New soliton solutions of the Burgers equation with additional time-dependent variable coefficient

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Abstract: In this article, we use the functional variational method to solve the Burgers equation with an additional time-dependent variable coefficient. The main advantage of the proposed method over other methods is that it allows to obtain more new solutions of the equation. Among the solutions obtained, new soliton solutions should be noted, which are of great importance for revealing the internal mechanism of physical phenomena. Three-dimensional graphs of solutions are constructed using the mathematical program Matlab. For a better understanding of the physical properties of some of the resulting solutions, their graphical representations are shown. This method is effective for finding exact solutions to many other similar wave equations.

Key-Words: Burgers equation, nonlinear evolution equations, variable coefficient, functional variable method, soliton solutions, ordinary differential equation.

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1 Introduction

The Burger equation is a fundamental partial differential equation in fluid mechanics. It is also a very important model used in several areas of applied mathematics such as heat conduction, acoustic waves, gas dynamics and traffic flow[1]. In[2], the Burgers equation was first introduced in 1915. This equation is expressed in the following basic form

$$u_t + uu_x - u_{xx} = 0.$$

It was later proposed by Burger as one of the classes of equations describing mathematical models of turbulence[3]. In 1951, Cole studied the Burgers equation and gave a theoretical solution to this equation[4].

In[5], the Burgers equation with variable coefficient

$$u_t + \alpha(t)uu_x - \beta(t)u_{xx} = 0$$

is investigated, where $\alpha(t)$ and $\beta(t)$ are given continuous differentiable functions. These variable coefficients of Burgers equation with the nonlinear $\alpha(t)$ and dispersion $\beta(t)$ can model propagation of a long shock-wave in a two-layer shallow liquid. In addition, this equation appears in ion acoustic waves in plasma[6], traffic flow[7, 8], dynamics of soil water[9], shock formation in elastic gas[10], turbulence in fluid dynamics[11, 12].

The shock wave, multi-shock solitons and voltera integral type wave solutions[13, 14] for the Burgers equation with variable coefficient are obtained by using Cole-Hopf transformation[15]. Zhang obtained not trivial and time dependent conservation laws with the presence of admissible transformation and Lie symmetry for the Burgers equation with variable coefficient[16]. Christov obtained the kink or shock type traveling waves solution of the Burgers equation with variable coefficient by using the Crank-Nicolson numerical scheme.

To find exact solutions to nonlinear evolution equations, many direct methods are used. For example: tanh-function method[17], functional variable method[18], Hirota method[19], Backlund transform method[20], G/G' expansion method[21] and extended tanh-method[22].

In this paper, we consider the Burgers equation with additional time-dependent variable coefficient

$$u_t + h_1(t)uu_x - h_2(t)u_{xx} + \omega(t)u_x = 0, \quad (1)$$

where $u(x,t)$ is an unknown function, $x \in R$, $t \geq 0$, $h_1(t) \neq 0$, $h_2(t) \neq 0$ and $\omega(t) \neq 0$ are given continuous differentiable functions.

The main aim of this paper is to find the exact soliton solutions of the Eq.1 via functional variable method. The main advantage of the proposed method over other methods is that it allows to obtain more new solutions of the equation. Among the solutions obtained, new soliton solutions should be noted, which are of great importance for revealing the internal mechanism of physical phenomena. Three-dimensional graphs of solutions are constructed using the mathematical program Matlab. For a better understanding of the physical properties of some of the resulting solutions, their graphical representations are shown. This method is effective for finding exact solutions to many other similar wave equations.

2 Description of the method

The basic idea of the functional variable method proposed in[23]. Let us consider the nonlinear differential equation with independent variables x, y, z, t and a dependent variable u

$$P(u,\, u_t,\, u_x,\, u_y,\, u_z,\, u_{xy},\, u_{yz},\, u_{xz},\, ...)=0, \quad (2)$$

where P is a polynomial in $u(t,\, x,\, y,\, z,\, ...)$ and its partial derivatives.

The following transformation

$$\xi=\sum_{i=0}^{p}\alpha_i\chi_i+\delta, \quad (3)$$

is used for the new wave variable.

