{\frac{q_2-q_1}{2}}[G(t)]^{\frac{q_1}{2}}\right\}$$
$$\geq C_6 G^{\frac{q_1}{2}}(t) \tag{44}$$

where $C_6 = C_5 \min\left\{1, G(0)^{\frac{q_2-q_1}{2}}\right\}$. Using (44) we can derive the following result

$$G^{\frac{q_1-2}{2}}(t) \geq \frac{1}{G^{\frac{2-q_1}{2}}(0) - \frac{q_1-2}{2} C_6 t}.$$

The above inequality implies that $G(t)$ blows up at finite time $T^* \leq \frac{2G^{\frac{2-q_1}{2}}(0)}{(q_1-2)C_6}$, which is a contraction.

3.2 Second Blow-Up Result

In this subsection, we will address problem (5) by establishing a blow-up criterion and obtaining bounds for the blow-up time of weak solutions through the use of differential inequality techniques. For our result, we need to consider the following auxiliary functions.

$$\alpha(t) = [\kappa \parallel \nabla u(t) \parallel_2^2 + (g \circ \nabla u)(t)]^{\frac{1}{2}}, \tag{45}$$

and for ε (a small positive number) and N (a precise positive constant) to be picked later;

$$\mathrm{A}(t) := H^{1-\alpha}(t) + \varepsilon \int_\Omega |u(t)|^2 \mathrm{d}x + \varepsilon N \mathrm{E}_1 t, \quad t \in 0, T), \tag{46}$$

and

$$\varphi(t) = \int_\Omega |u|^{q_2} \mathrm{d}x + (q_2+1)E(t) + (q_2+1)\int_\Omega \frac{1}{q(x)} |u|^{q(x)} \mathrm{d}x \tag{47}$$

Let B, α_1, α_0, c_* and E_1 be positive auxiliary constants satisfying

$$c_* = \max((2B)^{q_1}, (2B)^{q_2}), B = \sqrt{\kappa} c_*^{\frac{-1}{q_2}} B_1,$$
$$\alpha_1 = \left(\frac{q_1}{q_2} B_1^{-q_2}\right)^{\frac{1}{q_2-2}}$$
$$\alpha(0) = \alpha_0 = \kappa^{\frac{1}{2}} \|\nabla u_0\|_2, \quad \mathrm{E}_1 = \left(\frac{1}{2} - \frac{1}{q_2}\right)\alpha_1^2. \tag{48}$$

The second result of the blow-up is as follows.

Theorem 3 *Assuming that g, $p(.)$, and $q(.)$ satisfy conditions $(H1)-(H3)$ with $q_1 > 2$. Then the local solution of problem (5) under boundary conditions satisfying $E(0) < E_1$, $\kappa^{\frac{1}{2}}\|\nabla u_0\| > \alpha_1$ blows up in finite time T^*, which provide the following estimates*

$$\int_{\varphi(0)}^{+\infty} \frac{\mathrm{d}z}{\mathrm{c}\left(z^\delta + z^{\frac{\delta q_1}{q_2}} + z + z^{\frac{q_1}{q_2}} + 1\right)} \leq T^*$$
$$\leq \frac{1-\alpha}{\alpha \frac{\delta_1}{\delta_2} \mathrm{A}^{\frac{\alpha}{1-\alpha}}(0)},$$

where

$$0 < \alpha \leq \frac{q_1-2}{2q_1}, \tag{49}$$

c, δ, δ_1, and δ_2 are defined in (80), (78), (70) and (73), respectively.

Our desired result depends heavily on the following lemma 9.

Lemma 9 *Let $h: [0, +\infty) \to \mathbb{R}$ be defined by*

$$h(t) := h(\alpha) = \frac{1}{2}\alpha^2 - \frac{B_1^{q_2}}{q_1}\alpha^{q_2}, \tag{50}$$

then h has the following properties:

- **(i)** h is increasing for $0 < \alpha \leq \alpha_1$ and decreasing for $\alpha \geq \alpha_1$,
- **(ii)** $\lim_{\alpha \to +\infty} h(\alpha) = -\infty$ and $h(\alpha_1) = \mathrm{E}_1$,
- **(iii)** $\mathrm{E}(t) \geq h(\alpha(t))$,

where $\alpha(t)$ is given in (46), α_1 and E_1 are given in (49).

