Some Ostrowski Type Inequalities On time Scales Involving Functions of Two Independent Variables

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Abstract: In this paper, we establish some new Ostrowski type inequalities on time scales involving functions of two independent variables for multiple points, which unify continuous and discrete analysis, and some of which are sharp. The established results extend some known results in the literature, and can be used in the estimate of error bounds for some numerical quadrature formulae.

Key–Words: Ostrowski type inequality; Time scales; Numerical integration; Error bound; Sharp bound

1 Introduction

Since Ostrowski established the famous Ostrowski inequality in [1], various generalizations of the ostrowski inequality including continuous and discrete versions have been established in recent years (for example, see [2-13] and the references therein), which can be used to provide explicit error bounds for some known and some new numerical quadrature formulae. On the other hand, Hilger[14] initiated the theory of time scales as a theory capable of treating continuous and discrete analysis in a consistent way, based on which some authors have studied the Ostrowski type inequalities on time scales (see [15-23]). The established Ostrowski type inequalities on time scales unify continuous and discrete analysis to some extent.

Our aim in this paper is to establish some new Ostrowski type inequalities on time scales involving functions of two independent variables, which on one hand extend some known results in the literature, on the other hand unify continuous and discrete analysis.

We first give the following definition for further use.

Definition 1 $h_k : \mathbb{T}^2 \to \mathbb{R},\ k = 0, 1, 2, \dots$ *are defines by*

$$h_{k+1}(t,s) = \int_s^t h_k(\tau, s)\Delta\tau,\ \forall s,t \in \mathbb{T},$$

where $\mathbb{T}$ *is an arbitrary time scale, and* $h_0(t,s) = 1$.

Throughout this paper, $\mathbb{R}$ denotes the set of real numbers and $\mathbb{R}_+ = [0,\infty)$, while $\mathbb{Z}$ denotes the set of integers, and $\mathbb{N}_0$ denotes the set of nonnegative integers. For a function f and two integers $m_0,\ m_1$, we have $\sum_{s=m_0}^{m_1} f = 0$ provided $m_0 > m_1$. $\mathbb{T}_1,\ \mathbb{T}_2$ denote two arbitrary time scales, and for an interval $[a,b]$, $[a,b]_{\mathbb{T}_i} := [a,b] \bigcap \mathbb{T}_i,\ i = 1,2$. Finally, for the sake of convenience, we denote the forward jump operators on $\mathbb{T}_1,\ \mathbb{T}_2$ by σ uniformly.

2 Main Results

Theorem 2 *Let* $a,\ b \in \mathbb{T}_1,\ c,\ d \in \mathbb{T}_2,\ f :\in C_{rd}([a,b]_{\mathbb{T}_1} \times [c,d]_{\mathbb{T}_2}, \mathbb{R})$ *such that the partial delta derivative of order 2 exists and there exists a constant* K *with* $\sup_{a<s<b,\ c<t<d} |\frac{\partial^2 f(s,t)}{\Delta_1 s \Delta_2 t}| = K$. *Suppose that* $x_i \in [a,b]_{\mathbb{T}_1},\ y_i \in [c,d]_{\mathbb{T}_2},\ i = 0,1,...,n$, *where* $n \geq 1$ *is an integer.* $I_n : a = x_0 < x_1 < ... < x_{n-1} < x_n = b$ *is a division of the interval* $[a,b]_{\mathbb{T}_1}$, *while* $J_n : c = y_0 < y_1 < ... < y_{n-1} < y_n = d$ *is a division of the interval* $[c,d]_{\mathbb{T}_2}$. $\alpha_i \in [x_{i-1}, x_i]_{\mathbb{T}_1},\ \beta_i \in [y_{i-1}, y_i]_{\mathbb{T}_2},\ i = 1,2,...,n$. *Then we have*

