Analytical solutions of heat conduction problems for anisotropic solid body

ISTVÁN ECSEDI[1], ÁKOS JÓZSEF LENGYEL[2]*
[1,2]Institute of Applied Mechanics
[1,2]University of Miskolc
[1,2]H-3515, Miskolc-Egyetemváros, Miskolc
[1,2]HUNGARY

Abstract: Exact analytical solutions of heat conduction problems in anisotropic two-dimensional solid bodies are presented in this study. Time-dependent and steady-state problems are considered. A linear coordinate transformation is introduced which reduces the anisotropic heat conduction problem to an equivalent isotropic one. The solution of the anisotropic heat conduction problem is expressed in terms of solutions of the corresponding isotropic heat conduction problem. The connection of data of the applied linear coordinate transformation and the thermal material properties of anisotropic solid body is analysed. All result of the paper is based on the Fourier's theory of heat conduction in solid bodies. Examples illustrate the applications of the developed method.

Key-Words: anisotropic, heat conduction, steady-state, time-dependent

Received: April 28, 2021. Revised: August 4, 2021. Accepted: August 16, 2021. Published: September 19, 2021.

1 Introduction

Many materials in which the thermal conductivity varies with direction are called anisotropic materials. To date, relatively few reported results of temperature distribution or heat flux fields in anisotropic solid body have published. Some text books such as Carslaw and Jaeger [1] and Özisik [2] have devoted a considerable section of their contents to heat conduction problems in anisotropic bodies. The exact analytical solutions of anisotropic heat conduction is limited to very simple geometries [2] and for complicated geometries one has to resort to numerical procedures. For steady-state two-dimensional anisotropic heat conduction problem in the case of absence of internal heat generation a Monte Carlo solution is presented by Kowsary and Arabi [3]. For layered composite anisotropic bodies the heat conduction problems are solved with linear coordinate transformation in papers by Poon [4], Poon et al. [5], Ma and Chang [6], Yan et al. [7] and Hsieh and Ma [11]. Mulholland and Gupta studied the heat conduction in a three-dimensional anisotropic body by the use of coordinate transformation of the principal axes of conductivity tensor [8]. Clements and Budhi gave solutions for a class of steady-state heat conduction problems in anisotropic media by means of the boundary element method [9]. Xiangzhou developed a partition-matching technique to solve the heat conduction in a two-dimensional anisotropic strip with prescribed temperature on the boundary [10].

In this paper a method is presented to find some analytical solutions for steady-state and time dependent heat conduction problems in homogeneous anisotropic bodies. By the proposed method the existing solutions of heat conduction equation for isotropic bodies are employed to find solutions of the corresponding anisotropic problems. A linear coordinate transformation is used to develop the formulation. By this formulation some analytical solutions of heat conduction of anisotropic bodies are found which are remarkable and useful especially for benchmarking purposes. The present paper is a contribution to the existing exact benchmark solutions for steady-state and time-dependent heat conduction problems in anisotropic two-dimensional solid bodies. The first kind and the second kind boundary conditions are considered. Although the presented derivations concern to two-dimensional case they can easily be generalized to three-dimensional heat conduction problems.

2 A brief survey of the two-dimensional heat conduction problem

2.1 Isotropic body

Suppose in a coordinate system (whose components are x, y) there is a „two-dimensional" solid body occupying a plane domain A with the boundary curve ∂A. The temperature T of the body depends on the space coordinate x, y and the time coordinate Θ, that is $T = T(x, y, \Theta)$. The heat conduction problem of this two-dimensional isotropic homogeneous body according to the Fourier's theory leads to the following initial-boundary value problem [1, 2].

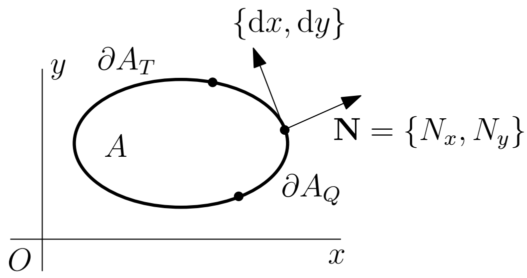

Figure 1: 2D isotropic homogeneous solid body.

$$K\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}\right)+R=MC\frac{\partial T}{\partial \Theta},\quad (x,y)\in A,\quad \Theta>0, \tag{1}$$

$$T(x,y,0)=T_0(x,y),\quad (x,y)\in A, \tag{2}$$

$$T(x,y,\Theta)=G(x,y,\Theta),\quad (x,y)\in\partial A_T,\quad \Theta>0, \tag{3}$$

$$K\left(\frac{\partial T}{\partial x}N_x+\frac{\partial T}{\partial y}N_y\right)=Q(x,y,\Theta),\quad (x,y)\in\partial A_Q,\quad \Theta>0, \tag{4}$$

where K is the thermal conductivity coefficient, G is the prescribed boundary temperature on ∂A_T and Q is the prescribed heat flux on ∂A_Q, R is the internal heat source in A, M is the mass density, C is the specific heat, N_x, N_y are the components of the unit normal vector of the boundary curve ∂A as shown in Fig. 1. We note that $\partial A = \partial A_T \cup \partial A_Q$ and $\partial A_T \cap \partial A_Q = \{0\}$. Let

$$F(x,y)=0 \tag{5}$$

be the equation of the boundary curve ∂A. It is evident that

$$\frac{\partial F}{\partial x}\mathrm{d}x+\frac{\partial F}{\partial y}\mathrm{d}y=\left\{\frac{\partial F}{\partial x};\frac{\partial F}{\partial y}\right\}\cdot\{\mathrm{d}x;\mathrm{d}y\}=0, \tag{6}$$

where the dot between the two vectors indicates their scalar product. From Eq. (6) it follows that (Fig. 1)

$$N_x=\frac{1}{N}\frac{\partial F}{\partial x},\quad N_y=\frac{1}{N}\frac{\partial F}{\partial y},\quad N=\sqrt{\left(\frac{\partial F}{\partial x}\right)^2+\left(\frac{\partial F}{\partial y}\right)^2}. \tag{7}$$

By the use of Eq. (7) we can reformulate the boundary condition (4) as

$$K\left(\frac{\partial T}{\partial x}\frac{\partial F}{\partial x}+\frac{\partial T}{\partial y}\frac{\partial F}{\partial y}\right)=NQ. \tag{8}$$

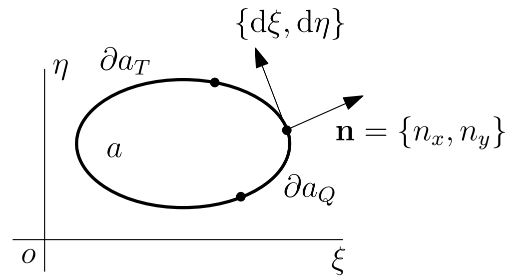

Figure 2: 2D anisotropic homogeneous solid body.

