Time Direct and Time Inverse Problems for Wave Energy and Steel Quenching Models, Solved Exactly and Approximately

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Abstract: - In this paper we develop mathematical models for 3-D hyperbolic heat equation. This equation describes the mathematical models for steel quenching in highly agitated water and wave power of energy by waves of ocean surface. 1-D model is obtained from 3-D problem by conservative averaging method. We construct their exact analytical solutions by the Green function method. We solve problem with constant initial conditions in the form of triple or one series. We solve time reverse problems for the determination of the initial heat flux. Approximate solution of 1-D hyperbolic heat equation is obtained by finite difference method, including solution with exact spectrum. The conservative averaging method till now was used with polynomial approximation, but here we use hyperbolic approximation.

Key-Words: - Hyperbolic Equation, Ocean Energy, Green Function, Exact Solution, Inverse Problem, Finite difference, Spectral problem, Exact spectrum, Ordinary differential system, Integral spline.

1 Introduction

The wave power is the transport of energy by ocean surface waves. Wave-power generation currently is not a widely employed commercial technology, although there have been attempts to use it since at least 1890 [1]. The production of electricity in sea or ocean from wave or wind energy is an interesting idea [2], [3], [14]. See also the papers [7], [8] and [32].

Contrary to traditional method the intensive quenching process uses environmentally friendly highly agitated water or low concentration of water/mineral salt solutions [4] - [6]. Traditionally for the mathematical description of the steel quenching the classical heat conduction equation is used. We have proposed to use hyperbolic heat equation [9]-[13], [15], [16] for more realistic description of the intensive quenching (IQ) process. Hyperbolic heat equation is widely used in different fields [29], [30].

The idea of the usage of hyperbolic heat equation can be easily transferred to completely different sector of application - to the generation of electricity in sea or ocean by the usage of wave energy. Here we describe the equipment in development of both - in time as well as in spatial arrangement of equipment using the three-dimensional hyperbolic heat equation. Therefore it is important to examine not only the development of equipment in time, but also the movement of its different components. We consider three-dimensional and one-dimensional statement for non-homogeneous equation with non-homogeneous boundary conditions.

Naturally we would like to use physically real parameters for numerical calculations, and that is why we choose parameters from the intensive steel quenching process. The problem for wave power energy is only in the investigation phase, without concrete physical parameters [2], [3] and [14].

We solve direct and time inverse problems with Green function methods. In the case of intensive steel quenching process constant initial conditions are very natural. In this paper we have solved 3-D problem as triple series solution and 1-D problem as single series solution.

2 3-D Problem for Wave Power

2.1 Problem Formulation
Here we offer to consider the description of system in time and space. We consider the following partial differential equation:

$$\tau \frac{\partial^2 \tilde{U}}{\partial t^2} + \frac{\partial \tilde{U}}{\partial t} = a^2 \left(\frac{\partial^2 \tilde{U}}{\partial x^2} + \frac{\partial^2 \tilde{U}}{\partial y^2} + \frac{\partial^2 \tilde{U}}{\partial z^2} \right) -$$

$$\tilde{C}\tilde{U} + \tilde{F}(x, y, z, t), a^2 = \frac{k}{c\rho}, \qquad (*)$$

$$x \in (0, l), y \in (0, b), z \in (0, w), t \in (0, T].$$

Here c is specific heat capacity, k - heat conductivity coefficient, ρ - density, τ - relaxation

time. The source term $\tilde{F}(x,y,z,t)$ can be from different parts of the same device or outer source, for example, wind source. As the first step we use well known substitution:

$$\tilde{U}(x,y,z,t) = \exp\left(-\frac{t}{2\tau}\right)U(x,y,z,t).$$

After transformation partial differential equation can be written in the form:

$$\frac{\partial^2 U}{\partial t^2} = a_\tau^2\left(\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2}\right) - CU +$$

$$F(x,y,z,t), C = \left(\tilde{C} - \frac{1}{4\tau}\right)\frac{1}{\tau}, a_\tau^2 = \frac{a^2}{\tau}, \quad (1)$$

$$F(x,y,z,t) = \exp\left(\frac{t}{2\tau}\right)\frac{1}{\tau}\tilde{F}(x,y,z,t).$$

The initial conditions for the function $U(x,y,z,t)$ are assumed in following form:

$$U\big|_{t=0} = U_0(x,y,z), \quad (2)$$

$$\frac{\partial U}{\partial t}\bigg|_{t=0} = U_1(x,y,z). \quad (3)$$

From the practical point of view in the steel quenching the condition (3) can be unrealistic. As additional condition we assume that the temperature distribution and the heat fluxes distribution at the end of process are given (known):

$$U\big|_{t=T} = U_T(x,y,z), \quad (4)$$

$$\frac{\partial U}{\partial t}\bigg|_{t=T} = U_T^1(x,y,z). \quad (5)$$

In the case of wave energy we can assume different non-homogeneous boundary conditions:

$$\left(\frac{\partial U}{\partial x} - \alpha_1 U\right)\bigg|_{x=0} = g_1(y,z,t), \alpha_i = \frac{h_i}{k}, \quad (6)$$

$$\left(\frac{\partial U}{\partial y} - \alpha_2 U\right)\bigg|_{y=0} = g_2(x,z,t), \quad (7)$$

$$\left(\frac{\partial U}{\partial z} - \alpha_3 U\right)\bigg|_{z=0} = g_3(x,y,t), i=1,2,3. \quad (8)$$

$$\left(\frac{\partial U}{\partial x} + \beta_1 U\right)\bigg|_{x=l} = g_4(y,z,t), \beta_i = \frac{h_i}{k}, \quad (9)$$

$$\left(\frac{\partial U}{\partial y} + \beta_2 U\right)\bigg|_{y=b} = g_5(x,z,t), \quad (10)$$

$$\left(\frac{\partial U}{\partial z} + \beta_3 U\right)\bigg|_{z=w} = g_6(x,y,t), i=1,2,3. \quad (11)$$

Here h_i is heat exchange coefficient. In fact it is possible to look at other types of boundary conditions: first (Dirichlet) and second (Neumann) type.

2.2. Solution of 3-D Problem

Firstly we assume that we have non-homogeneous Klein-Gordon equation-with source term: $C \geq 0$. The solution in three-dimensional problem is in following form:

$$U(x,y,z,t) = H(x,y,z,t) +$$

$$\int_0^l d\xi \int_0^w d\varsigma \int_0^b U_1(\xi,\eta,\varsigma)G(x,y,z,\xi,\eta,\varsigma,t)d\eta + \quad (12)$$

$$\int_0^l d\xi \int_0^w d\varsigma \int_0^b U_0(\xi,\eta,\varsigma)\frac{\partial}{\partial t}G(x,y,z,\xi,\eta,\varsigma,t)d\eta$$

Here

$$H(x,y,z,t) =$$

$$-a_\tau^2\int_0^t d\tau \int_0^w d\varsigma \int_0^b g_1(\eta,\varsigma,\tau)G(x,y,z,0,\eta,\varsigma,t-\tau)d\eta$$

$$-a_\tau^2\int_0^t d\tau \int_0^w d\varsigma \int_0^l g_2(\xi,\varsigma,\tau)G(x,y,z,\xi,0,\varsigma,t-\tau)d\xi$$

$$-a_\tau^2\int_0^t d\tau \int_0^b d\eta \int_0^l g_3(\xi,\eta,\tau)G(x,y,z,\xi,\eta,0,t-\tau)d\xi$$

$$+a_\tau^2\int_0^t d\tau \int_0^w d\varsigma \int_0^b g_4(\eta,\varsigma,\tau)G(x,y,z,l,\eta,\varsigma,t-\tau)d\eta$$

$$+a_\tau^2\int_0^t d\tau \int_0^w d\varsigma \int_0^l g_5(\xi,\varsigma,\tau)G(x,y,z,\xi,b,\varsigma,t-\tau)d\xi$$

$$+a_\tau^2\int_0^t d\tau \int_0^b d\eta \int_0^l g_6(\xi,\eta,\tau)G(x,y,z,\xi,\eta,w,t-\tau)d\xi +$$

$$\int_0^t d\tau \int_0^l d\xi \int_0^w d\varsigma \int_0^b F(\xi,\eta,\varsigma,\tau)G(x,y,z,\xi,\eta,\varsigma,t-\tau)d\eta.$$