$$
\begin{aligned}
&|\sum_{i=1}^{n-1}\sum_{j=1}^{n-1}(h_k(x_i,\alpha_i)-h_k(x_i,\alpha_{i+1}))(h_k(y_j,\beta_j)\\
&-h_k(y_j,\beta_{j+1}))f(x_i,y_j)+\sum_{i=1}^{n-1}(h_k(x_i,\alpha_i)-h_k(x_i,\alpha_{i+1}))\\
&h_k(y_n,\beta_n)f(x_i,y_n)-\sum_{i=1}^{n-1}(h_k(x_i,\alpha_i)-h_k(x_i,\alpha_{i+1}))\\
&h_k(y_0,\beta_1)f(x_i,y_0)+\sum_{j=1}^{n-1}h_k(x_n,\alpha_n)(h_k(y_j,\beta_j)\\
&-h_k(y_j,\beta_{j+1}))f(x_n,y_j)+h_k(x_n,\alpha_n)h_k(y_n,\beta_n)f(x_n,y_n)\\
&-\sum_{j=1}^{n-1}h_k(x_0,\alpha_1)(h_k(y_j,\beta_j)-h_k(y_j,\beta_{j+1}))f(x_0,y_j)\\
&-h_k(x_0,\alpha_1)h_k(y_n,\beta_n)f(x_0,y_n)\\
&-h_k(x_n,\alpha_n)h_k(y_0,\beta_1)f(x_n,y_0)\\
&+h_k(x_0,\alpha_1)h_k(y_0,\beta_1)f(x_0,y_0)\\
&-\sum_{i=0}^{n-1}\int_{x_i}^{x_{i+1}}[h_k(y_n,\beta_n)h_{k-1}(s,\alpha_{i+1})f(\sigma(s),y_n)\\
&-h_k(y_0,\beta_1)h_{k-1}(s,\alpha_{i+1})f(\sigma(s),y_0)]\Delta_1 s\\
&-\sum_{i=0}^{n-1}\sum_{j=1}^{n-1}\int_{x_i}^{x_{i+1}}[h_k(y_j,\beta_j)h_{k-1}(s,\alpha_{i+1})\\
&-h_k(y_j,\beta_{j+1})h_{k-1}(s,\alpha_{i+1})]f(\sigma(s),y_j)\Delta_1 s\\
&-\sum_{j=0}^{n-1}\int_{y_j}^{y_{j+1}}[h_k(x_n,\alpha_n)h_{k-1}(t,\beta_{j+1})f(x_n,\sigma(t))\\
&-h_k(x_0,\alpha_1)h_{k-1}(t,\beta_{j+1})f(x_0,\sigma(t))]\Delta_2 t\\
&-\sum_{i=1}^{n-1}\sum_{j=0}^{n-1}\int_{y_j}^{y_{j+1}}[h_k(x_i,\alpha_i)h_{k-1}(t,\beta_{j+1})\\
&-h_k(x_i,\alpha_{i+1})h_{k-1}(t,\beta_{j+1})]f(x_i,\sigma(t))\Delta_2 t\\
&+\sum_{i=0}^{n-1}\sum_{j=0}^{n-1}\int_{x_i}^{x_{i+1}}\int_{y_j}^{y_{j+1}}h_{k-1}(s,\alpha_{i+1})h_{k-1}(t,\beta_{j+1})\\
&f(\sigma(s),\sigma(t))\Delta_2 t\Delta_1 s|\\
&\le K\{\sum_{i=0}^{n-1}[(-1)^{k+1}h_{k+1}(x_i,\alpha_{i+1})+h_{k+1}(x_{i+1},\alpha_{i+1})]\}\\
&\times\{\sum_{j=0}^{n-1}[(-1)^{k+1}h_{k+1}(y_j,\beta_{j+1})+h_{k+1}(y_{j+1},\beta_{j+1})]\}.
\end{aligned}
\tag{1}
$$

The inequality (1) is sharp in the sense that the right side of (1) can not be replaced by a smaller one.

The following lemma will be used to prove Theorem 2.