Proof. $h(\alpha)$ is continuous and differentiable in $[0, +\infty)$,

$$h'(\alpha) = \alpha\left(1 - B_1^{q_2}\alpha^{q_2-2}(t)\right)\begin{cases} > 0, \alpha \in (0, \alpha_1) \\ < 0, \alpha \in (\alpha_1, +\infty), \end{cases}$$

which means that

$$h(\alpha) \text{isstrictlyincreasingin} (0, \alpha_1),$$
$$h(\alpha) \text{isstrictlydecreasingin} (\alpha_1, +\infty). \tag{51}$$

Then **(i)** follows. Since $q_2 - 2 > 0$, we have $\lim_{\alpha \to +\infty} h(\alpha) = -\infty$. A simple computation yields to $h(\alpha_1) = \mathrm{E}_1$. Then **(ii)** holds valid. By Lemma 3

$$\int_\Omega |u(.)|^{q(x)} \mathrm{d}x \leq \max\left\{\|u\|_{q(.)}^{q_1}, \|u\|_{q(.)}^{q_2}\right\}$$
$$\leq c_* \max\left(\left(\int_{\{\|\nabla u\|_2 \geq 1\}} |\nabla u(t)|^2 \mathrm{d}x\right)^{q_1}, \left(\int_{\{\|\nabla u\|_2 \geq 1\}} |\nabla u(t)|^2 \mathrm{d}x\right)^{q_2}\right)$$
$$= c_* \left(\int_{\{\|\nabla u\|_2 \geq 1\}} |\nabla u(t)|^2 \mathrm{d}x\right)^{q_2}.$$
$$\leq c_* \left(\int_\Omega |\nabla u(t)|^2 \mathrm{d}x\right)^{q_2}$$

Using (H1), (15) and Lemma 3, we have

$$\begin{aligned}\mathrm{E}(t) \geq & \frac{1}{2}\left(1-\int_0^t g(s)\mathrm{d}s\right)\times \\ & \| \nabla u(t) \|_2^2+\frac{1}{2}(g\circ\nabla u)(t) \\ & -\frac{1}{q_1}\int_\Omega |u(t)|^{q(x)}\mathrm{d}x \\ \geq & \frac{1}{2}\kappa \| \nabla u(t) \|_2^2+\frac{1}{2}(g\circ\nabla u)(t) \\ & -\frac{1}{q_1}B^{q_2}c_*\left(\int_\Omega |\nabla u(t)|^2\mathrm{d}x\right)^{q_2} \\ \geq & \frac{1}{2}[\kappa \| \nabla u(t) \|_2^2+(g\circ\nabla u)(t)] \\ & -\frac{B_1^{q_2}}{q_1}[\kappa \| \nabla u(t) \|_2^2+(g\circ\nabla u)(t)]^{\frac{q_2}{2}} \\ = & \frac{1}{2}\alpha^2(t)-\frac{B_1^{q_2}}{q_1}\alpha^{q_2}(t)=h(\alpha(t)).\end{aligned}$$

Then **(iii)** holds true.

Lemma 10 *Assuming the conditions in Theorem 8 are fulfilled, there is a positive constant* $\alpha_2 > \alpha_1$ *such that*

$$\alpha(t) \geq \alpha_2 > \alpha_1, t \geq 0; \tag{52}$$

$$\varrho(u) \geq B_1^{q_2}\alpha_2^{q_2}, \tag{53}$$

where α_1, B_1 and E_1 are given in (49).

Proof. Since $\mathrm{E}(0) < \mathrm{E}_1$ and $h(\alpha)$ is a continuous function, there exist α_2' and α_2 with $\alpha_2' < \alpha_1 < \alpha_2$ such that $h(\alpha_2') = h(\alpha_2) = \mathrm{E}(0)$ which join with Lemma 9 give:

$$h(\alpha_0) \leq \mathrm{E}(0) = h(\alpha_2). \tag{54}$$

From Lemma 9**(i)**, we infer that:

$$\alpha_0 \geq \alpha_2, \tag{55}$$

so (51) holds for $t = 0$.

Now we prove (53), we proceed by contradiction and assume there exist $t^* > 0$ such that $\alpha(t^*) < \alpha_2$, then we distinguish two cases,

Case 1. If $\alpha_2' < \alpha(t^*) < \alpha_2$, we know through Lemma 9 and (52) that

$$h(\alpha(t^*)) > E(0) \geq \mathrm{E}(t^*),$$

which contradicts Lemma 9**(iii)**.