The Green function [19] - [21] for initial-boundary problem for Klein-Gordon equation ($C \geq 0$) is known; see [21]:

$$G(x,y,z,\xi,\eta,\zeta,t) = 8\times$$

$$\sum_{m=1}^\infty \sum_{n=1}^\infty \sum_{k=1}^\infty \frac{\sin(\lambda_m x + \varepsilon_m)\sin(\mu_n y + \varepsilon_n)\sin(\nu_k z + \varepsilon_k)}{E_{mnk}\sqrt{\rho_{mnk}}}$$

$$\times \sin\left(\lambda_m \xi + \varepsilon_m\right)\sin\left(\mu_n \eta + \varepsilon_n\right)\sin\left(\nu_k \zeta + \varepsilon_k\right)$$

$$\times \sin\left(t\sqrt{\rho_{mnk}}\right), \varepsilon_m = arctg\left(\frac{\lambda_m}{l}\right), \varepsilon_n = arctg\left(\frac{\mu_n}{b}\right),$$

$$\rho_{mnk} = a_\tau^2\left(\lambda_m^2 + \mu_n^2 + \nu_k^2\right) + C, \varepsilon_k = arctg\left(\frac{\nu_k}{w}\right),$$

$$E_{mnk} = \left[l + \frac{\left(\alpha_1\beta_1 + \lambda_m^2\right)\left(\alpha_1 + \beta_1\right)}{\left(\alpha_1^2 + \lambda_m^2\right)\left(\beta_1^2 + \lambda_m^2\right)}\right] \times$$

$$\left[b + \frac{\left(\alpha_2\beta_2 + \mu_n^2\right)\left(\alpha_2 + \beta_2\right)}{\left(\alpha_2^2 + \mu_n^2\right)\left(\beta_2^2 + \mu_n^2\right)}\right] \times$$

$$\left[w + \frac{\left(\alpha_3\beta_3 + \nu_k^2\right)\left(\alpha_3 + \beta_3\right)}{\left(\alpha_3^2 + \nu_k^2\right)\left(\beta_3^2 + \nu_k^2\right)}\right].$$

The eigenvalues λ_m, μ_n, ν_k are positive roots of the transcendental equations:

$$\lambda = \frac{\lambda^2 - \alpha_1\beta_1}{\alpha_1 + \beta_1} tg\left(l\lambda\right), \mu = \frac{\mu^2 - \alpha_2\beta_2}{\alpha_2 + \beta_2} tg\left(b\mu\right),$$

$$\nu = \frac{\nu^2 - \alpha_3\beta_3}{\alpha_3 + \beta_3} tg\left(w\nu\right).$$

There is an interesting situation, if both additional conditions (4), (5) are known. In this case we introduce new time argument by formula

$$\tilde{t} = T - t. \tag{13}$$

The formulation for new time variable is following:

$$\frac{\partial^2 U}{\partial \tilde{t}^2} = a_\tau^2\left(\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2}\right) - CU +$$

$$+ F(x, y, z, T - \tilde{t}), a_\tau^2 = a^2 / \tau,$$

$$\left(\frac{\partial U}{\partial x} - \alpha_1 U\right)\bigg|_{x=0} = g_1\left(y, z, T - \tilde{t}\right),$$

$$\left(\frac{\partial U}{\partial x} + \beta_1 U\right)\bigg|_{x=l} = g_4\left(y, z, T - \tilde{t}\right),$$

$$\left(\frac{\partial U}{\partial y} - \alpha_2 U\right)\bigg|_{y=0} = g_2\left(x, z, T - \tilde{t}\right),$$

$$\left(\frac{\partial U}{\partial y} + \beta_2 U\right)\bigg|_{y=b} = g_5\left(x, z, T - \tilde{t}\right),$$

$$\left(\frac{\partial U}{\partial z} - \alpha_3 U\right)\bigg|_{z=0} = g_3\left(x, y, T - \tilde{t}\right), \tag{14}$$

$$\left(\frac{\partial U}{\partial z} + \beta_3 U\right)\bigg|_{z=w} = g_6\left(x, y, T - \tilde{t}\right),$$

$$U\big|_{\tilde{t}=0} = U_T(x, y, z), \frac{\partial U}{\partial \tilde{t}}\bigg|_{\tilde{t}=0} = -U_T^1(x, y, z).$$

Similar to (12) the solution of inverse problem looks like:

$$U(x, y, z, \tilde{t}) = H\left(x, y, z, \tilde{t}\right) -$$

$$\int_0^l d\xi \int_0^w d\varsigma \int_0^b U_T^1(\xi, \eta, \varsigma) G(x, y, z, \xi, \eta, \varsigma, \tilde{t}) d\eta$$

$$+ \int_0^l d\xi \times \tag{15}$$

$$\int_0^w d\varsigma \int_0^b U_T(\xi, \eta, \varsigma) \frac{\partial}{\partial \tilde{t}} G(x, y, z, \xi, \eta, \varsigma, \tilde{t}) d\eta.$$

We transform the expression for $H\left(x, y, z, \tilde{t}\right)$ in form:

$$H\left(x, y, z, \tilde{t}\right) =$$

$$-a_\tau^2 \int_{T-\tilde{t}}^T d\tau \int_0^w d\varsigma \int_0^b g_1(\eta, \varsigma, \tau) G(x, y, z, 0, \eta, \varsigma, T - \tau) d\eta$$

$$-a_\tau^2 \int_{T-\tilde{t}}^T d\tau \int_0^w d\varsigma \int_0^l g_2(\xi, \varsigma, \tau) G(x, y, z, \xi, 0, \varsigma, T - \tau) d\xi$$

$$-a_\tau^2 \int_{T-\tilde{t}}^T d\tau \int_0^b d\eta \int_0^l g_3(\xi, \eta, \tau) G(x, y, z, \xi, \eta, 0, T - \tau) d\xi$$

$$+a_\tau^2 \int_{T-\tilde{t}}^T d\tau \int_0^w d\varsigma \int_0^b g_4(\eta, \varsigma, \tau) G(x, y, z, l, \eta, \varsigma, T - \tau) d\eta$$

$$+a_\tau^2 \int_{T-\tilde{t}}^T d\tau \int_0^w d\varsigma \int_0^l g_5(\xi, \varsigma, \tau) G(x, y, z, \xi, b, \varsigma, T - \tau) d\xi$$

$$+a_\tau^2 \int_{T-\tilde{t}}^T d\tau \int_0^b d\eta \int_0^l g_6(\xi, \eta, \tau) G(x, y, z, \xi, \eta, w, T - \tau) d\xi$$

$$+ \int_{T-\tilde{t}}^T d\tau \times$$

$$\int_0^l d\xi \int_0^w d\varsigma \int_0^b F(\xi, \eta, \varsigma, \tau) G(x, y, z, \xi, \eta, \varsigma, T - \tau) d\eta.$$