2.2 Anisotropic body

Suppose in a coordinate system (whose components are ξ, η) there is a „two-dimensional" solid body occupying a plane domain with the boundary curve ∂a. The temperature t of the anisotropic body depends on the space coordinate ξ, η and the time coordinate ϑ, that is $t = t(\xi,\eta,\vartheta)$. The heat conduction problem of this two-dimensional homogeneous anisotropic body leads to the following initial-boundary value problem [1, 2]

$$k_{11}\frac{\partial^2 t}{\partial \xi^2}+2k_{12}\frac{\partial^2 t}{\partial\xi\partial\eta}+k_{22}\frac{\partial^2 t}{\partial\eta^2}+r=mc\frac{\partial t}{\partial\vartheta},\quad (\xi,\eta)\in a,\quad \vartheta>0, \tag{9}$$

$$t(\xi,\eta,0)=t_0(\xi,\eta),\quad (\xi,\eta)\in a, \tag{10}$$

$$t(\xi,\eta,\vartheta)=g(\xi,\eta,\vartheta),\quad (\xi,\eta)\in\partial a_t,\quad \vartheta>0, \tag{11}$$

$$k_{11}\frac{\partial t}{\partial\xi}n_\xi+k_{12}\left(\frac{\partial t}{\partial\eta}n_\xi+\frac{\partial t}{\partial\xi}n_\eta\right)+k_{22}\frac{\partial t}{\partial\eta}n_\eta==q(\xi,\eta,\vartheta),\quad (\xi,\eta)\in\partial a_q,\quad \vartheta>0. \tag{12}$$

Here k_{11}, $k_{12} = k_{21}$, k_{22} are the conductivity coefficients that form a second order positive definite symmetric tensor which is called the thermal conductivity tensor, g is the prescribed boundary temperature defined on ∂a_t, q is the prescribed boundary heat flux defined on ∂a_q, r is the internal heat source, m is the mass density, c is the specific heat and n_ξ, n_η are the components of the unit normal vector of boundary curve ∂a as shown in Fig. 2. We note that $\partial a = \partial a_t \cup \partial a_q$ and $\partial a_t \cap \partial a_q = \{0\}$. Let

$$f(\xi,\eta)=0 \tag{13}$$

be the equation of the boundary curve ∂a then we have

$$\frac{\partial f}{\partial\xi}\mathrm{d}\xi+\frac{\partial f}{\partial\eta}\mathrm{d}\eta=\left\{\frac{\partial f}{\partial\xi};\frac{\partial f}{\partial\eta}\right\}\cdot\{\mathrm{d}\xi;\mathrm{d}\eta\}=0,\ \text{on } \partial a. \tag{14}$$

From Eq. (14) it follows that (Fig. 2)

$$n_\xi = \frac{1}{n}\frac{\partial f}{\partial \xi}, \quad n_\eta = \frac{1}{n}\frac{\partial f}{\partial \eta}, \quad n = \sqrt{\left(\frac{\partial f}{\partial \xi}\right)^2 + \left(\frac{\partial f}{\partial \eta}\right)^2}. \tag{15}$$

By the use of Eq. (15) we can reformulate the heat flux boundary condition as

$$k_{11}\frac{\partial t}{\partial \xi}\frac{\partial f}{\partial \xi} + k_{12}\left(\frac{\partial t}{\partial \eta}\frac{\partial f}{\partial \xi} + \frac{\partial t}{\partial \xi}\frac{\partial f}{\partial \eta}\right) + k_{22}\frac{\partial t}{\partial \eta}\frac{\partial f}{\partial \eta} = nq, \quad (\xi,\eta) \in \partial a_q, \quad \vartheta > 0. \tag{16}$$

3 Theory

We introduce the next linear coordinate transformation

$$x = \alpha\xi + \beta\eta, \quad y = \gamma\xi + \delta\eta, \quad \Theta = \lambda\vartheta, \tag{17}$$

where

$$\varepsilon = \alpha\delta - \beta\gamma \neq 0, \quad \lambda > 0. \tag{18}$$

The coefficients of above defined linear transformations are unit free.

Theorem 1. Let

$$f(\xi,\eta) = F(\alpha\xi + \beta\eta, \gamma\xi + \delta\eta) \tag{19}$$

be. Assuming that the conductivity coefficients of anisotropic two-dimensional body have the form

$$k_{11} = \frac{\beta^2+\delta^2}{(\alpha\delta-\beta\gamma)^2}K, \quad k_{12} = k_{21} = -\frac{\alpha\beta+\gamma\delta}{(\alpha\delta-\beta\gamma)^2}K, \quad k_{22} = \frac{\alpha^2+\gamma^2}{(\alpha\delta-\beta\gamma)^2}K, \tag{20}$$

and let

$$t_0(\xi,\eta) = T_0(\alpha\xi + \beta\eta, \gamma\xi + \delta\eta), \tag{21}$$

$$g(\xi,\eta,\vartheta) = G(\alpha\xi + \beta\eta, \gamma\xi + \delta\eta, \lambda\vartheta), \tag{22}$$

$$q(\xi,\eta,\vartheta) = BQ(\alpha\xi + \beta\eta, \gamma\xi + \delta\eta, \lambda\vartheta), \tag{23}$$

where

$$B = \left\{\frac{k_{11}\left(\frac{\partial f}{\partial \xi}\right)^2 + 2k_{12}\frac{\partial f}{\partial \xi}\frac{\partial f}{\partial \eta} + k_{22}\left(\frac{\partial f}{\partial \eta}\right)^2}{K\left[\left(\frac{\partial f}{\partial \xi}\right)^2 + \left(\frac{\partial f}{\partial \eta}\right)^2\right]}\right\}^{\frac{1}{2}} \tag{24}$$

and

$$mc = \frac{MC}{\lambda} \tag{25}$$

be then we have

$$t(\xi,\eta,\vartheta) = T(\alpha\xi + \beta\eta, \gamma\xi + \delta\eta, \lambda\vartheta). \tag{26}$$