Case 2. If $\alpha(t^*) \leq \alpha_2'$, then $\alpha(t^*) \leq \alpha_2' < \alpha_2$. Setting $\lambda(t) = \alpha(t) - \frac{\alpha_2+\alpha_2'}{2}$, then $\lambda(t)$ is a continuous function, $\lambda(t^*) < 0$ and by applying (56), $\lambda(0) > 0$. Hence, there exists $t_0 \in (0, t^*)$ such that $\lambda(t_0) = 0$, that means $\alpha(t_0) = \frac{\alpha_2+\alpha_2'}{2}$, which signifies

$$h(\alpha(t_0)) > E(0) \geq \mathrm{E}(t_0).$$

This contradicts to Lemma 9**(iii)**, hence (51) follows. By (15), we have

$$\frac{1}{2}\left[\left(1-\int_0^t g(s)\mathrm{d}s\right) \| \nabla u(t) \|_2^2+(g\circ\nabla u)(t)\right] \leq \mathrm{E}(t)+\frac{1}{q_1}\int_\Omega |u(t)|^{q(x)}\mathrm{d}x,$$

which give

$$\frac{1}{q_1}\int_\Omega |u(t)|^{q(x)}\mathrm{d}x \geq \frac{1}{2}\left[\left(1-\int_0^t g(s)\mathrm{d}s\right) \| \nabla u(t) \|_2^2+(g\circ\nabla u)(t)\right] -\mathrm{E}(t) \geq \frac{1}{2}[\kappa \| \nabla u(t) \|_2^2+(g\circ\nabla u)(t)]-\mathrm{E}(0) \geq \frac{1}{2}\alpha_2^2-h(\alpha_2)=\frac{B_1^{q_2}}{q_1}\alpha_2^{q_2},$$

then the second inequality in (54) holds. Let

$$H(t) = \mathrm{E}_1 - \mathrm{E}(t) \text{for} t \geq 0. \tag{56}$$

The following lemma hold

Lemma 11 *Under the assumptions of Theorem 3, if* $0 \leq E(0) < E_1$, *the functional* $H(t)$ *defined in (57) satisfies the following estimates:*

$$0 < H(0) \leq H(t) \leq \int_\Omega \frac{1}{q(x)}|u(t)|^{q(x)}\mathrm{d}x \leq \frac{1}{q_1}\varrho(u),\ t \geq 0. \tag{57}$$

Proof. Lemma 3 ensure that $H(t)$ is nondecreasing in t. Thus

$$H(t) \geq H(0) = \mathrm{E}_1 - \mathrm{E}(0) > 0, \qquad t \geq 0. \tag{58}$$

By (49) and Lemma 10, we have

$$\begin{aligned} & \mathrm{E}_1-\left[\begin{array}{c}\frac{1}{2}\left(1-\int_0^t g(s)\mathrm{d}s\right) \| \nabla u(t) \|_2^2 \\ +\frac{1}{2}(g\circ\nabla u)(t)\end{array}\right] \\ \leq \quad & \mathrm{E}_1-\left[\frac{1}{2}(\kappa \| \nabla u(t) \|_2^2+(g\circ\nabla u)(t))\right] \\ = \quad & \mathrm{E}_1-\frac{1}{2}\alpha^2(t)=-\frac{1}{p_2}\alpha_1^2<0,\end{aligned}$$

for all $t \in 0, T)$, which gives

$$H(t) = \mathrm{E}_1 - \left[\frac{1}{2}\left(1-\int_0^t g(s)\mathrm{d}s\right) \| \nabla u(t) \|_2^2 + \frac{1}{2}(g \circ \nabla u)(t)\right] + \int_\Omega \frac{1}{q(x)}\left|u(t)\right|^{q(x)}\mathrm{d}x \le \int_\Omega \frac{1}{q(x)}\left|u(t)\right|^{q(x)}\mathrm{d}x. \le \frac{1}{q_1}\varrho(u) \quad (59)$$

(57) follows from (58) and (59).

Lemma 12 *Assume that the conditions in Theorem 3 hold, then there exists a positive constant C such that*

$$\| \nabla u(t) \|_2^2 \le \mathrm{C}\varrho(u). \quad (60)$$

for all $t \in 0, T)$.

Proof. By Lemma 10 and $\alpha_2 > \alpha_1$, we have

$$\varrho(u) \ge B_1\alpha_2^{q_2} > B_1\alpha_1^{q_2-2}\alpha_1^2 = \frac{q_1}{q_2}B_1^{1-q_2}\alpha_1^2,$$

which combining with (49) imply

$$\mathrm{E}_1 \le B_1^{1-q_2}\frac{q_2}{q_1}\left(\frac{1}{2}-\frac{1}{q_2}\right)\varrho(u). \quad (61)$$