For the heat flux in time we have the expression:

$$\frac{\partial}{\partial \tilde{t}} U(x, y, z, \tilde{t}) = \frac{\partial}{\partial \tilde{t}} H\left(x, y, z, \tilde{t}\right) -$$

$$\int_0^l d\xi \int_0^w d\varsigma \int_0^b V_T(\xi, \eta, \varsigma) \frac{\partial}{\partial \tilde{t}} G(x, y, z, \xi, \eta, \varsigma, \tilde{t}) d\eta +$$

$$\int_0^l d\xi \int_0^w d\varsigma \int_0^b U_T(\xi,\eta,\varsigma)\frac{\partial^2}{\partial \tilde{t}^2}G(x,y,z,\xi,\eta,\varsigma,\tilde{t})d\eta.$$

From last expression at $\tilde{t}=T$ we have solution for the time inverse problem:

$$U_1(x,y,z)=-\frac{U_0(x,y,z)}{2\tau_r}+\frac{\partial}{\partial \tilde{t}}H\left(x,y,z,\tilde{t}\right)\Big|_{\tilde{t}=T}$$

$$-\int_0^l d\xi \times$$

$$\int_0^w d\varsigma \int_0^b U_T^1(\xi,\eta,\varsigma)\frac{\partial}{\partial \tilde{t}}G(x,y,z,\xi,\eta,\varsigma,\tilde{t})\Big|_{\tilde{t}=T} d\eta \quad (16)$$

$$+\int_0^l d\xi \times$$

$$\int_0^w d\varsigma \int_0^b U_T(\xi,\eta,\varsigma)\frac{\partial^2}{\partial \tilde{t}^2}G(x,y,z,\xi,\eta,\varsigma,\tilde{t})\Big|_{\tilde{t}=T} d\eta$$

If only one additional condition of (4) and (5) is given from the solution (15) we obtain 1st kind Fredholm integral equation for the determination of unknown initial heat flux [15].

2.3. Solution for Constant Initial Conditions

Intensive steel quenching process with constant initial conditions is very natural [9]-[13]. We have homogeneous equation (1) and homogeneous boundary conditions (6)-(11). We would like to finish the three dimensional solution with a simplification for constant initial conditions:

$$U\big|_{t=0}=U_0, U_0=const,$$

$$\frac{\partial U}{\partial t}\Big|_{t=0}=U_1, U_1=const. \quad (17)$$

Solutions of time direct problem is these, see (12):

$$U(x,y,z,t)=U_1 J_0 + U_0 J_1 =$$

$$U_1\int_0^l d\xi \int_0^w d\varsigma \int_0^b G(x,y,z,\xi,\eta,\varsigma,t)d\eta + \quad (18)$$

$$U_0\int_0^l d\xi \int_0^w d\varsigma \int_0^b \frac{\partial}{\partial t}G(x,y,z,\xi,\eta,\varsigma,t)d\eta.$$

We can integrate both integrals. For J_0:

$$J_0(x,y,z,t)=8\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{\sin\left(\lambda_m x+\varepsilon_m\right)}{E_{mnk}\sqrt{\rho_{mnk}}}$$

$$\sin\left(\mu_n y+\varepsilon_n\right)\sin\left(\nu_k z+\varepsilon_k\right)\sin\left(t\sqrt{\rho_{mnk}}\right)$$

$$\times\left[\cos\left(\varepsilon_m\right)-\cos\left(\lambda_m l+\varepsilon_m\right)\right]/\lambda_m$$

$$\times\left[\cos\left(\varepsilon_n\right)-\cos\left(\mu_n b+\varepsilon_n\right)\right]/\mu_n \times$$

$$\times\left[\cos\left(\varepsilon_k\right)-\cos\left(\nu_k w+\varepsilon_k\right)\right]/\nu_k. \quad (19)$$

Similarly we can integrate second integral J_1:

$$J_1(x,y,z,t)=8\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{\sin\left(\lambda_m x+\varepsilon_m\right)}{E_{mnk}\lambda_m\mu_n\nu_k}$$

$$\sin\left(\mu_n y+\varepsilon_n\right)\sin\left(\nu_k z+\varepsilon_k\right)\cos\left(t\sqrt{\rho_{mnk}}\right)$$

$$\times\left[\cos\left(\varepsilon_m\right)-\cos\left(\lambda_m l+\varepsilon_m\right)\right] \quad (20)$$

$$\times\left[\cos\left(\varepsilon_n\right)-\cos\left(\mu_n b+\varepsilon_n\right)\right]$$

$$\times\left[\cos\left(\varepsilon_k\right)-\cos\left(\nu_k w+\varepsilon_k\right)\right].$$

We have the solution in the form of two triple series (18), (19). For the heat flux we have an expression:

$$\frac{\partial}{\partial t}U(x,y,z,t)=U_0\int_0^l d\xi \int_0^b \frac{\partial^2}{\partial t^2}G(x,y,z,\xi,\eta,\varsigma,t)d\eta$$

$$+V_0\int_0^l d\xi \int_0^b \frac{\partial}{\partial t}G(x,y,z,\xi,\eta,\varsigma,t)d\eta=U_0 J_2 + V_0 J_1.$$

Finally for J_2 we have:

$$J_2(x,y,z,t)=-8\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{\sin\left(\lambda_m x+\varepsilon_m\right)}{E_{mnk}\lambda_m\mu_n\nu_k}$$

$$\sin\left(\mu_n y+\varepsilon_n\right)\sin\left(\nu_k z+\varepsilon_k\right)\sin\left(t\sqrt{\rho_{mnk}}\right)\times$$

$$\times\sqrt{\rho_{mnk}}\left[\cos\left(\varepsilon_m\right)-\cos\left(\lambda_m l+\varepsilon_m\right)\right]$$

$$\times\left[\cos\left(\varepsilon_n\right)-\cos\left(\mu_n b+\varepsilon_n\right)\right]$$

$$\times\left[\cos\left(\varepsilon_k\right)-\cos\left(\nu_k w+\varepsilon_k\right)\right].$$

3 Solution of One Dimensional Problem for IQP

3.1. Solution for One Dimensional Problem

We will start with a formulation of the mathematical model of the steel plate which is relatively thin in y and z directions: $w<<l, b<<l$. They show that rectangle is thin and narrow. In accordance with the conservative averaging method [17], [18], we introduce the following integral averaged value (one space-dimensional function):

$$u(x,t)=(bw)^{-1}\int_0^w dz\int_0^b U(x,y,z,t)dy, \quad (21)$$

$$\widehat{f}(x,t)=(bw)^{-1}\int_0^z dz\int_0^b F(x,y,z,t)dy.$$

Additionally:

$$u_0(x) = (bw)^{-1} \int_0^w dz \int_0^b U_0(x,y,z)dy,$$

$$v_0(x) = (bw)^{-1} \int_0^w dz \int_0^b U_1(x,y,z)dy,$$

$$g_i(t) = (bw)^{-1} \int_0^w dz \int_0^b g_i(y,z,t)dy, i=1,4,$$

$$g_i(t) = (lw)^{-1} \int_0^w dz \int_0^l g_i(x,z,t)dx, i=2,5,$$

$$g_i(t) = (lb)^{-1} \int_0^b dy \int_0^l g_i(x,y,t)dx, i=3,6.$$

Integrating the right hand side of 3-D equation (1) in the directions y and z, we obtain:

$$\frac{1}{bw} \int_0^w dz \int_0^b \left[a_\tau^2 \left(\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2} \right) - CU \right] dy =$$

$$a_\tau^2 \frac{\partial^2 u}{\partial x^2} - Cu + \hat{f}(x,t) + \frac{a_\tau^2}{bw} \int_0^w dz \int_0^b \left(\frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2} \right) dy.$$