Lemma 3 *(Generalized Montgomery Identity) Let*

$$
H(s,t,I_n,J_n)=h_k(s,\alpha_{i+1})h_k(t,\beta_{j+1}),
$$

$$
(s,t)\in[x_i,x_{i+1})\times[y_j,y_{j+1}),\ i,\ j=0,1,...,n-1. \tag{2}
$$

Then we have

$$
\begin{aligned}
&\int_a^b\int_c^d H(s,t,I_n,J_n)\frac{\partial^2 f(s,t)}{\Delta_1 s\Delta_2 t}\Delta_2 t\Delta s\\
&=\sum_{i=1}^{n-1}\sum_{j=1}^{n-1}(h_k(x_i,\alpha_i)-h_k(x_i,\alpha_{i+1}))(h_k(y_j,\beta_j)\\
&-h_k(y_j,\beta_{j+1}))f(x_i,y_j)+\sum_{i=1}^{n-1}(h_k(x_i,\alpha_i)\\
&-h_k(x_i,\alpha_{i+1}))h_k(y_n,\beta_n)f(x_i,y_n)\\
&-\sum_{i=1}^{n-1}(h_k(x_i,\alpha_i)-h_k(x_i,\alpha_{i+1}))h_k(y_0,\beta_1)f(x_i,y_0)\\
&+\sum_{j=1}^{n-1}h_k(x_n,\alpha_n)(h_k(y_j,\beta_j)-h_k(y_j,\beta_{j+1}))f(x_n,y_j)\\
&+h_k(x_n,\alpha_n)h_k(y_n,\beta_n)f(x_n,y_n)\\
&-\sum_{j=1}^{n-1}h_k(x_0,\alpha_1)(h_k(y_j,\beta_j)-h_k(y_j,\beta_{j+1}))f(x_0,y_j)\\
&-h_k(x_0,\alpha_1)h_k(y_n,\beta_n)f(x_0,y_n)\\
&-h_k(x_n,\alpha_n)h_k(y_0,\beta_1)f(x_n,y_0)\\
&+h_k(x_0,\alpha_1)h_k(y_0,\beta_1)f(x_0,y_0)\\
&-\sum_{i=0}^{n-1}\int_{x_i}^{x_{i+1}}[h_k(y_n,\beta_n)h_{k-1}(s,\alpha_{i+1})f(\sigma(s),y_n)\\
&-h_k(y_0,\beta_1)h_{k-1}(s,\alpha_{i+1})f(\sigma(s),y_0)]\Delta_1 s\\
&-\sum_{i=0}^{n-1}\sum_{j=1}^{n-1}\int_{x_i}^{x_{i+1}}[h_k(y_j,\beta_j)h_{k-1}(s,\alpha_{i+1})\\
&-h_k(y_j,\beta_{j+1})h_{k-1}(s,\alpha_{i+1})]f(\sigma(s),y_j)\Delta_1 s\\
&-\sum_{j=0}^{n-1}\int_{y_j}^{y_{j+1}}[h_k(x_n,\alpha_n)h_{k-1}(t,\beta_{j+1})f(x_n,\sigma(t))
\end{aligned}
$$

$$-h_k(x_0,\alpha_1)h_{k-1}(t,\beta_{j+1})f(x_0,\sigma(t))]\Delta_2 t$$

$$-\sum_{i=1}^{n-1}\sum_{j=0}^{n-1}\int_{y_j}^{y_{j+1}}[h_k(x_i,\alpha_i)h_{k-1}(t,\beta_{j+1})$$

$$-h_k(x_i,\alpha_{i+1})h_{k-1}(t,\beta_{j+1})]f(x_i,\sigma(t))\Delta_2 t$$

$$+\sum_{i=0}^{n-1}\sum_{j=0}^{n-1}\int_{x_i}^{x_{i+1}}\int_{y_j}^{y_{j+1}}h_{k-1}(s,\alpha_{i+1})h_{k-1}(t,\beta_{j+1})$$

$$f(\sigma(s),\sigma(t))\Delta_2 t\Delta_1 s. \quad (3)$$

Proof. We have the following observation

$$\int_a^b\int_c^d H(s,t,I_n,J_n)\frac{\partial^2 f(s,t)}{\Delta_1 s\Delta_2 t}\Delta_2 t\Delta_1 s=$$

$$\sum_{i=0}^{n-1}\sum_{j=0}^{n-1}\int_{x_i}^{x_{i+1}}\int_{y_j}^{y_{j+1}}h_k(s,\alpha_{i+1})h_k(t,\beta_{j+1})\frac{\partial^2 f(s,t)}{\Delta_1 s\Delta_2 t}\Delta_2 t\Delta_1 s$$

$$=\sum_{i=0}^{n-1}\sum_{j=0}^{n-1}\int_{x_i}^{x_{i+1}}h_k(s,\alpha_{i+1})[h_k(y_{j+1},\beta_{j+1})\frac{\partial f(s,y_{j+1})}{\Delta_1 s}-$$

$$h_k(y_j,\beta_{j+1})\frac{\partial f(s,y_j)}{\Delta_1 s}-\int_{y_j}^{y_{j+1}}h_{k-1}(t,\beta_{j+1})\frac{\partial f(s,\sigma(t))}{\Delta_1 s}\Delta_2 t]\Delta_1 s$$