combining (57), (61) and the definition of $H(t)$, we have

$$\begin{aligned}
&\frac{1}{2}\kappa \| \nabla u(t) \|_2^2 \\
&\le \frac{1}{2}\left(1-\int_0^t g(s)\mathrm{d}s\right) \| \nabla u(t) \|_2^2 \\
&= \mathrm{E}(t) - \frac{1}{2}(g \circ \nabla u)(t) + \int_\Omega \frac{1}{q(x)}|u(t)|^{q(x)}\mathrm{d}x \\
&\le B_1^{1-q_2}\frac{q_2}{q_1}\left(\frac{1}{2}-\frac{1}{q_2}\right)\varrho(u) - H(t) \\
&-\frac{1}{2}(g \circ \nabla u)(t) + \frac{1}{p_1}\varrho(u) \\
&= \left(B_1^{1-q_2}\frac{q_2}{q_1}\left(\frac{1}{2}-\frac{1}{q_2}\right)+\frac{1}{q_1}\right)\varrho(u) - H(t) \\
&\quad -\frac{1}{2}(g \circ \nabla u)(t) \\
&\le \left(B_1^{1-q_2}\frac{q_2}{q_1}\left(\frac{1}{2}-\frac{1}{q_2}\right)+\frac{1}{q_1}\right)\varrho(u). \quad (62)
\end{aligned}$$

Then the desired result, with $C = \frac{\left(B_1^{1-q_2}\frac{q_2}{q_1}\left(1-\frac{2}{q_2}\right)+\frac{2}{q_1}\right)}{\kappa}$.

The proof of Theorem 3 is shown below, based on the lemmas presented above

Proof of Theorem 3 .

Case 1. If $0 \le \mathrm{E}(0) < \mathrm{E}_1$, then by differentiating (47), we get

$$\mathrm{A}'(t) = (1-\alpha)H^{-\alpha}(t)H'(t) + 2\varepsilon\int_\Omega uu_t\mathrm{d}x + N\mathrm{E}_1.$$

Integrating by parts on Ω, recalling Eq (5), we obtain

$$\begin{aligned}
&-\int_0^t g(t-s)|\nabla u|^{p(x)-2}\nabla u(x,s)\nabla u(t)\mathrm{d}s \\
&= \frac{1}{2}g(t)\int_\Omega |\nabla u|^2\mathrm{d}x - \frac{1}{2}(g' \circ \nabla u)(t) \\
&+ \frac{1}{2}\frac{d}{dt}(g \circ \nabla u)(t) \\
&-\frac{1}{2}\frac{d}{dt}\left(\int_0^t g(s)ds\int_\Omega |\nabla u|^2\mathrm{d}x\right)
\end{aligned}$$

P

utting (13) in (17), we get

$$\begin{aligned}
\mathrm{A}'(t) &= (1-\alpha)H^{-\alpha}(t)H'(t) - 2\varepsilon \| \nabla u(t) \|_2^2 \\
&+\varepsilon\int_0^t g(t-s)\int_\Omega \nabla u(t)\Delta_{p(x)}u(s)\mathrm{d}x\mathrm{d}s \\
&+\varepsilon\int_\Omega |u(t)|^{q(x)}\mathrm{d}x + \varepsilon N\mathrm{E}_1 \\
&= (1-\alpha)H^{-\alpha}(t)H'(t) + \varepsilon\|u_t\|_2^2 \\
&- \varepsilon \| \nabla u(t) \|_2^2 \\
&+\varepsilon\int_0^t g(t-s)\int_\Omega \nabla u(t)(|\nabla u|^{p(x)-2}\nabla u(x,s) \\
&\quad - \nabla u(t))\mathrm{d}x\mathrm{d}s \\
&+\varepsilon\int_0^t g(t-s)\int_\Omega |\nabla u(t)|^2\mathrm{d}x\mathrm{d}s \\
&+\varepsilon\int_\Omega |u(t)|^{q(x)}\mathrm{d}x + \varepsilon N\mathrm{E}_1. \quad (63)
\end{aligned}$$

Employing Young inequality, we can obtain:

$$\begin{aligned}
&\left|\int_0^t g(t-s)\int_\Omega \nabla u(t)(|\nabla u|^{p(x)-2}\nabla u(x,s) - \nabla u(t))\mathrm{d}x\mathrm{d}s\right| \\
&\le \tau\int_0^t g(t-s) \| |\nabla u|^{p(x)-2}\nabla u(x,s) - \nabla u(t) \|_2^2\,\mathrm{d}s + \frac{1}{4\tau}\int_0^t g(s)\mathrm{d}s \| \nabla u(t) \|_2^2 \\
&= \tau(g \circ \nabla u)(t) + \frac{1}{4\tau}\int_0^t g(s)\mathrm{d}s \| \nabla u(t) \|_2^2 \\
&\text{for any } \tau > 0 \quad (64)
\end{aligned}$$