The last term can be transformed as follows:

$$\frac{a_\tau^2}{bw} \int_0^w dz \int_0^b \left(\frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2} \right) dy = \frac{a_\tau^2}{b} \frac{\partial U}{\partial y} \Big|_{y=0}^{y=b} + \frac{a_\tau^2}{w} \frac{\partial U}{\partial z} \Big|_{z=0}^{z=w}.$$

Taking into account the heat exchange with environment, i.e. boundary conditions (7), (8), (10) and (11), we obtain:

$$\frac{1}{b} \frac{\partial U}{\partial y} \Big|_{y=0}^{y=b} + \frac{1}{w} \frac{\partial U}{\partial z} \Big|_{z=0}^{z=w} = \frac{g_3 + g_4}{b} + \frac{g_5 + g_6}{w}.$$

This allows us to use the simplest approximation – by constant – for the function $U(x,y,z,t)$ in the y,z – directions: $U(x,y,z,t) = u(x,t)$

As a result we obtain a 1-D differential equation:

$$\frac{\partial^2 u}{\partial t^2} = a_\tau^2 \frac{\partial^2 u}{\partial x^2} - cu + f(x,t), x \in (0,l), \quad (22)$$

$$t \in (0,T], c = C + a_\tau^2 \left(\frac{\alpha_2 + \beta_2}{b} + \frac{\alpha_3 + \beta_3}{w} \right),$$

where the source term is as follows:

$$f(x,t) = \hat{f}(x,t) + a_\tau^2 \times$$

$$\left[\frac{g_3(x,t) - g_4(x,t)}{b} + \frac{g_5(x,t) - g_6(x,t)}{w} \right].$$

Initial conditions are as follows:

$$u\big|_{t=0} = u_0(x), \quad (23)$$

$$\frac{\partial u}{\partial t} \Big|_{t=0} = v_0(x). \quad (24)$$

The boundary conditions (6) and (9) remain in the same form:

$$\left(\frac{\partial u}{\partial x} - \alpha_1 u \right) \Big|_{x=0} = g_1(t), \quad (25)$$

$$\left(\frac{\partial u}{\partial x} + \beta_1 u \right) \Big|_{x=l} = g_4(t). \quad (26)$$

It is important to mark that one-dimensional approach is exact only if the solution is approximated with a constant in the second and third space directions.

Foremost, we assume that we have a non-homogeneous Klein-Gordon [21], [31] equation with a source term $c \geq 0$. Solution of this one-dimensional direct problem (22)-(26) is well known [19], [20]. Green function is given in [21]:

$$u(x,t) = \int_0^l v_0(\xi) G(x,\xi,t)d\xi +$$

$$\frac{\partial}{\partial t} \int_0^l u_0(\xi) G(x,\xi,t)d\xi + H(x,t),$$

$$H(x,t) = -a_\tau^2 \int_0^t g_1(\tau) G(x,0,t-\tau)d\tau \quad (27)$$

$$+ \int_0^t d\tau \int_0^l f(\xi,\tau) G(x,\xi,t-\tau)d\xi$$

$$+ a_\tau^2 \int_0^t g_4(\tau) G(x,l,t-\tau)d\tau.$$

The Green function has the following representation ([19]-[21]):

$$G(x,\xi,t) = \sum_{i=1}^\infty \frac{\varphi_i(x)\varphi_i(\xi)\sin(t\mu_i)}{\|\varphi_i\|^2 \mu_i}, \quad (28)$$

$$\varphi_i(x) = \cos(\lambda_i x) + \frac{\alpha_1}{\lambda_i} \sin(\lambda_i x),$$

$$\|\varphi_i\|^2 = \frac{\alpha_1}{2\lambda_i^2} + \frac{l}{2}\left(1 + \frac{\alpha_1^2}{\lambda_i^2}\right) + \frac{\beta_1}{2\lambda_i^2} \frac{\lambda_i^2 + \alpha_1^2}{(\lambda_i^2 + \beta_1^2)},$$

$$\mu_i = \sqrt{a_\tau^2 \lambda_i^2 + c}.$$

The eigenvalues λ_i are positive roots of the transcendental equation:

$$\lambda = \frac{\lambda^2 - \alpha_1 \beta_1}{\alpha_1 + \beta_1} \tan(\lambda l). \quad (29)$$

It is easy to write out, so called, "wave energy" [31]:

$$I_w(t) = \sum_{i=1}^{\infty} \frac{\sin^2(t\mu_i)}{\mu_i}. \qquad (30)$$

Right now we look at case $c < 0, a_\tau^2 \lambda_i^2 + c < 0$. In this case the Green function has different form:

$$G(x,\xi,t) = \sum_{i=1}^{m} \frac{\varphi_i(x)\varphi_i(\xi)\sinh\left(t\sqrt{|a_\tau^2\lambda_i^2 + c|}\right)}{\|\varphi_i\|^2 \sqrt{|a_\tau^2\lambda_i^2 + c|}}$$

$$+ \sum_{i=m+1}^{\infty} \frac{\varphi_i(x)\varphi_i(\xi)\sin\left(t\sqrt{a_\tau^2\lambda_i^2 + c}\right)}{\|\varphi_i\|^2 \sqrt{a_\tau^2\lambda_i^2 + c}}. \qquad (31)$$

Here the natural number m in the both sums is given by inequalities:

$$a_\tau^2\lambda_i^2 + c < 0, i = \overline{1,m},$$

$$a_\tau^2\lambda_i^2 + c \geq 0, i = \overline{m+1,\infty}.$$

In this case "wave energy" [31] has equality:

$$I_0(t) = \sum_{i=1}^{m} \frac{\sinh^2\left(t\sqrt{|a_\tau^2\lambda_i^2 + c|}\right)}{\sqrt{|a_\tau^2\lambda_i^2 + c|}} + \qquad (32)$$

$$+ \sum_{i=m+1}^{\infty} \frac{\sin^2\left(t\sqrt{a_\tau^2\lambda_i^2 + c}\right)}{\sqrt{a_\tau^2\lambda_i^2 + c}}.$$

As it was told earlier, initial condition (24) is unrealizable from the experimental point of view, and the $v_0(x)$ must be calculated theoretically. The differentiation of the solution (23) gives:

$$\frac{\partial}{\partial t} u(x,t) = \int_0^l u_0(\xi)\frac{\partial^2}{\partial t^2} G(x,\xi,t)d\xi +$$

$$+ \int_0^l v_0(\xi)\frac{\partial}{\partial t} G(x,\xi,t)d\xi + \frac{\partial}{\partial t} H(x,t). \qquad (33)$$

The additional conditions (4) and (5) at the end of the process regarding $u(x,t)$ are as follows:

$$u|_{t=T} = v_T(x),$$

$$v_T(x) = (bw)^{-1} \int_0^b dy \int_0^w U_T(x,y,z)dz, \qquad (34)$$

and

$$\left.\frac{\partial u}{\partial t}\right|_{t=T} = w_T(x),$$

$$w_T(x) = (bw)^{-1} \int_0^b dy \int_0^w U_T^1(x,y,z)dz. \qquad (35)$$

The solution (27) at the final moment $t = T$ gives:

$$v_T(x) = \int_0^l u_0(\xi)\frac{\partial}{\partial t} G(x,\xi,t)\Big|_{t=T} d\xi +$$

$$+ \int_0^l v_0(\xi)G(x,\xi,T)d\xi + H(x,T)$$

or:

$$\int_0^l K(x,\xi)v_0(\xi)d\xi = f_0(x). \qquad (36)$$

Here

$$f_0(x) = v_T(x) - \int_0^l u_0(\xi)\frac{\partial}{\partial t} G(x,\xi,t)\Big|_{t=T} d\xi -$$

$$-H(x,T), K(x,\xi) = G(x,\xi,T).$$

We have to use regularization method for this ill-posed problem.