Proof. The proof of this theorem is based on the following equations

$$\frac{\partial T}{\partial x} = \frac{1}{\alpha\delta-\beta\gamma}\left(\delta\frac{\partial t}{\partial \xi} - \gamma\frac{\partial t}{\partial \eta}\right), \tag{27}$$

$$\frac{\partial T}{\partial y} = \frac{1}{\alpha\delta-\beta\gamma}\left(-\beta\frac{\partial t}{\partial \xi} + \alpha\frac{\partial t}{\partial \eta}\right), \tag{28}$$

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{(\alpha\gamma-\beta\delta)^2}\left[\delta^2\frac{\partial^2 t}{\partial \xi^2} - 2\delta\gamma\frac{\partial^2 t}{\partial \xi\partial\eta} + \gamma^2\frac{\partial^2 t}{\partial \eta^2}\right], \tag{29}$$

$$\frac{\partial^2 T}{\partial y^2} = \frac{1}{(\alpha\gamma-\beta\delta)^2}\left[\beta^2\frac{\partial^2 t}{\partial \xi^2} - 2\alpha\beta\frac{\partial^2 t}{\partial \xi\partial\eta} + \alpha^2\frac{\partial^2 t}{\partial \eta^2}\right], \tag{30}$$

$$\begin{aligned} K\left(\frac{\partial T}{\partial x}\frac{\partial F}{\partial x} + \frac{\partial T}{\partial y}\frac{\partial F}{\partial y}\right) &= \\ = \frac{K}{(\alpha\delta-\beta\gamma)^2}\Big[\left(\beta^2+\delta^2\right)\frac{\partial f}{\partial \xi}\frac{\partial t}{\partial \xi} &- (\alpha\beta+\gamma\delta)\left(\frac{\partial f}{\partial \xi}\frac{\partial t}{\partial \eta} + \frac{\partial f}{\partial \eta}\frac{\partial t}{\partial \xi}\right) + \left(\alpha^2+\gamma^2\right)\frac{\partial f}{\partial \eta}\frac{\partial t}{\partial \eta}\Big] = k_{11}\frac{\partial f}{\partial \xi}\frac{\partial t}{\partial \xi} + \\ k_{12}\left(\frac{\partial f}{\partial \xi}\frac{\partial t}{\partial \eta} + \frac{\partial f}{\partial \eta}\frac{\partial f}{\partial \xi}\right) &+ k_{22}\frac{\partial f}{\partial \eta}\frac{\partial t}{\partial \eta}, \end{aligned} \tag{31}$$

$$\sqrt{\left(\frac{\partial F}{\partial x}\right)^2 + \left(\frac{\partial F}{\partial y}\right)^2} = \left\{\frac{1}{K}\left[k_{11}\left(\frac{\partial f}{\partial \xi}\right)^2 + 2k_{12}\frac{\partial f}{\partial \xi}\frac{\partial f}{\partial \eta} + k_{22}\left(\frac{\partial f}{\partial \eta}\right)^2\right]\right\}^{\frac{1}{2}}, \tag{32}$$

$$\frac{\partial T}{\partial \Theta} = \frac{1}{\lambda}\frac{\partial t}{\partial \vartheta}. \tag{33}$$

Substitution of Eqs. (27–33) into Eqs. (1–4) leads to the initial-boundary value problem formulated by Eqs. (9–12) under the conditions (19–25) assuming that Eq. (26) is valid. From the Cauchy-Schwarz inequality

$$\left(\alpha^2+\gamma^2\right)\left(\beta^2+\delta^2\right) \geq (\alpha\beta+\gamma\delta)^2 \quad (34)$$

and the definition of k_{11}, k_{22} and $k_{12}=k_{21}$ it follows that the thermal conductivity coefficients defined by Eq. (20) satisfy the next inequalities

$$k_{11}>0, \quad k_{22}>0, \quad k_{11}k_{22}-k_{12}^2>0 \quad (35)$$

with arbitrary real numbers α, β, γ, δ which satisfy Eq. (18). From Eq. (35) it follows that the thermal conductivity tensor generated by Eq. (20) is a positive definite symmetric two-dimensional second order tensor.

4 Examples

4.1 Example 1

The two-dimensional steady-state heat conduction problem for the rectangle whose vertex points P_0, P_1, P_2, P_3 (Fig. 3) is defined by the next equations

$$K\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}\right)+R=0, \quad (x,y)\in A, \quad (R=\text{constant}), \quad (36)$$

$$T=T_0=\text{constant on } \overline{P_0P_3}, \quad T=T_1=\text{constant on } \overline{P_1P_2}, \quad (37)$$

$$Q=0 \text{ on } \overline{P_0P_1} \text{ and } \overline{P_3P_2}. \quad (38)$$

The corresponding anisotropic heat conduction problem is defined as

$$k_{11}\frac{\partial^2 t}{\partial \xi^2}+2k_{12}\frac{\partial^2 t}{\partial\xi\partial\eta}+k_{22}\frac{\partial^2 t}{\partial \eta^2}+r=0, \quad (x,y)\in a, \quad r=R, \quad (39)$$

$$t=t_0=T_0 \text{ on } \overline{p_0p_3}, \quad (40)$$

$$t=t_1=T_1 \text{ on } \overline{p_1p_2}, \quad (41)$$

$$q=0 \text{ on } \overline{p_0p_1} \text{ and } \overline{p_3p_2}. \quad (42)$$

In the present problem the anisotropic two-dimensional body is a parallelogram with vertex points $p_0(0,0)$, $p_1(\delta L/\varepsilon,-\gamma L/\varepsilon)$, $p_2\left((\delta L-H\beta)/\varepsilon,(\alpha H-\gamma L)/\varepsilon\right)$, $p_3(-\beta H/\varepsilon,\alpha H/\varepsilon)$ as shown in Fig. 4. The solution of the isotropic boundary value problem formulated by Eqs. (36–38) is as follows

$$T(x,y)=T_1+\left(\frac{T_2-T_1}{L}+\frac{RL}{2K}\right)x-\frac{R}{2K}x^2, \quad (x,y)\in A. \quad (43)$$

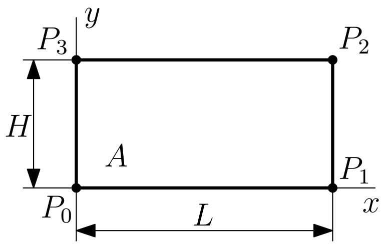

Figure 3: Rectangular domain.