Now, we can introduce the following transformation for the travelling wave solution of Eq.2

$$u(\chi_0,\ \chi_1,\, ...)=u(\xi), \quad (4)$$

and the chain rule

$$\frac{\partial u}{\partial\chi_i}=\alpha_i\frac{du}{d\xi},\quad \frac{\partial^2u}{\partial\chi_i\partial\chi_j}=\alpha_i\alpha_j\frac{d^2u}{d\xi^2},.... \quad (5)$$

After this transformation, the Eq.2 is transformed into an ordinary differential equation(ODE) of the form

$$Q(u,\, u',u'',u''',...)=0, \quad (6)$$

where Q is a polynomial in $u(\xi)$ and its total derivatives, while $u'=\frac{du}{d\xi}$.

Let us make a transformation in which the unknown function u is considered as a functional variable of the form

$$u'=F(u), \quad (7)$$

then, the solution can be found by the relation

$$\int\frac{du}{F(u)}=\xi+C, \quad (8)$$

Some successive differentiations of u in terms of F are given as

$$\begin{aligned}&u''=\tfrac{dF(u)}{du}\tfrac{du}{d\xi}=\tfrac{dF(u)}{du}F(u)=\tfrac{1}{2}\tfrac{d(F^2(u))}{du},\\&u'''=\tfrac{1}{2}\tfrac{d^2(F^2(u))}{du^2}\sqrt{F^2(u)},\\&u^{(IV)}=\tfrac{1}{2}\left[\tfrac{d^3(F^2(u))}{du^3}F^2(u)+\tfrac{d^2(F^2(u))}{du^2}\tfrac{d(F^2(u))}{du}\right],\\&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\end{aligned} \quad (9)$$

The Eq.6 can be reduced in terms of u, F and its derivatives upon using the expressions of Eq.7 and Eq.9 into Eq.2 gives

$$R(u,\frac{dF(u)}{du},\frac{d^2F(u)}{du^2},\frac{d^3F(u)}{du^3},...)=0. \quad (10)$$

After integration, Eq.10 provides the expression of $F(u)$ and this, together with Eq.7, give appropriate solutions to the being considered problem.

3 Algorithm for finding solutions

Let us consider the Burgers equation with additional time-dependent variable coefficient

$$u_t+h_1(t)uu_x-h_2(t)u_{xx}+\omega(t)u_x=0. \quad (11)$$

This equation appears in many physical problems, including the behavior of waves in nonlinear optics, plasma or liquids, water waves, ion-acoustic waves in collisions, and less commonly in plasma. The first term u_t represents the evolution term and the second represents the dispersion term.

The wave variable

$$\xi=a(t)+b(t)x \quad (12)$$

will convert Eq.11 to the following form

$$u'_t+(a_t(t)+b_t(t))\,u'_\xi+h_1(t)b(t)uu'_\xi-$$
$$-h_2(t)b^2(t)u''_\xi+\omega(t)b(t)u'_\xi=0, \quad (13)$$

where $a(t)$ and $b(t)$ are an unknown time-dependent functions, to be determined later.

Let $a(t)$, $b(t)$, $h_1(t)$, $h_2(t)$ and $\omega(t)$ are constant functions in Eq.11, that is, $a(t)=a$, $b(t)=b$, $h_1(t)=h_1$, $h_2(t)=h_2$ and $\omega(t)=\omega$. In this case, the following transformation

$$\xi=a+bx. \quad (14)$$

is used in Eq.11 to get an ordinary differential equation of the form

$$h_1buu'_\xi-h_2b^2u''_\xi+\omega bu'_\xi=0. \quad (15)$$

Integrating Eq.15, we have

$$\frac{h_1 b}{2}u^2 - h_2 b^2 u'_\xi + \omega b u = 0. \tag{16}$$

From Eq.16 follows the expression for the function u'_ξ:

$$u'_\xi = su + nu^2, \tag{17}$$

where $s = \frac{\omega}{h_2 b}$, $n = \frac{h_1}{2h_2 b}$.

Now we move on to the question of determining unknown functions $a(t)$ and $b(t)$. To do this, we search the solution of Eq.11 in the form

$$u(t,\xi) = \sum_{k=0}^{m} q_k(t)\Phi^k(\xi), \tag{18}$$

where Φ satisfies Eq.19

$$\Phi' = \lambda\Phi + \mu\Phi^2, \tag{19}$$

where λ and μ are free parameters and m is an undetermined integer and $q_k(t)$ are coefficients to be determined later.