There is an interesting situation if both additional conditions (4) and (5) are known – we may introduce a new time argument (13).

The main differential equation (22) remains in its form, only the source term changes:

$$\frac{\partial^2 u}{\partial \tilde{t}^2} = a_\tau^2 \frac{\partial^2 u}{\partial x^2} - cu + f(x, T - \tilde{t}), \qquad (37)$$

$$x \in (0,l), \tilde{t} \in (0,T].$$

The boundary conditions (25), (26) change similarly:

$$\left(\frac{\partial u}{\partial x} - \alpha_1 u\right)\Big|_{x=0} = g_1(T - \tilde{t}),$$

$$\left(\frac{\partial u}{\partial x} + \beta_1 u\right)\Big|_{x=l} = g_4(T - \tilde{t}). \qquad (38)$$

Both additional conditions transform to initial conditions for the equation (27):

$$u|_{\tilde{t}=0} = v_T(x),$$

$$\left.\frac{\partial u}{\partial \tilde{t}}\right|_{\tilde{t}=0} = -w_T(x). \qquad (39)$$

The solution of the direct problem (37)-(39) is similar to the solution (23):

$$u(x,\tilde{t}) = \int_0^l v_T(\xi)\frac{\partial}{\partial \tilde{t}} G(x,\xi,\tilde{t})d\xi$$

$$- \int_0^l w_T(\xi)G(x,\xi,\tilde{t})d\xi + H(x,\tilde{t}). \qquad (40)$$

The last term can be written in the following form:

$$H(x,\tilde{t}) = -a_\tau^2 \int_{T-\tilde{t}}^{T} g_1(\tau)G(x,0,\tilde{t}-T+\tau)d\tau$$

$$+ a_\tau^2 \int_{T-\tilde{t}}^{T} g_4(\tau)G(x,l,\tilde{t}-T+\tau)d\tau +$$

$$\int_{T-\tilde{t}}^{T} d\tau \int_{0}^{l} f(\xi,\tau) G(x,\xi,\tilde{t}-T+\tau) d\xi. \quad (41)$$

For the heat flux we have an expression similar to the formula (27):

$$\frac{\partial}{\partial \tilde{t}} u(x,\tilde{t}) = \int_{0}^{l} u_T(\xi) \frac{\partial^2}{\partial \tilde{t}^2} G(x,\xi,\tilde{t}) d\xi$$

$$-\int_{0}^{l} v_T(\xi) \frac{\partial}{\partial \tilde{t}} G(x,\xi,\tilde{t}) d\xi + \frac{\partial}{\partial \tilde{t}} H(x,\tilde{t}).$$

From here, a nice explicit representation of the necessary initial heat flux immediately follows:

$$v_0(x) = -\int_{0}^{l} v_T(\xi) \frac{\partial}{\partial \tilde{t}} G(x,\xi,\tilde{t}) \Big|_{\tilde{t}=T} d\xi +$$

$$\int_{0}^{l} u_T(\xi) \frac{\partial^2}{\partial \tilde{t}^2} G(x,\xi,\tilde{t}) \Big|_{\tilde{t}=T} d\xi + \frac{\partial}{\partial \tilde{t}} H(x,\tilde{t}) \Big|_{\tilde{t}=T}$$
(42)

We would like to finish the section with a comparison remark about the obtained solutions of the time inverse problem and the solution from paper [9]. The main distinction is between Green functions. In the paper [9], authors have used the Green function for the classical (parabolic) heat equation, but here – we have used the Green function for the wave (hyperbolic) equation.

3.2. Simplifications for Homogeneous Initial Conditions

We would like to finish the one dimensional solution with a simplification for constant initial conditions:

$$u\big|_{t=0} = u_0(x) = u_0 = const,$$

$$\frac{\partial u}{\partial t}\Big|_{t=0} = v_0(x) = v_0 = const. \quad (43)$$

The solution of the time direct problem is the following (see (27)):

$$u(x,t) = u_0 \int_{0}^{l} \frac{\partial}{\partial t} G(x,\xi,t) d\xi + \quad (44)$$

$$v_0 \int_{0}^{l} G(x,\xi,t) d\xi + H(x,t).$$

Intensive steel quenching process with initial conditions (43) is very natural [4]-[6], [9] - [13], [15], [16]. We have the homogeneous equation (31) and the homogeneous boundary conditions. It means that we have:

$$H(x,t) = 0.$$

The solution (27) can be simplified as follows:

$$u(x,t) = u_0 \int_{0}^{l} \frac{\partial}{\partial t} G(x,\xi,t) d\xi + v_0 \int_{0}^{l} G(x,\xi,t) d\xi$$

$$= I_1 + I_0.$$

We can integrate both integrals. For the I_0:

$$I_0 = v_0 \sum_{i=1}^{\infty} \frac{\varphi_i(x) \sin(t\mu_i)}{\|\varphi_i\|^2 \mu_i} \int_{0}^{l} \varphi_i(\xi) d\xi$$

$$= v_0 \sum_{i=1}^{\infty} \frac{\varphi_i(x) \sin(t\mu_i)}{\|\varphi_i\|^2 \lambda_i \mu_i} B_i,$$

$$B_i = \left[\sin(\lambda_i \xi) - \frac{\alpha_1}{\lambda_i} \cos(\lambda_i \xi) \right]_{\xi=0}^{\xi=l}. \quad (45)$$

Similarly, we can integrate the second integral I_1:

$$I_1 = u_0 \sum_{i=1}^{\infty} \frac{\varphi_i(x) \cos(t\mu_i)}{\|\varphi_i\|^2 \lambda_i} B_i. \quad (46)$$

We can use representations (37) to test our proposed method. For IQP we have positive initial temperature (hot steel sample) and negative heat flux: $u_0 > 0, v_0 < 0$. The temperature field is given by formula:

$$u(x,t) = u_0 \sum_{i=1}^{\infty} \frac{\varphi_i(x) \cos(t\mu_i)}{\|\varphi_i\|^2 \lambda_i} B_i +$$

$$+ v_0 \sum_{i=1}^{\infty} \frac{\varphi_i(x) \sin(t\mu_i)}{\|\varphi_i\|^2 \lambda_i \mu_i} B_i. \quad (47)$$

We have the following expression for heat flux:

$$\frac{\partial}{\partial t} u(x,t) = u_0 \int_{0}^{l} \frac{\partial^2}{\partial t^2} G(x,\xi,t) d\xi +$$

$$+ v_0 \int_{0}^{l} \frac{\partial}{\partial t} G(x,\xi,t) d\xi = I_2 + I_0,$$

$$I_2 = u_0 \int_{0}^{l} \frac{\partial^2}{\partial t^2} G(x,\xi,t) d\xi =$$

$$= -u_0 \sum_{i=1}^{\infty} \frac{\varphi_i(x) \mu_i \sin(t\mu_i)}{\|\varphi_i\|^2 \lambda_i} B_i.$$

Finally, we have:

$$\frac{\partial}{\partial t} u(x,t) = -u_0 \sum_{i=1}^{\infty} \frac{\varphi_i(x) \mu_i \sin(t\mu_i)}{\|\varphi_i\|^2 \lambda_i} B_i +$$

$$+ v_0 \sum_{i=1}^{\infty} \frac{\varphi_i(x) \sin(t\mu_i)}{\|\varphi_i\|^2 \lambda_i \mu_i} B_i. \quad (48)$$

We can calculate $u(x,T) = u_T(x)$ and $\frac{\partial}{\partial t} u(x,T) = v_T(x)$ by selecting arbitrary $u_0 > 0, v_0 < 0$. We find $\overline{u}_0 > 0, \overline{v}_0 < 0$ by solving time reverse problem (34).