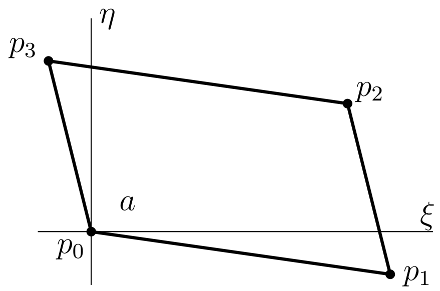

Figure 4: Parallelogram shape domain.

According to Theorem 1 the solution of the corresponding anisotropic heat conduction problem is

$$t(\xi,\eta)=t_1+\left(\frac{t_2-t_1}{L}+\frac{rL}{2K}\right)(\alpha\xi+\beta\eta)- -\frac{r}{2K}(\alpha\xi+\beta\eta)^2, \quad (\xi,\eta)\in a. \quad (44)$$

4.2 Example 2

The two-dimensional time dependent problem for an isotropic body shown in Fig. 5 defined as

$$K\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}\right)+R=MC\frac{\partial T}{\partial\Theta}, \quad (x,y)\in A, \quad (R=\text{constant}), \quad (45)$$

$$T=0, \text{ on } \overline{P_0P_3}\cup\overline{P_1P_2}, \quad (46)$$

$$Q=0, \text{ on } \overline{P_0P_1}\cup\overline{P_3P_2}, \quad (47)$$

$$T(x,y,0)=T_0=\text{constant}, \quad (x,y)\in A. \quad (48)$$

The solution of this initial-boundary value problem can be written in the form [12]

$$T(x,y,\Theta)=\frac{RL^2}{K}\left[\frac{1}{2}\left(1-\left(\frac{x}{L}\right)^2\right)- -2\sum_{n=0}^{\infty}\frac{(-1)^n}{(\nu_n L)^3}\exp\left(-\frac{K}{MC}\nu_n^2\Theta\right)\cos\nu_n x\right], \quad (x,y)\in A, \quad \Theta\geq 0, \quad (49)$$

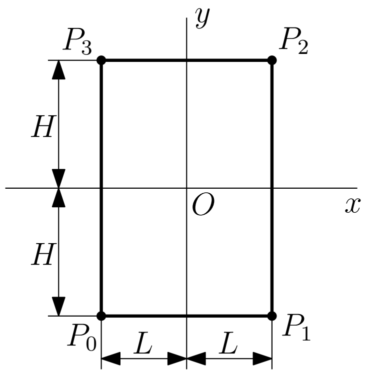

Figure 5: Transient heat conduction in isotropic rectangular domain.

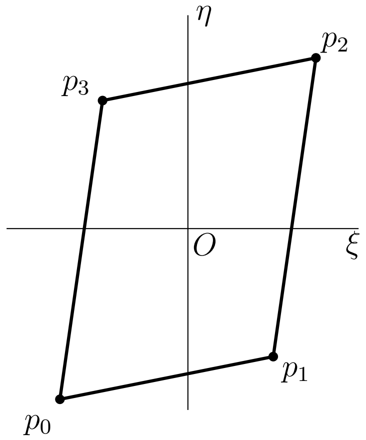

Figure 6: Transient heat conduction in anisotropic parallelogram domain.

where

$$\nu_n = \frac{2n+1}{2L}\pi, \quad (n = 0, 1, 2 \ldots). \tag{50}$$

The domain of the corresponding anisotropic body is shown in Fig. 6. The field equation is given by Eq. (9) with $r = R =$ constant and we have the following boundary and initial conditions for the anisotropic heat conduction problem

$$t = 0 \text{ on } \overline{p_0 p_3} \cup \overline{p_1 p_2}, \tag{51}$$

$$q = 0 \text{ on } \overline{p_0 p_1} \cup \overline{p_3 p_2}, \tag{52}$$

$$t(\xi, \eta, 0) = t_0 = T_0 = \text{constant}, \quad (\xi, \eta) \in a. \tag{53}$$

The positions of the vertex points of the two-dimensional anisotropic body are as follows (Fig. 6) $p_0(-\alpha L - \beta H, -\gamma L - \delta H), p_1(\alpha L - \beta H, \gamma L - \delta H)$, $p_2(\alpha L + \beta H, \gamma L + \delta H), p_3(-\alpha L + \beta H, -\gamma L + \delta H)$.

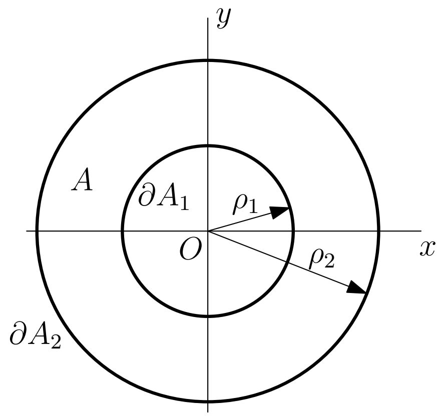

Figure 7: Hollow circular domain.

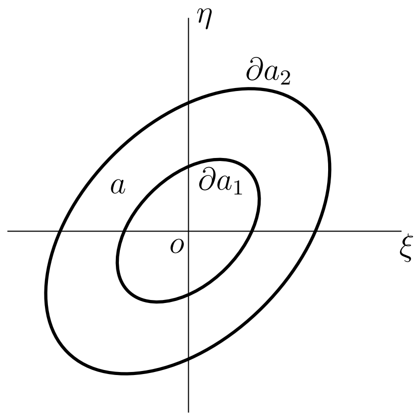

Figure 8: Hollow elliptical domain.