One of the most useful ways to obtain the m parameter is in Eq.18 is the homogeneous balance method. Substituting Eq.18 into Eq.13 and by making balance between the linear term u''_ξ and the nonlinear term uu'_ξ, we will get that $2m+1 = m+2$, this in turn gives $m = 1$, and the solution Eq.18 takes the form

$$u(t,\xi) = q_0(t) + q_1(t)\Phi(\xi). \tag{20}$$

Now, we substitute Eq.20 into Eq.13 and set each coefficient of $\Phi^k(\Phi')^p$ ($k = 0,\,1,\,2$ and $p = 0,\,1$) to zero to obtain a set of algebraic equations for $q_0(t)$, $q_1(t)$, $a(t)$ and $b(t)$:

$$\begin{cases} q_{0_t}(t) = 0, q_{1_t}(t) = 0, \\ h_1(t)b(t)q_1(t) - 2\mu h_2(t)b^2(t) = 0, \\ a_t(t) + b_t(t)x + h_1(t)b(t)q_0(t) - \\ -\lambda h_2(t)b^2(t) + \omega(t)b(t) = 0. \end{cases} \tag{21}$$

Solving the system of algebraic equations, we obtain

$$q_0(t) = const, \quad q_1(t) = const, \tag{22}$$

$$b(t) = \frac{q_1(t)}{2\mu}\frac{h_1(t)}{h_2(t)}, a(t) = \int_0^t (\lambda h_2(\tau)b^2(\tau) -$$

$$-b_\tau(\tau)x + h_1(\tau)b(\tau)q_0(\tau) - \omega(\tau)b(\tau))d\tau. \tag{23}$$

Putting determined parameters $q_0(t)$, $q_1(t)$, $a(t)$ and $b(t)$ into Eq.20 and taking into account Eq.12 and Eq.19 we get the soliton solution of Eq.1:

$$u(x,t) = Acth\left(\frac{a(t)+b(t)x}{2}\right), \tag{24}$$

where $b(t) = \frac{q_1(t)h_1(t)}{2\mu h_2(t)}$, $q_0(t) = -A = const$, $q_1(t) = -2A = const$, $A > 0$,
$a(t) = \int_0^t (\lambda h_2(\tau)b^2(\tau) - b_\tau(\tau)x + h_1(\tau)b(\tau)q_0(\tau) -$
$-\omega(\tau)b(\tau))d\tau$.

4 Example

We illustrate the application of algorithm to solving the eq.11. Exact soliton solution of the eq.11 can be defined explicitly for exact values of $h_1(t) = t$, $h_2(t) = t$, $\omega(t) = -2t$, $\lambda = 1$, $\mu = 1$. In this case, the soliton solution of the eq.11 has the form

$$u(x,t) = cth\left(\frac{t^2+x}{2}\right). \tag{25}$$

This solution of Eq.11 is verified and 3D plots of the solutions obtained using Matlab mathematical software are shown. Soliton wave solutions are an important class of solutions to nonlinear partial differential equations, since many nonlinear partial differential equations have been found to have different soliton wave solutions(Figure 1.).

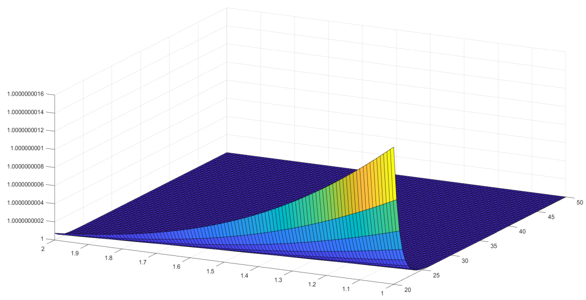

Figure 1: Soliton wave solution of the eq.11 for $h_1(t) = t$, $h_2(t) = t$, $\omega(t) = -2t$, $\lambda = 1$, $\mu = 1$.

5 Conclusion

In this work, the functional variational method is successfully used to solve the Burgers equation with additional variables and time-dependent coefficients. Soliton solutions of this equation and three-dimensional graphs of the resulting solutions are found. The main advantage of the proposed method over other methods is that it gives more accurate traveling wave solutions. It is concluded that exact solutions are of great importance for revealing the internal mechanism of physical phenomena.

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Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

The original contributions presented in the study are included in the article/supplementary material, further inquiries can be directed to the corresponding author/s.

Conflict of Interest

The author declare that the research was conducted in the absence of any commercial or financial relationships that could be construed as a potential conflict of interest.

Contribution of individual authors to the creation of a scientific article (ghostwriting policy)

Bazar Babajanov and Fakhriddin Abdikarimov conceived of the presented idea. Bazar Babajanov developed the theory and performed the computations. Fakhriddin Abdikarimov verified the analytical methods. Both authors discussed the results and contributed to the final manuscript. Both authors contributed to the article and approved the submitted version.

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