3.3. Numerical Results for Exact Solution

We would like to use physically real parameters, and that is why we choose parameters from the intensive steel quenching process. Let us take typical steel parameters in our model and homogeneous initial conditions:

$$c_m = 477 \frac{J}{kg \cdot K}, \ \rho = 7900 \frac{kg}{m^3}, \ k = 14.9 \frac{W}{m \cdot K},$$

$$l = 1\,m, \ b = 0.2\,m, \ w = 0.1\,m, h_i = 100,$$

$$\tau_r = 1.5\,s, u_0 = 600°C, \ v_0 = -500\frac{K}{s}. \quad (49)$$

We obtain the solution from formulas (47), (48).

In general, temperature and flux are x-dependent, but calculations showed that values are almost constant regarding space dimension. The next figures show temperature and flux distribution in time.

If we take $T = 1$ as the final time, the values at this moment are the following:

$$u_T = u(x,T) = 8, \ v_T = \frac{\partial u}{\partial t}(x,T) = -640.$$
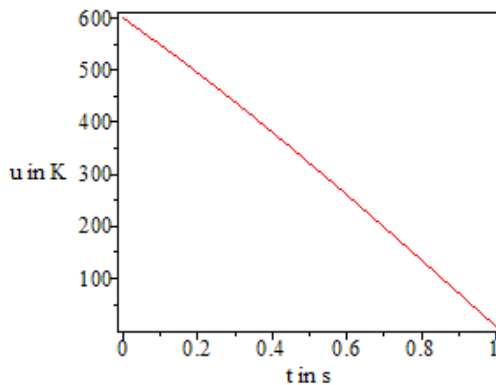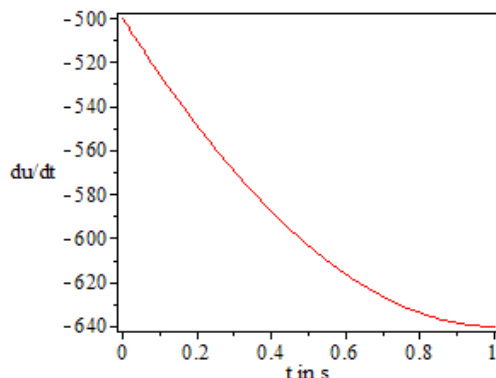

Fig. 1. The temperature in the time.

Fig. 2. The time flux as the time co-ordinate

With the usage of the solution (40) of time reverse problem with these values as initial conditions:

$$u\big|_{\tilde{t}=0} = u_T, \ \frac{\partial u}{\partial \tilde{t}}\bigg|_{\tilde{t}=0} = -v_T.$$

We return to our initial conditions (43) now $\tilde{t} = T$. Calculation errors were negligible in our example ($\Delta u \simeq 0.2$ K).

In the case of wave energy device is important to fixe devices interaction with the surrounding. It is shown in the Fig. 3.
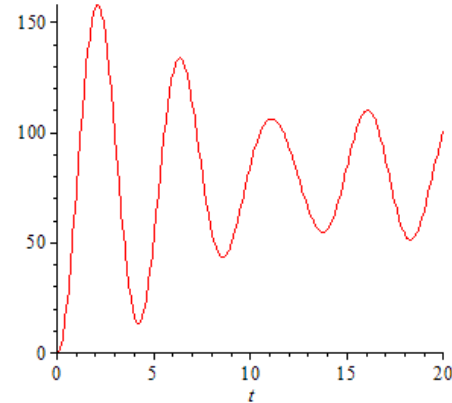

Fig. 3. The wave energy [31] for the final time $T = 20$.

The wave type energy change in the time is very interesting process, which can be useful for the increase of wave energy power.

These numerical results were obtained by Dr. Math. R. Vilums. Authors express their thanks to Dr. Vilums.

4 Hyperbolic One Dimensional Problem

Here we examine a one-dimensional hyperbolic heat equation, analogical to equation (*) from the beginning of paper. This equation may be connected with wave energy [14]:

$$\tau \frac{\partial^2 T}{\partial t^2} + \frac{\partial T}{\partial t} = \bar{k} \frac{\partial^2 T}{\partial x^2} - \gamma^* T + f(x,t), \quad (50)$$

$$0 < x < l, 0 < t < t_f.$$

Here t_f is the final time, $\bar{k} = k/c\rho$, γ^* - is the coefficients obtained with application of the conservative averaging method for the reduction of the 3-D problem to the 1-D problem.

Initial and boundary conditions are as follows:

$$T\big|_{t=0} = T_0(x), \ \frac{\partial T}{\partial t}\bigg|_{t=o} = V_0(x),$$

$$\frac{\partial T}{\partial x}\bigg|_{x=0} = 0, \left(\frac{\partial T}{\partial x} + \sigma T\right)\bigg|_{x=0} = 0, \sigma = \frac{\alpha_x}{k_x}. \quad (51)$$

For the intensive steel quenching the function $V_0(x)$ is unknown and instead of two final conditions (4), (5) we can use only one additional condition:

$$T\left(x,t_f\right)=T_f\left(x\right). \qquad (52)$$

4.1. Reduction of Hyperbolic Problem to the System of Ordinary Differential Equations

To reduce the problem to ordinary differential equation we use finite difference method with uniform grid $x_j = jh, j = \overline{0,N}, Nh = l$. We use second order approximation for partial derivation of second order respect to x and for initial problem for hyperbolic heat equation (50), we obtain system of ordinary differentia equations of second order in the following matrix form:

$$\tau\ddot{U}(t)+\dot{U}(t)+\overline{k}AU(t)+\gamma^*U(t)=F(t),$$
$$U(0)=U_0,\dot{U}(0)=V_0. \qquad (53)$$

In formula (53) we have the column-vectors of $N+1$ order $U(t),\dot{U}(t),\ddot{U}(t),V_0,U_0,F(t)$ with elements

$$u_j(t)\approx T(x_j,t),\dot{u}_j(t)\approx\frac{\partial T(x_j,t)}{\partial t}, \qquad (54)$$

$$\ddot{u}_j(t)\approx\frac{\partial^2 T(x_j,t)}{\partial t^2}, v_j(0)=V_0(x_j),$$

$$u_j(0)=U_0(x_j), f_j\left(t\right)=f(x_j,t), j=\overline{0,N}.$$

Here A is the 3-diagonal of $N+1$ order in the form:

$$A=\frac{1}{h^2}\begin{pmatrix} 2-2 & 0\ldots & 0 & 0 & 0 \\ -1 & 2-1\ldots & 0 & 0 & 0 \\ \ldots\ldots\ldots\ldots\ldots \\ 0 & 0 & 0\ldots-1 & 2 & -1 \\ 0 & 0 & 0\ldots 0 & -2 & 2+h\sigma \end{pmatrix}.$$

4.2. The Discrete Spectral Problem

The 3-diagonal matrix A of $N+1$ order can be represented with following difference operator of second order approximation [27]:

$$Ay=\begin{cases} -2\left(y_1-y_0\right)/h^2, j=0, \\ -\left(y_{j+1}-2y_j+y_{j-1}\right)/h^2, j=\overline{1,N-1}, \\ -2\left(y_{N-1}-y_N\right)/h^2+2\sigma y_N/h, j=N. \end{cases} \qquad (55)$$

Using two vectors y^m, y^n scalar product:

$$\left[y^m,y^n\right]=\sum_{j=1}^{N-1}y_j^m y_j^n+\frac{1}{2}\left(y_0^m y_0^n+y_N^m y_N^n\right).$$

We can easy prove: the operator A is symmetric and $\left[Ay,y\right]\geq 0$ [27].