Application of Theorem 1 gives the result

$$t(\xi, \eta, \Theta) = \frac{rL^2}{K}\left[\frac{1}{2}\left(1 - \left(\frac{\alpha\xi + \beta\eta}{L}\right)^2\right) - \right.$$
$$\left. -2\sum_{n=0}^{\infty} \frac{(-1)^n}{(\nu_n L)^3} \exp\left(-\frac{K}{\lambda mc}\nu_n \vartheta\right) \cos\left(\nu_n(\alpha\xi + \beta\eta)\right)\right],$$
$$(\xi, \eta) \in a, \quad \vartheta \geq 0. \tag{54}$$

4.3 Example 3

We consider the next two-dimensional problem of the steady-state heat conduction (Fig. 7)

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0 \text{ in}$$
$$A = \left\{(x, y) | \rho_1^2 \leq x^2 + y^2 \leq \rho_2^2\right\}, \quad \rho_1 > 0, \tag{55}$$

$$T = T_1 \text{ on } \partial A_1 = \left\{(x, y) | x^2 + y^2 = \rho_1^2\right\}, \tag{56}$$

$$T = T_2 \text{ on } \partial A_2 = \left\{(x, y) | x^2 + y^2 = \rho_2^2\right\}. \tag{57}$$

Here T_1 and T_2 are constants. The solution of the boundary value problem formulated by Eqs. (55–57) is

$$T(x,y) = T_1 + (T_2 - T_1)\frac{\ln \frac{x^2+y^2}{\rho_1^2}}{\ln \frac{\rho_2^2}{\rho_1^2}}, \quad (x,y) \in A \cup \partial A_1 \cup \partial A_2. \tag{58}$$

The corresponding anisotropic heat conduction problem to the isotropic heat conduction problem formulated by Eqs. (55–57) is as follows

$$k_{11}\frac{\partial^2 t}{\partial \xi^2} + 2k_{12}\frac{\partial^2 t}{\partial \xi \partial \eta} + k_{22}\frac{\partial^2 t}{\partial \eta^2} = 0 \text{ in } a, \tag{59}$$

$$t = t_1 = T_1 \text{ on } \partial a_1, \tag{60}$$

$$t = t_2 = T_2 \text{ on } \partial a_2, \tag{61}$$

where (Fig. 8)

$$a = \left\{(\xi,\eta) | \rho_1^2 \leq \left(\alpha^2+\gamma^2\right)\xi^2 + +2(\alpha\beta+\gamma\delta)\xi\eta + \left(\beta^2+\delta^2\right)\eta^2 \leq \rho_2^2\right\}, \tag{62}$$

$$\partial a_i = \left\{(\xi,\eta) | \left(\alpha^2+\gamma^2\right)\xi^2 + 2(\alpha\beta+\gamma\delta)\xi\eta + + \left(\beta^2+\delta^2\right)\eta^2 = \rho_i^2, i=1,2\right\}. \tag{63}$$

Application of Theorem 1 gives

$$t(\xi,\eta) = t_1 + (t_2 - t_1)\frac{\ln\left[\frac{(\alpha\xi+\beta\eta)^2+(\gamma\xi+\delta\eta)^2}{\rho_1^2}\right]}{\ln\left(\frac{\rho_1^2}{\rho_2^2}\right)}. \tag{64}$$

4.4 Example 4

Steady-state conduction in a sector of a circular ring (Fig. 9) is prescribed by the next equations

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0 \text{ in } A, \tag{65}$$

$$T = T_1 = \text{constant on } \partial A_1 \text{ and } T = T_2 = \text{constant on } \partial A_2, \tag{66}$$

$$Q = 0 \text{ on } \partial A_3 \cup \partial A_4. \tag{67}$$

The solution of this problem is

$$T(x,y) = T_1 + \frac{T_2 - T_1}{\phi}\tan^{-1}\frac{y}{x}, \quad 0 \leq \phi \leq \frac{\pi}{2}. \tag{68}$$

The corresponding steady-state heat conduction problem can be written in the form

$$k_{11}\frac{\partial^2 t}{\partial \xi^2} + 2k_{12}\frac{\partial^2 t}{\partial \xi \partial \eta} + k_{22}\frac{\partial^2 t}{\partial \eta^2} = 0 \text{ in } a, \tag{69}$$

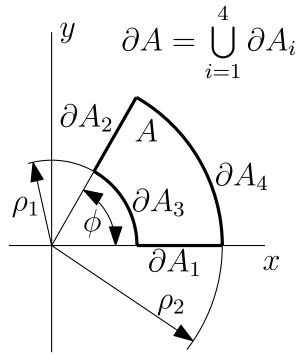

Figure 9: Sector of a circular ring.

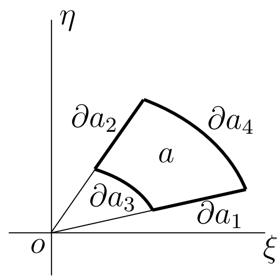

Figure 10: Sector of an elliptical ring.

$$t = t_1 = T_1 \text{ on } \partial a_1, \quad t = t_2 = T_2 \text{ on } \partial a_2, \tag{70}$$

$$q = 0 \text{ on } \partial a_3 \cup \partial a_4. \tag{71}$$

The two-dimensional domain a and its boundary contour are shown in Fig. 10. The equation of the boundary segment ∂a_i $(i = 1, 2)$ is

$$\left(\alpha^2+\gamma^2\right)\xi^2 + 2(\alpha\beta+\gamma\delta)\xi\eta + + \left(\beta^2+\delta^2\right)\eta^2 = \rho_i^2, \quad (i=1,2) \tag{72}$$

and the equation of the boundary segment ∂a_i $(i = 3, 4)$ is

$$\eta = p_i\xi, \quad (i=3,4), \quad p_3 = -\frac{\gamma}{\delta}, \quad p_4 = \frac{\alpha\tan\phi - \gamma}{\delta - \beta\tan\phi}. \tag{73}$$

By the application of Theorem 1 we obtain

$$t(\xi,\eta) = t_1 + \frac{t_2 - t_1}{\phi}\tan^{-1}\frac{\gamma\xi+\delta\eta}{\alpha\xi+\beta\eta}. \tag{74}$$

4.5 Example 5

The transient heat conduction problem for the isotropic two-dimensional domain whose boundary

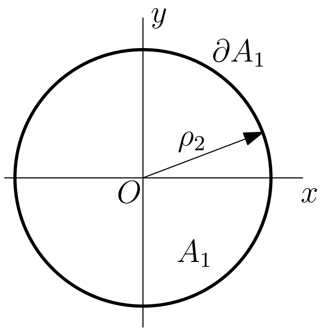

Figure 11: Solid circular domain.