By substituting (65) in (64) and then applying (15), we can choose $\tau > 0$ such that $0 < \tau < \frac{q_1}{2}$, we can deduce

$$\begin{aligned}
&\mathrm{A}'(t) \ge \\
&(1-\alpha)H^{-\alpha}(t)H'(t) + \varepsilon\|u_t\|_2^2 - \varepsilon \| \nabla u(t) \|_2^2 \\
&+\int_0^t g(s)\mathrm{d}s \| \nabla u(t) \|_2^2 - \tau\varepsilon(g \circ \nabla u)(t)
\end{aligned}$$

$$-\frac{1}{4\tau}\varepsilon\int_0^t g(s)\mathrm{d}s \times$$
$$\| \nabla u(t) \|_2^2+ \varepsilon p_1(H(t)-\mathrm{E}_1)$$
$$+\frac{q_1}{2}\varepsilon(g\circ\nabla u)(t)$$
$$+\frac{q_1}{2}\varepsilon\left(1-\int_0^t g(s)\mathrm{d}s\right)\| \nabla u(t) \|_2^2$$
$$\geq (1-\alpha)H^{-\alpha}(t)H'(t)+\varepsilon\left(\frac{q_1}{2}-\tau\right)(g\circ\nabla u)(t)$$
$$+\varepsilon(N-p_1)\mathrm{E}_1+\varepsilon p_1 H(t)$$
$$+\varepsilon\left[\left(\frac{q_1}{2}-1\right)-\left(\frac{q_1}{2}-1+\frac{1}{4\tau}\right)\int_0^\infty g(s)\mathrm{d}s\right]$$
$$\| \nabla u(t) \|_2^2. \tag{65}$$

By combining (11) and (66), we obtain:
$$\mathrm{A}'(t)\geq (1-\alpha)H^{-\alpha}(t)H'(t)+a_1\varepsilon(g\circ\nabla u)(t)$$
$$+a_2\varepsilon \| \nabla u(t) \|_2^2+\varepsilon(N-q_1)\mathrm{E}_1+\varepsilon q_1 H(t), \tag{66}$$

Where
$$a_1=\left(\frac{q_1}{2}-\tau\right)>0,$$
$$a_2=\left(\frac{q_1}{2}-1\right)-\left(\frac{q_1}{2}-1+\frac{1}{4\tau}\right)\int_0^\infty g(s)\mathrm{d}s>0.$$
obviously $H(t)\geq$
$$\mathrm{E}_1-\frac{1}{2}\| \nabla u(t) \|_2^2-\frac{1}{2}(g\circ\nabla u)(t)+\frac{1}{q_2}\varrho(u), \tag{67}$$

Using (67) in (66) and rewriting q_1 as $q_1=q_1-2a_3+2a_3$,with $0<a_3<min\left(a_1,a_2,\frac{q_1}{2}\right)$ produce
$$\mathrm{A}'(t)\geq(1-\alpha)H^{-\alpha}(t)H'(t)$$
$$+\varepsilon(a_1-a_3)(g\circ\nabla u)(t)+\varepsilon(a_2-a_3)\| \nabla u(t) \|_2^2$$
$$+\varepsilon\big(N-(q_1-2a_3)\big)\mathrm{E}_1+\varepsilon(q_1-$$
$$2a_3)H(t)+\varepsilon\frac{2}{q_2}a_3\varrho(u).$$

At this point, we choose N that is sufficiently large so that
$$N-(q_1-2a_3)>0.$$

After determining a fixed value for N, we select a small enough ε to meet the necessary conditions
$$\mathrm{A}(0)=H^{1-\alpha}(0)+\varepsilon\int_\Omega |u_0|^2\mathrm{d}x>0,$$
$$since\ H(0)>0. \tag{68}$$

Then there is a constant δ_1 satisfying
$$0<\delta_1\leq\min\begin{Bmatrix}\frac{q_1}{2}+1-a_3, a_1-a_3\\ ,\frac{2}{q_2}a_3, q_1-2a_3\end{Bmatrix}, \tag{69}$$

and
$$\mathrm{A}'(t)\geq\delta_1\varepsilon[\|u_t\|_2^2+(g\circ\nabla u)(t)+$$
$$\| \nabla u(t) \|_2^2+H(t)+\varrho(u)], \tag{70}$$

which combining with (69) infer
$$\mathrm{A}(t)\geq\mathrm{A}(0)>0,\forall t\in 0,T).$$
Choosing $\varepsilon>0$ such that $\varepsilon<\frac{1}{T}\left(\frac{\alpha_2}{\alpha_1}\right)^{q_2}$, recalling Lemma 10 and then, we have
$$|\varepsilon N\mathrm{E}_1 T|^{\frac{1}{1-\alpha}}\leq\left(\frac{\alpha_2}{\alpha_1}\right)^{p_2}N\mathrm{E}_1\leq\frac{N\mathrm{E}_1}{B_1\alpha_1^{p_2}}\varrho(u). \tag{71}$$