The corresponding spectral problem

$$Ay^n=\mu_n y^n, n=\overline{1,N+1}$$

has following solution [27]:

$$\begin{cases} y^n=C_n^{-1}\left(\frac{1}{\sqrt{2}}y_0^n,y_1^n,\ldots,y_{N-1}^n,\frac{1}{\sqrt{2}}y_N^n\right), \\ \mu_n=\frac{4}{h^2}\sin^2\left(\frac{p_n h}{2}\right). \end{cases} \qquad (56)$$

Here

$$y_j^n=\frac{\sin\left(p_n h\right)}{h}\cos\left(p_n x_j\right), j=\overline{0,N}.$$

p_n are positive roots of the transcendental equation:

$$\cot\left(p_n l\right)=\frac{\sin^2\left(p_n h\right)}{h\sigma\sin\left(p_n h\right)}, n=\overline{1,N+1}. \qquad (57)$$

The constants $C_n^2=\left[y^n,y^n\right]$ can be obtained in following form:

$$C_n^2=h\left[A_1^2 S_1+0.5\left(A_1^2\left(1+\cos^2\left(p_n l\right)\right)\right)\right],$$

$$A_1=\sin\left(p_n h\right)/h, S_1=\sum_{j=1}^{N-1}\cos^2\left(p_n x_j\right)=$$

$$=0.5\left[N-1+\frac{\cos\left(p_n l\right)\sin\left(p_n\left(l-h\right)\right)}{\sin\left(p_n h\right)}\right].$$

Calculated $\left[y^n,y^n\right]$ finally we have the orthonormal eigenvectors y^n, y^m with the scalar product $\left[y^n,y^m\right]=\delta_{n,m}$. Here $\delta_{n,m}$ is the Kronecker symbol. It means, that we have matrix A, which can be represented in form:

$$A=PDP^T. \qquad (58)$$

Here the column of the matrix P and the diagonal matrix D contains M orthonormal eigenvectors y^n and eigenvalues $\mu_n, n=\overline{1,M}$ correspond, where $M=N+1$. From $P^T P=E$ follows that $P^{-1}=P^T$.

Now we examine spectral problem for differential equation and finite difference scheme with exact spectrum [26], [28]. The solution of the spectral problem for differential equations

$$y''(x)+\lambda^2 y(x)=0, x\in\left(0,l\right),$$

$$y'(0)=0, y'(l)+\sigma y(l)=0$$

is in form

$$y_n(x)=C_n^{-1}\left[\lambda_n\cos\left(\lambda_n x\right)\right],$$

$$C_n^2=\frac{1}{2}\left[l\lambda_n^2+\frac{\sigma\lambda_n^2}{\lambda_n^2+\sigma^2}\right]. \qquad (59)$$

We have

$$\left(y_n, y_m \right) = \int_0^l y_n(x) y_m(x) dx = \delta_{n,m}.$$

The eigenvalues λ_n are positive roots of the transcendental equations:

$$\cot \left(\lambda_n l \right) = \frac{\lambda_n}{\sigma}, n \geq 1. \qquad (60)$$

For the scalar product $\left[y^n, y^m \right]$ the integral $\left(y_n, y_m \right)$ is approximated with trapezoidal formula and in the limit case if $h \to 0$ then from (57), (60) follow $\mu_n = \lambda_n^2$.

For the difference scheme with exact spectrum [26], [28] the matrix A is in the form (58) and the diagonal matrix D contains the first $N+1$ eigenvalues $\delta_k = \lambda_k^2, k = \overline{1, N+1}$ from the differential operator $\dfrac{\partial^2}{\partial x^2}$ correspondingly.

4.3. Solution of the Discrete Problem (53)

The solution of system (53) we achieve by usage of the representation of matrix $A = PDP^T$. From transformation $W = P^T U \left(U = PW \right)$ follows the separate system of ordinary differential system:

$$\begin{cases} \tau \ddot{W}(t) + \dot{W}(t) + \overline{k} D W(t) + \gamma^* W(t) = G(t), \\ W(0) = P^T U_0, \dot{W}(0) = P^T V_0, G(t) = P^T F(t) \end{cases} \quad (60)$$

Where $W(t), G(t)$ are the column-vectors of M order with elements $w_k(t), g_k(t), k = \overline{1, M}$.

The solution of this system is the function:

$$w_k(t) = \frac{1}{\tau \kappa_k} \int_0^t e^{-\frac{t-\xi}{2\tau}} \sinh \left[\kappa_k (t-\xi) \right] g_k(\xi) d\xi + e^{-\frac{t}{2\tau}} \times$$

$$\left\{ \frac{\sinh\left(\kappa_k t\right)}{\kappa_k} \left[\dot{w}_k(0) + \frac{w_k(0)}{2\tau} \right] + \cosh\left(\kappa_k t\right) w_k(0) \right\}.$$

(61)

Here $\kappa_k = \sqrt{\dfrac{1}{4\tau^2} - \dfrac{\overline{k} d_k}{\tau}}, d_k = \delta_k + \dfrac{\gamma^*}{\overline{k}}, \delta_k = \mu_k$.

If $4\overline{k} d_k \tau > 1$, then the hyperbolic functions are replaced with the trigonometrical and the parameter $\kappa_k = \sqrt{\dfrac{\overline{k} d_k}{\tau} - \dfrac{1}{4\tau^2}}$.

If $\kappa_k = 0$, then

$$w_k(t) = \frac{1}{\tau} \int_0^t e^{-\frac{t-\xi}{2\tau}} (t-\xi) g_k(\xi) d\xi +$$

$$e^{-\frac{t}{2\tau}} \left\{ t \left[\dot{w}_k(0) + \frac{w_k(0)}{2\tau} \right] + w_k(0) \right\}.$$

4.4. The solving of the inverse problem

For the inverse problem the vector V_0 in the formula (53) is unknown. We have additional condition (52):

$$U\left(t_f \right) = u_f.$$

Here u_f is the vectors- column with elements

$$u_{f,k} = T_f(x_k), k = \overline{1, M}.$$

The analytical solution of this problem can be obtain from (60) replacing the second initial condition $W(0) = P^T V_0$ with $W(t_f) = P^T u_f$.

The solution is:

$$w_k(t) = e^{-\frac{t}{2\tau}} \left\{ \frac{\sinh\left(\kappa_k t\right)}{\sinh\left(\kappa_k t_f\right)} \left[e^{\frac{t_f}{2\tau}} w_k\left(t_f\right) - w_k(0) \times \right. \right.$$

$$\left. \cosh\left(\kappa_k t_f\right) - \frac{1}{\tau\kappa_k} \int_0^{t_f} e^{\frac{\xi}{2\tau}} \sinh\left[\kappa_k \left(t_f - \xi\right) \right] g_k(\xi) d\xi \right]$$

$$+ \frac{1}{\tau\kappa_k} \int_0^t e^{\frac{\xi}{2\tau}} \sinh \kappa_k \left[(t-\xi) \right] g_k(\xi) d\xi +$$

$$\cosh\left(\kappa_k t\right) w_k(0) \bigg\}.$$

(62)

Here $w_k\left(t_f\right)$ are the components of vector $W\left(t_f\right)$. From representation (62) follows that the second condition is in the form:

$$\dot{w}_k(0) = \frac{\kappa_k}{\sinh\left(\kappa_k t_f\right)} \left[e^{\frac{t_f}{2\tau}} w_k\left(t_f\right) - w_k(0) \times \right.$$

$$\left. \cosh\left(\kappa_k t_f\right) - \frac{1}{\tau\kappa_k} \int_0^{t_f} e^{\frac{\xi}{2\tau}} \sinh\left[\kappa_k \left(t_f - \xi\right) \right] g_k(\xi) d\xi \right]$$

$$- w_k(0) / 2\tau, V_0 = P\dot{W}(0).$$

When replaced this expression in (61) we obtain the solution (62).