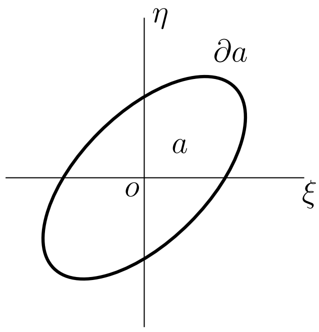

Figure 12: Solid elliptical domain.

curve is a circle of radius ρ_2 (Fig. 11) is defined by the equation

$$K\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}\right)=MC\frac{\partial T}{\partial \Theta}, \qquad (x,y)\in A, \quad \Theta>0, \tag{75}$$

$$T(x,y,0)=T_0=\text{constant}, \quad (x,y)\in A, \tag{76}$$

$$T(x,y,\Theta)=0, \quad (x,y)\in \partial A_1, \quad \Theta>0. \tag{77}$$

The solution of this initial-boundary value problem [2, 12]

$$T(x,y,\Theta)=2\frac{T_0}{\rho_2}\sum_{i=1}^{\infty}\frac{J_0\left(\omega_i\sqrt{x^2+y^2}\right)}{\omega_i J_1(\omega_i\rho_2)} \exp\left(-\frac{K}{MC}\omega_i^2\Theta\right), \tag{78}$$

where $J_0(z)$ and $J_1(z)$ are Bessel functions of the first kind and of order zero and one, respectively. Furthermore, ω_i $(i=1,2,\ldots)$ is the positive root of the transcendent equation

$$J_0(\rho_2\tilde{\omega})=0. \tag{79}$$

The associated transient anisotropic heat conduction problem is formulated by the following equations in $a\cup\partial a$ (Fig. 12)

$$k_{11}\frac{\partial^2 t}{\partial \xi^2}+2k_{12}\frac{\partial^2 t}{\partial \xi\partial\eta}+k_{22}\frac{\partial^2 t}{\partial \eta^2}=mc\frac{\partial t}{\partial \vartheta}, \qquad (\xi,\eta)\in a, \quad \vartheta>0. \tag{80}$$

$$t(\xi,\eta,0)=t_0=T_0=\text{constant}, \quad (\xi,\eta)\in a, \tag{81}$$

$$t(\xi,\eta,\Theta)=0, \quad (\xi,\eta)\in \partial a, \quad \vartheta>0. \tag{82}$$

Here the equation of the boundary curve ∂a (Fig. 12) is

$$\left(\alpha^2+\gamma^2\right)\xi^2+2(\alpha\beta+\gamma\delta)\xi\eta+ +\left(\beta^2+\delta^2\right)\eta^2-\rho_2^2=0. \tag{83}$$

From Theorem 1 for this case we have

$$t(\xi,\eta,\vartheta)=2\frac{t_0}{\rho_2}\sum_{i=1}^{\infty}\frac{b}{\omega_i J_1(\omega_i\rho_2)}\exp\left(-\frac{K}{mc}\omega_i^2\vartheta\right), \tag{84}$$

where

$$b=J_0\left\{\omega_i\left[\left(\alpha^2+\gamma^2\right)\xi^2+ +2(\alpha\beta+\gamma\delta)\xi\eta+\left(\beta^2+\delta^2\right)\eta^2\right]^{0.5}\right\} \tag{85}$$

5 Relationship for the thermal resistances

Let us consider a hollow two-dimensional body in the plane Oxy which is bounded by two closed curves ∂A_1 ans ∂A_2 as shown in Fig. 13. The steady-state heat conductance in this body is determined by constant boundary temperatures according to next equations

$$T=T_1=\text{constant on } \partial A_1, \tag{86}$$

$$T=T_2=\text{constant on } \partial A_2. \tag{87}$$

It is known the heat flow Q_{12} between the boundary curves ∂A_1 and ∂A_2 for isotropic homogeneous body can be obtained as $(T_1>T_2)$

$$Q_{12}=K(T_1-T_2)\int\limits_{\partial A_1}\left(\frac{\partial\tilde{T}}{\partial x}N_x+\frac{\partial\tilde{T}}{\partial y}N_y\right)\mathrm{d}S= =K(T_1-T_2)\int\limits_{A}\left[\left(\frac{\partial\tilde{T}}{\partial x}\right)^2+\left(\frac{\partial\tilde{T}}{\partial y}\right)^2\right]\mathrm{d}A, \tag{88}$$

where $\tilde{T}=\tilde{T}(x,y)$ is the solution of the following boundary value problem

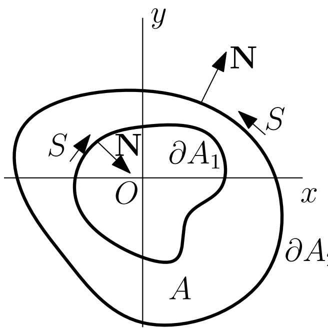

Figure 13: Hollow two-dimensional isotropic body.