By utilizing Holder's and Young's inequalities, and keeping in mind the embedding $L^{q(\cdot)}(\Omega)\hookrightarrow L^2(\Omega)$, it can be observed that:
$$\left|\int_\Omega |u|^2\mathrm{d}x\right|^{\frac{1}{1-\alpha}}\leq\| u \|_2^{\frac{1}{1-\alpha}}\|u\|_2^{\frac{1}{1-\alpha}}$$
$$\leq\quad(1+|\Omega|)^{\frac{q_1-2}{q_1(1-\alpha)}}\| u \|_{q(x)}^{\frac{1}{1-\alpha}}\|u\|_2^{\frac{1}{1-\alpha}}$$
$$\leq\quad c_4\left(\|u\|_2^2+\| u \|_{q(x)}^{\frac{2}{1-2\alpha}}\right)$$
$$\leq\quad c_4\|\nabla u\|_2^2+c_4\max\begin{Bmatrix}\left(\left(\int_\Omega |u|^{q(x)}\mathrm{d}x\right)^{\frac{2}{(1-2\alpha)q_1}},\right.\\ \left(\left(\int_\Omega |u|^{q(x)}\mathrm{d}x\right)^{\frac{2}{(1-2\alpha)q_2}}\right.\end{Bmatrix}$$
$$\leq\quad(c_4\mathrm{C}+c_5)\int_\Omega |u|^{q(x)}\mathrm{d}x, \tag{72}$$

where:
$$c_4=(1+|\Omega|)^{\frac{q_1-2}{q_1(1-\alpha)}}$$
$$c_5=c_4\max\begin{Bmatrix}(q_1H(0))^{\frac{2}{(1-2\alpha)q_1}-1}\\ ,(q_1H(0))^{\frac{2}{(1-2\alpha)q_2}-1}\end{Bmatrix},$$
C as in (61).

Let δ_2 be a positive constant such:
$$\delta_2=2^{1/(1-\alpha)+1}\max\left(1,\varepsilon^{\frac{1}{1-\alpha}},c_4\mathrm{C},c_5+\frac{N\mathrm{E}_1}{B_1\alpha_1^{p_2}}\right). \tag{73}$$

Using (47), (72), (73), and Cauchy-Schwarz's inequality,
$$\mathrm{A}^{\frac{1}{1-\alpha}}(t)\leq\quad 2^{1/(1-\alpha)+1}\begin{pmatrix}H(t)+\varepsilon^{\frac{1}{1-\alpha}}\left|\int_\Omega |u|^2\mathrm{d}x\right|^{\frac{1}{1-\alpha}}\\ +\varepsilon^{\frac{1}{1-\alpha}}(N\mathrm{E}_1T)^{\frac{1}{1-\alpha}}\end{pmatrix}$$
$$\leq\qquad\delta_2[H(t)+\varrho(u)]. \tag{74}$$

We join (72) and (73) with (71), it result

$$\mathrm{A}'(t) \geq \frac{\delta_1}{\delta_2}\mathrm{A}^{\frac{1}{1-\alpha}}(t), \text{for all } t \geq 0, \quad (75)$$

After integrating (65) over the interval $(0,t)$, we can deduce that:

$$\mathrm{A}^{\frac{\alpha}{1-\alpha}}(t) \geq \frac{1}{\mathrm{A}^{\frac{\alpha}{1-\alpha}}(0)-\frac{\alpha}{1-\alpha}\frac{\delta_1}{\delta_2}t}. \quad (76)$$

Consequently, $\mathrm{A}(t)$ blows up in a finite time $\hat{T}$,

$$\hat{T} \leq \frac{1-\alpha}{\alpha\frac{\delta_1}{\delta_2}\mathrm{A}^{\frac{\alpha}{1-\alpha}}(0)}.$$

Since $\mathrm{A}(0) > 0$, (77) shows that $\lim_{t\to T^*}\mathrm{A}(t) = \infty$, where $T^* = \frac{1-\alpha}{\alpha\frac{\delta_1}{\delta_2}\mathrm{A}^{\frac{\alpha}{1-\alpha}}(0)}$. This ends the proof.

Case 2. If $E(0) < 0$, we can use Lemma 12 by setting $H(t) = -E(t)$ to obtain a result similar to Lemma 12. Before this, we had $0 < -E(0) = H(0) \leq H(t)$ and $H(t) \leq \frac{1}{q_1}\varrho(u)$. By taking $N = 0$ in (47) and using the same approach as in **Case1**, we can reach our desired outcome.