If $w_k\left(t_f\right) = g_k(t) = 0$, then

$$\dot{w}_k(0) = -\frac{\kappa_k}{\cosh\left(\kappa_k t_f\right)} w_k(0) - \frac{w_k(0)}{2\tau}.$$

For Fourier method, we obtain the Fourier coefficients $w_k(t), \dot{w}_k(0)$ from formula (62), where

$$w_k(t_f) = (u_f, y_k), V_0(x) = \sum_{k=1}^{\infty} \dot{w}_k(0) y_k(x).$$

5 Some Numerical Results

Here we look the intensive steel quenching for Carbon steel with these physical properties:

$$k = 60.5 \frac{W}{mC}, \rho = 7870 \frac{kg}{m^3}, c = 434 \frac{J}{kgC},$$

$$\alpha = 30, \overline{k} = 0.000017713, \tau = 0.1; 0.5,$$

$$\gamma = 0; 10; 3 \cdot 10^6, N = 20, l = 0.1; 1.$$

Here we give only one result. This is comparison to the approximation by function from integral parabolic and exponential spline [22]-[25] for two different lengths of interval $[0, l]$.

As example for comparison we use solution of such boundary value problem:

$$k_z \frac{\partial}{\partial z} \left(\frac{\partial T(z)}{\partial z} \right) - b^2 T(z) = f_a, z \in (0, L_z), \quad (63)$$

$$\frac{\partial T(0)}{\partial z} = 0, k_z \frac{\partial T(L_z)}{\partial z} + \alpha_z T(L_z) = 0.$$

Here $b = \pi \sqrt{\frac{k_x}{L_x^2} + \frac{k_y}{L_y^2}}, f_a = const.$

The exact solution is:

$$T(z) = C \cosh(b_1 z) - \frac{f_a}{k_z b_1^2}, b_1 = \frac{b}{\sqrt{k_z}},$$

$$C = \frac{\alpha_z f_a / b_1^2}{b_1 k_z \sinh(b_1 L_z) + \alpha_z \cosh(b_1 L_z)}.$$

The averaged value is:

$$T^a = L_z^{-1} \int_0^{L_z} T(z) dz = \frac{C \sinh(b_1 L_z)}{b_1 L_z} - \frac{f_a}{k_z b_1^2}.$$

The hyperbolic approximation with parameter a_z we use such function:

$$U(z) = T^a + m L_z \frac{\sinh\left[a_z \left(z - \frac{L_z}{2} \right) \right]}{2 \sinh\left(a_z \frac{L_z}{2} \right)} +$$

$$+ \frac{eG}{4} \left[\frac{\sinh^2 \left[a_z \left(z - \frac{L_z}{2} \right) \right]}{\sinh^2 \left(a_z \frac{L_z}{2} \right)} - A_0 \right], \quad (64)$$

$$A_0 = \frac{\dfrac{\sinh(a_z L_z)}{a_z L_z} - 1}{\cosh(a_z L_z) - 1}, G = \frac{L_z}{k_z}.$$

We have:

$$m k_z = e = a_z T^a / \left(G a_z (0.5 + A_1) + 2 d_z \right), d_z =$$

$$0.5 L_z a_z \cosh(0.5 a_z L_z), A_1 = 0.25(1 - A_0),$$

$$T^a = f_a / \left(B_z g_z + b^2 \right), B_z = 2 d_z / L_z.$$

Integral parabolic spline for one segment has form [22]-[25]:

$$U(z) = T^a + m(z - \frac{L_z}{2}) + eG \left[\frac{(z - \frac{L_z}{2})^2}{L_z^2} - \frac{1}{12} \right], \quad (65)$$

$$G = L_z / k_z > 0.$$

Boundary conditions give the same values for coefficients m, e, and $A_1 = 1/6, d_z = 1, a_z = 0$.

We can use also the hyperbolic spline with two parameters a_z and a_{z0} replaced (64) in the multiplicative for mG the parameter a_z with a_{z0}. Then the maximal error δ is equal to zero for every values of the parameters (63), if $a_z = b, a_{z0} = b/2$.

In the following figures 3 and 4 we show that we have approximations (64) and (65) approach the solution, obtained for $k_z = 6, b = 81, a_z = 30$, $f_a = 80000, L_z = 0.1$ (fig. 3, with one parameter $a_z = b, a_{z0} = b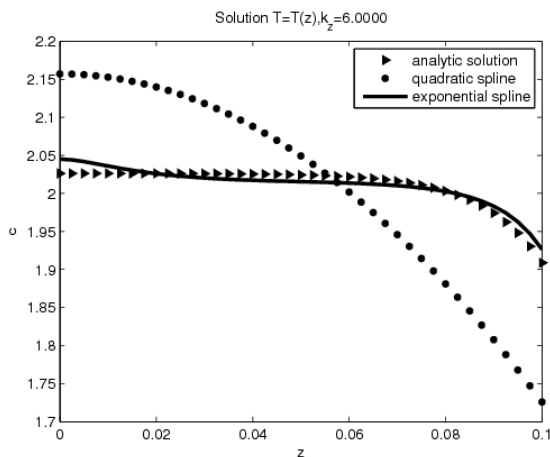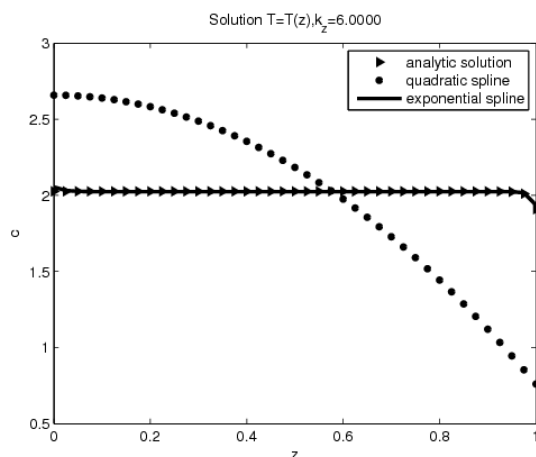/2$ and 0.0188 corresponding for parabolic and hyperbolic splines), $L_z = 1$ (fig.4, for two parameters $a_z = b, a_{z0} = b/2, \delta = 1.157$, for hyperbolic spline with one parameter $a_z = b, \delta = 0.0288$).

The next two figures show that hyperbolic approximation is really much better as approximation with polynomial function.

Solution T=T(z),k_z=6.0000

Fig. 3. $L_z = 0.1, a_z = 81.$

Solution T=T(z),k_z=6.0000

Fig. 4. $L_z = 1, a_z = b, a_{z0} = b/2.$

6 Conclusions

We have constructed some solutions for direct and time inverse problems for hyperbolic heat equation. The solutions for determination of initial heat flux are obtained in closed analytical form: as triple series. In second part of paper we investigate the hyperbolic problem for wave equation. As was mentioned earlier the wave energy change in the time can be very useful for invertors to maximise the wave energy power. By finite difference method we reduced this equation to system of ordinary differential equations. The results of calculations show, that hyperbolic approximation is better than integral parabolic spline. We will explain in the future our results for different boundary conditions and make errors analyse. Bigger evolution is to develop approximation for integral splines with all three types of boundary conditions for partial differential equations.

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