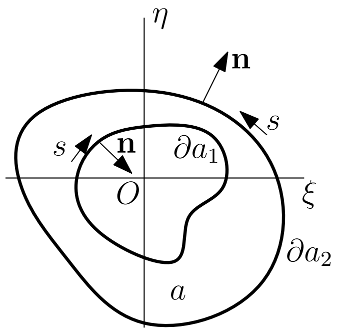

Figure 14: Hollow two-dimensional anisotropic body.

$$\frac{\partial^2 \tilde{T}}{\partial x^2} + \frac{\partial^2 \tilde{T}}{\partial y^2} = 0, \quad (x,y) \in A, \tag{89}$$

$$\begin{aligned} \tilde{T}(x,y) &= 1, \quad (x,y) \in \partial A_1, \\ \tilde{T}(x,y) &= 0, \quad (x,y) \in \partial A_2. \end{aligned} \tag{90}$$

The thermal resistance of this two-dimensional body is defined as [1]

$$\begin{aligned} \Omega &= \frac{1}{K \int\limits_{\partial A_1} \left(\frac{\partial \tilde{T}}{\partial x} N_x + \frac{\partial \tilde{T}}{\partial y} N_y \right) \mathrm{d}s} = \\ &= \frac{1}{K \int\limits_{A} \left[\left(\frac{\partial \tilde{T}}{\partial x} \right)^2 + \left(\frac{\partial \tilde{T}}{\partial y} \right)^2 \right] \mathrm{d}A}. \end{aligned} \tag{91}$$

By the same way we can define the thermal resistance of the hollow two-dimensional anisotropic domain shown in Fig. 14. At first we consider a steady-state heat conduction problem defined by the next equations

$$t = t_1 = \text{constant on } \partial a_1, \tag{92}$$

$$t = t_2 = \text{constant on } \partial a_2. \tag{93}$$

The heat flow q_{12} from the inner boundary curve ∂a_1 to the outer boundary curve ∂a_2 $(t_1 > t_2)$ can be computed as

$$\begin{aligned} q_{12} &= (t_1 - t_2) \int\limits_{\partial a_1} \left[\left(k_{11} \frac{\partial \tilde{t}}{\partial \xi} + k_{12} \frac{\partial \tilde{t}}{\partial \eta} \right) n_\xi + \right. \\ &\quad \left. + \left(k_{21} \frac{\partial \tilde{t}}{\partial \xi} + k_{22} \frac{\partial \tilde{t}}{\partial \eta} \right) n_\eta \right] \mathrm{d}s = \\ &= (t_1 - t_2) \int\limits_{a} \left[k_{11} \left(\frac{\partial \tilde{t}}{\partial \xi} \right)^2 + \right. \\ &\quad \left. + 2k_{12} \frac{\partial \tilde{t}}{\partial \xi} \frac{\partial \tilde{t}}{\partial \eta} + k_{22} \left(\frac{\partial \tilde{t}}{\partial \eta} \right)^2 \right] \mathrm{d}a, \end{aligned} \tag{94}$$

where $\tilde{t} = \tilde{t}(\xi, \eta)$ is the solution of the next boundary value problem

$$\begin{gathered} k_{11} \frac{\partial^2 \tilde{t}}{\partial \xi^2} + 2k_{12} \frac{\partial^2 \tilde{t}}{\partial \xi^2} \frac{\partial^2 \tilde{t}}{\partial \xi \partial \eta} + k_{22} \frac{\partial^2 \tilde{t}}{\partial \eta^2} = 0 \\ (\xi, \eta) \in a, \end{gathered} \tag{95}$$

$$\begin{aligned} \tilde{t}(\xi, \eta) &= 1, \quad (\xi, \eta) \in \partial a_1, \\ \tilde{t}(\xi, \eta) &= 0, \quad (\xi, \eta) \in \partial a_2. \end{aligned} \tag{96}$$

The thermal resistance of the two-dimensional hollow anisotropic body shown in Fig. 14 is defined by the next equation

$$\omega = \frac{1}{W} \tag{97}$$

where

$$\begin{aligned} W &= \int\limits_{\partial a_1} \left[\left(k_{11} \frac{\partial \tilde{t}}{\partial \xi} + k_{12} \frac{\partial \tilde{t}}{\partial \eta} \right) n_\xi + \right. \\ &\quad \left. + \left(k_{11} \frac{\partial \tilde{t}}{\partial \xi} + k_{12} \frac{\partial \tilde{t}}{\partial \eta} \right) n_\xi \right] \mathrm{d}s = \\ &= \int\limits_{a} \left[k_{11} \left(\frac{\partial \tilde{t}}{\partial \xi} \right)^2 + 2k_{12} \frac{\partial \tilde{t}}{\partial \xi} \frac{\partial \tilde{t}}{\partial \eta} + k_{22} \left(\frac{\partial \tilde{t}}{\partial \eta} \right)^2 \right] \mathrm{d}a. \end{aligned} \tag{98}$$

Assuming that Eqs. (17–20) are valid, then we have

$$\omega = |\varepsilon| \Omega. \tag{99}$$

The validity of Eq. (99) follows from the formulas of Ω and ω and from Eqs. (27–28). It is evident if $|\varepsilon| = 1$ then the isotropic hollow two-dimensional domain and the corresponding anisotropic hollow two-dimensional domain have the same thermal resistance. The thermal resistance of the isotropic hollow circular domain shown in Fig. 7 is

$$\Omega = \frac{\ln \frac{\rho_2}{\rho_1}}{2\pi K}. \tag{100}$$

In this case the thermal resistance of the corresponding anisotropic hollow two-dimensional body which is shown in Fig. 13 can be computed as according to Eq. (99)

$$\omega = |\varepsilon| \frac{\ln \frac{\rho_2}{\rho_1}}{2\pi K}. \tag{101}$$

6 Determination of the coefficients of space coordinate transformation in terms of heat conductivity coefficients

At first, we define the unit-free thermal conductivity coefficients by the following equations

$$\overline{k}_{11} = \frac{k_{11}}{K}, \quad \overline{k}_{22} = \frac{k_{22}}{K}, \quad \overline{k}_{12} = k_{21} = \frac{k_{12}}{K}. \tag{102}$$

The linear coordinate transformation (17) can be considered as a combination of a pure rotation with a stretching and shrinking which can be described as