We still need to determine an upper bound of the blowing-up time, we can calculate it as follows; Using (15), (16) and Lemma 3, the derivative of (48) give:

$$\begin{aligned}
\varphi'(t) &= q_2\int_\Omega |u|^{q_2-2}uu_t\mathrm{d}x + (q_2+1)E'(t) \\
&+\frac{q_2+1}{q_1}\int_\Omega |u|^{q(x)-2}uu_t\mathrm{d}x \leq q_2\int_\Omega |u|^{2q_2-2}\mathrm{d}x \\
&+q_2\int_\Omega |u_t|^2\mathrm{d}x + (q_2+1)E'(t) \\
&+(q_2+1)^2\int_\Omega |u|^{2q(x)-2} + \int_\Omega |u_t|^2\mathrm{d}x \\
&\leq q_2\int_\Omega |u|^{2q_2-2}\mathrm{d}x \\
&+ (q_2+1)^2\left(\begin{array}{c}\int_\Omega |u|^{2q_1-2}\mathrm{d}x \\ +\int_\Omega |u|^{2q_2-2}\mathrm{d}x\end{array}\right) \\
&\leq q_2\int_\Omega |u|^{2q_2-2}\mathrm{d}x \\
&+(q_2+1)^2\left(\frac{2q_1-2}{2q_2-2}\int_\Omega |u|^{2q_2-2}\mathrm{d}x \right.\\
&\left.+\int_\Omega |u|^{2q_2-2}\mathrm{d}x + \frac{q_2-q_1}{q_2-1}\right) \\
&= \left(q_2 + \frac{q_1+q_2-2}{q_2-1}(q_2+1)^2\right)\int_\Omega |u|^{2q_2-2}\mathrm{d}x \\
&+(q_2+1)^2\frac{q_2-q_1}{q_2-1} \quad (77)
\end{aligned}$$

To estimate the term on the right-hand side of the inequality above, we need to analyze the following three scenarios

Case.1. $n < 3$. The inequality embedding has led us to

$$\begin{aligned}
\int_\Omega |u|^{2q_2-2}\mathrm{d}x &\leq \hat{B}^{2q_2-2} \parallel \nabla u \parallel_2^{2(q_2-1)} \\
&\leq \hat{B}^{2q_2-2}\left(\parallel \nabla u \parallel_2^2 + \int_\Omega |u|^{q_2}\mathrm{d}x\right)^{q_2-1}.
\end{aligned}$$

Case.2. $2 < q_2 < \frac{2n}{n-1}$, $n \geq 3$. Using Hölder's and embedding inequalities, we have

$$\int_\Omega |u|^{2q_2-2}\mathrm{d}x = \int_\Omega |u|^{2q_2-4}u^2\mathrm{d}x$$

$$\begin{aligned}
&\leq \left(\int_\Omega |u|^{n(q_2-2)}\mathrm{d}x\right)^{\frac{2}{n}}\left(\int_\Omega |u|^{\frac{2n}{n-2}}\mathrm{d}x\right)^{1-\frac{2}{n}} \\
&\leq |\Omega|^{\frac{2}{n}-\frac{2(q_2-2)}{q_2}} \parallel u \parallel_{\frac{2n}{n-2}}^2 \left(\int_\Omega |u|^{q_2}\mathrm{d}x\right)^{\frac{2(q_2-2)}{q_2}} \\
&\leq B_1^2|\Omega|^{\frac{2}{n}-\frac{2(q_2-2)}{q_2}} \parallel \nabla u \parallel_2^2 \left(\int_\Omega |u|^{q_2}\mathrm{d}x\right)^{\frac{2(q_2-2)}{q_2}} \\
&\leq B_1^2|\Omega|^{\frac{2}{n}-\frac{2(q_2-2)}{q_2}}\left(\parallel \nabla u \parallel_2^2 + \int_\Omega |u|^{q_2}\mathrm{d}x\right)^{\frac{3q_2-4}{q_2}}.
\end{aligned}$$

Case.3. $\frac{2n}{n-1} \leq q_2 < \frac{2n-2}{n-2}$, $n \geq 3$. Through the simulation of **Case 2**, we have obtained the following results.

$$\begin{aligned}
\int_\Omega |u|^{2q_2-2}\mathrm{d}x &= \int_\Omega |u|^{2q_2-4}u^2\mathrm{d}x \\
&\leq B^2|\Omega|^{\frac{2}{n}-\frac{2(q_2-2)}{p_2}}\left(\parallel \nabla u \parallel_2^2 + \int_\Omega |u|^{q_2}\mathrm{d}x\right)^{\frac{3q_2-4}{q_2}}.
\end{aligned}$$

Hence, we get :

$$\int_\Omega |u|^{2q_2-2}\mathrm{d}x \leq c^*\left(\int_\Omega |u|^{q_2}\mathrm{d}x + \int_\Omega |\nabla u|^2\mathrm{d}x\right)^{\delta}. \quad (78)$$

Where $\delta > 1$ equals $q_2 - 1, \frac{3q_2-4}{q_2}$ for the cases mentioned above.

Using (48) and $\mathrm{E}(t)$ definition, we can see that:

$$\int_\Omega |\nabla u|^2 \mathrm{d}x \le \frac{2}{\kappa}\mathrm{E}(0) + \frac{2}{\kappa}\frac{1}{q_1}\int_\Omega |u|^{q(x)}\mathrm{d}x \le \frac{2}{\kappa}\mathrm{E}(0) + \frac{2}{\kappa}\frac{c_6}{q_1}\left(\varphi(t) + \varphi^{\frac{q_1}{q_2}}(t)\right) \tag{79}$$

where $c_6 = (1+|\Omega|)^{\frac{q_2-q_1}{q_1}}$. Joining (78)-(79), taking into account that $E(t) + \int_\Omega \frac{1}{q(x)}|u|^{q(x)}\mathrm{d}x \ge 0$, which means $\varphi(t) \ge \int_\Omega |u|^{q_2}\mathrm{d}x$; we get

$$\varphi'(t) \le c^*\left(q_2 + \frac{q_1+q_2-2}{q_2-1}(q_2+1)^2\right) \times \left(\int_\Omega \left|u\right|^{q_2}\mathrm{d}x + \int_\Omega \left|\nabla u\right|^2 \mathrm{d}x\right)^\delta + (q_2+1)^2\frac{q_2-q_1}{q_2-1}$$

$$\le c^*\left(q_2 + \frac{q_1+q_2-2}{q_2-1}(q_2+1)^2\right) \times \left(\frac{2}{\kappa}\mathrm{E}(0) + \left(1+\frac{2c_6}{\kappa q_1}\right)\varphi(t) + \frac{2}{\kappa}\frac{c_6}{q_1}\varphi^{\frac{q_1}{q_2}}(t)\right)^\delta + (q_2+1)^2\frac{q_2-q_1}{q_2-1}$$

$$\le \mathrm{c}\left(\varphi^\delta(t) + \varphi^{\frac{q_1}{q_2}\delta}(t) + \varphi(t) + \varphi^{\frac{q_1}{q_2}}(t) + 1\right),$$

Where

$$\mathrm{C}=2^{\delta-1}\max\left(\left(c^*\left(q_2+\frac{q_1+q_2-2}{q_2-1}(q_2+1)^2\right)\frac{2}{\kappa}\mathrm{E}(0)\right)^\delta, \left(c^*\left(q_2+\frac{q_1+q_2-2}{q_2-1}(q_2+1)^2\right)\left(1+\frac{2c_6}{\kappa q_1}\right)^\delta\right), .\left(\frac{2c_6}{\kappa q_1}\right)^\delta, (q_2+1)^2\frac{q_2-q_1}{q_2-1}\right) \tag{80}$$

By the definition of T^*;

$$\lim_{t\to T^*}\int_\Omega |u|^{q_2}\mathrm{d}x = +\infty,$$

we obtain that

$$\int_{\varphi(0)}^{+\infty}\frac{\mathrm{d}z}{\mathrm{c}\left(z^\delta + z^{\delta\frac{q_1}{q_2}} + z + z^{\frac{q_1}{q_2}} + 1\right)} \le T^*.$$

The proof is complete.

Acknowledgements:
The authors would like to thank the anonymous referee(s) and the handling editor(s) for their kind comments.

4 General Comments and Issues

This paper is devoted to studying new class of mixed pseudo parabolic p(.)-Laplacian type equation with viscoelastic term on a bounded and regular domain (5), which that equations appear in dynamics of viscoelastic structures, besides the most common case is the evolution p(.)-Laplace equation, where the exponent p(.) is dependent on the external electromagnetic field.

We provide a blow-up threshold resulting in a finite time of solutions, yielding a new blow-up criterion. The upper bound estimate of the blow-up time is also derived. We show that blow-up may occur under appropriate smallness conditions on the initial datum, in which case we also establish a lower bound estimate.

The significance of this study is that it will determine a new criterion and upper and lower bounds estimate of the blow-up time, which have not stayed vocalised in either case for the value of q(.) (constant or variable) for this type of equation.

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Contribution of Individual Authors to the Creation of a Scientific Article (Ghostwriting Policy)

Touil Nadji and Abita Rahmoune wrote the main manuscript text. All authors reviewed the manuscript.

Sources of Funding for Research Presented in a Scientific Article or Scientific Article Itself

No funding was received for conducting this study.

Conflict of Interest

The authors have no conflicts of interest to declare.