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} c_1 & 0 \\ 0 & c_2 \end{bmatrix} \begin{bmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix} \begin{bmatrix} \xi \\ \eta \end{bmatrix}, \tag{103}$$

that is we have

$$\begin{aligned} \alpha = c_1 \cos\varphi, \quad \beta = -c_1 \sin\varphi, \\ \gamma = c_2 \sin\varphi, \quad \delta = c_2 \cos\varphi, \end{aligned} \tag{104}$$

$$\varepsilon = c_1 c_2. \tag{105}$$

From Eqs. $(20)_{1,2,3}$ and (104) it follows that

$$\overline{k}_{11} = \frac{1}{2c_1^2c_2^2}\left(c_1^2 + c_2^2 + \left(c_2^2 - c_1^2\right)\cos 2\varphi\right), \tag{106}$$

$$\overline{k}_{22} = \frac{1}{2c_1^2c_2^2}\left(c_1^2 + c_2^2 + \left(c_1^2 - c_2^2\right)\cos 2\varphi\right), \tag{107}$$

$$\overline{k}_{12} = \frac{1}{2c_1^2c_2^2}\left(c_1^2 - c_2^2\right)\sin 2\varphi. \tag{108}$$

Subtracted Eq. (106) from Eq. (107) we obtain

$$\overline{k}_{22} - \overline{k}_{11} = \frac{1}{c_1^2c_2^2}\left(c_1^2 - c_2^2\right)\cos 2\varphi. \tag{109}$$

Combination of Eq. (109) with Eq. (108)

$$\tan 2\varphi = \frac{2\overline{k}_{12}}{\overline{k}_{22} - \overline{k}_{11}}, \tag{110}$$

that is

$$\varphi = \frac{1}{2}\arctan\left(\frac{2\overline{k}_{12}}{\overline{k}_{22} - \overline{k}_{11}}\right). \tag{111}$$

Next, we will use the following equations to obtain the expressions of c_1 and c_2 in terms of $\overline{k}_{11}$, $\overline{k}_{22}$ and $\overline{k}_{12}$

$$\overline{k}_{11} + \overline{k}_{22} = \frac{1}{c_1^2} + \frac{1}{c_2^2}, \tag{112}$$

$$\begin{aligned} \cos 2\varphi &= \frac{1}{\sqrt{1 + \tan^2 2\varphi}} = \\ &= \frac{\overline{k}_{22} - \overline{k}_{11}}{\sqrt{\left(\overline{k}_{22} - \overline{k}_{11}\right)^2 + 4\overline{k}_{12}^2}}. \end{aligned} \tag{113}$$

The combination of Eq. (110) with Eqs. (112–113) leads to the formulas of c_1 and c_2

$$c_1 = \sqrt{\frac{2}{\overline{k}_{11} + \overline{k}_{22} + \sqrt{\left(\overline{k}_{22} - \overline{k}_{11}\right)^2 + 4\overline{k}_{12}^2}}}, \tag{114}$$

$$c_2 = \sqrt{\frac{2}{\overline{k}_{11} + \overline{k}_{22} - \sqrt{\left(\overline{k}_{22} - \overline{k}_{11}\right)^2 + 4\overline{k}_{12}^2}}}. \tag{115}$$

7 Conclusions

Steady-state and time dependent heat conduction problems in homogeneous anisotropic solid bodies are studied by using an analytical method. The temperature distribution for the considered anisotropic two-dimensional body is derived from the solution of an isotropic two-dimensional body. A linear coordinate transformation is used for the spatial and time coordinates to generate the mapping between the isotropic and anisotropic heat conduction problems. Examples illustrate the application of the developed method. The main result of the paper is a contribution to the existing exact benchmark solutions for conduction of heat in anisotropic solid bodies.

References:

[1] H. S. Carslaw, J. C. Jaeger, *Conduction of Heat in Solids*, Oxford University Press, UK, 1959.

[2] M. N. Özisik, *Heat Conduction*, Wiley, New York, 1993.

[3] F. Kowsary, M. Arabi, Monte Carlo Solution of Anisotropic Heat Conduction, *International Communications in Heat and Mass Transfer*, Vol.26, No.8, 1999, pp. 1163-1173.

[4] K. C. Poon, Transformation of Heat Conduction Problems in Layered-Composites from Anisotropic to Orthotropic, *Letters in Heat and Mass Transfer*, Vol.6, No.6, 1979, pp. 503-511.

[5] K. C. Poon, R. C. H. Tsou, Y. P. Chang, Solution of Anisotropic Problems of First Class by Coordinate-Transformation, *Journal of Heat Transfer*, Vol.101, No.2, 1979, pp. 340-345.

[6] C. C. Ma, S. W. Chang, Analytical Exact Solutions of Heat Conduction Problems for Anisotropic Multi-Layered Media, *International Journal of Heat and Mass Transfer*, Vol.47, No.8-9, 2004, pp. 1643-1655.

[7] L. Yan, A. Haji-Sheikh, J. V. Beck, Thermal Characteristics of Two-Layered Bodies with Embedded Thin-Film Heat Source, *Journal of Electronis Packaging*, Vol.115, No.3, 1993, pp. 276-283.

[8] G. P. Mulholland, B. P. Gupta, Heat Transfer in a Three-Dimensional Anisotropic Solid of Arbitrary Shape, *Journal of Heat Transfer*, Vol.99, No.1, 1977, pp. 135-137.

[9] D. L. Clements, W. S. Budhi, A Boundary Element Method for the Solution of a Class of Steady-State Problems for Anisotropic Media, *Journal of Heat Transfer*, Vol.121, No.2, 1999, pp. 462-465.

[10] Z. Xiangzhou, Steady-State Temperatures in an Anisotropic Strip, *Journal of Heat Transfer*, Vol.112, No.1, 1990, pp. 16-20.

[11] M. H. Hsieh, C. C. Ma, Analytical Investigations for Heat Conduction Problems in Anisotropic Thin-Layer Media with Embedded Heat Sources, *International Journal of Heat and Mass Transfer*, Vol.45, No.20, 2002, pp. 4117-4132.

[12] R. B. Hetnarski, M. R. Eslami, *Thermal Stresses – Advanced Theory and Applications*, Springer, New York, 2010.

Sources of funding for research presented in a scientific article or scientific article itself

This research was supported by the National Research, Development and Innovation Office – NKFIH, K115701. This research was also carried out as part of the EFOP-3.6.1-16-2016-00011 Younger and Renewing University – Innovative Knowledge City – institutional development of the University of Miskolc aiming at intelligent specialisation" project implemented in the framework of the Szechenyi 2020 program. The realization of this project is supported by the European Union, co-financed by the European Social Fund.