$$=\sum_{i=0}^{n-1}\sum_{j=0}^{n-1}\{[h_k(x_{i+1},\alpha_{i+1})f(x_{i+1},y_{j+1})$$

$$-h_k(x_i,\alpha_{i+1})f(x_i,y_{j+1})]h_k(y_{j+1},\beta_{j+1})$$

$$-[h_k(x_{i+1},\alpha_{i+1})f(x_{i+1},y_j)-h_k(x_i,\alpha_{i+1})f(x_i,y_j)]h_k(y_j,\beta_{j+1})$$

$$-\int_{x_i}^{x_{i+1}}h_{k-1}(s,\alpha_{i+1})[h_k(y_{j+1},\beta_{j+1})f(\sigma(s),y_{j+1})$$

$$-h_k(y_j,\beta_{j+1})f(\sigma(s),y_j)]\Delta_1 s$$

$$-\int_{y_j}^{y_{j+1}}h_{k-1}(t,\beta_{j+1})[h_k(x_{i+1},\alpha_{i+1})f(x_{i+1},\sigma(t))$$

$$-h_k(x_i,\alpha_{i+1})f(x_i,\sigma(t))]\Delta_2 t+$$

$$\int_{x_i}^{x_{i+1}}\int_{y_j}^{y_{j+1}}h_{k-1}(s,\alpha_{i+1})h_{k-1}(t,\beta_{j+1})f(\sigma(s),\sigma(t))\Delta_2 t\Delta_1 s\}$$

$$=\sum_{i=1}^{n-1}\sum_{j=0}^{n-1}(h_k(x_i,\alpha_i)-h_k(x_i,\alpha_{i+1}))h_k(y_{j+1},\beta_{j+1})f(x_i,y_{j+1})$$

$$+\sum_{j=0}^{n-1}h_k(x_n,\alpha_n)h_k(y_{j+1},\beta_{j+1})f(x_n,y_{j+1})$$

$$-\sum_{j=0}^{n-1}h_k(x_0,\alpha_1)h_k(y_{j+1},\beta_{j+1})f(x_0,y_{j+1})$$

$$-\sum_{i=1}^{n-1}\sum_{j=0}^{n-1}(h_k(x_i,\alpha_i)-h_k(x_i,\alpha_{i+1}))h_k(y_j,\beta_{j+1})$$

$$f(x_i,y_j)-\sum_{j=0}^{n-1}h_k(x_n,\alpha_n)h_k(y_j,\beta_{j+1})f(x_n,y_j)$$

$$+\sum_{j=0}^{n-1}h_k(x_0,\alpha_1)h_k(y_j,\beta_{j+1})f(x_0,y_j)$$

$$-\sum_{i=0}^{n-1}\int_{x_i}^{x_{i+1}}[h_k(y_n,\beta_n)h_{k-1}(s,\alpha_{i+1})f(\sigma(s),y_n)$$

$$-h_k(y_0,\beta_1)h_{k-1}(s,\alpha_{i+1})f(\sigma(s),y_0)]\Delta_1 s$$

$$-\sum_{i=0}^{n-1}\sum_{j=1}^{n-1}\int_{x_i}^{x_{i+1}}[h_k(y_j,\beta_j)h_{k-1}(s,\alpha_{i+1})$$

$$-h_k(y_j,\beta_{j+1})h_{k-1}(s,\alpha_{i+1})]f(\sigma(s),y_j)\Delta_1 s$$

$$-\sum_{j=0}^{n-1}\int_{y_j}^{y_{j+1}}[h_k(x_n,\alpha_n)h_{k-1}(t,\beta_{j+1})f(x_n,\sigma(t))$$

$$-h_k(x_0,\alpha_1)h_{k-1}(t,\beta_{j+1})f(x_0,\sigma(t))]\Delta_2 t$$

$$-\sum_{i=1}^{n-1}\sum_{j=0}^{n-1}\int_{y_j}^{y_{j+1}}[h_k(x_i,\alpha_i)h_{k-1}(t,\beta_{j+1})$$

$$-h_k(x_i,\alpha_{i+1})h_{k-1}(t,\beta_{j+1})]f(x_i,\sigma(t))\Delta_2 t$$

$$+\sum_{i=0}^{n-1}\sum_{j=0}^{n-1}\int_{x_i}^{x_{i+1}}\int_{y_j}^{y_{j+1}}h_{k-1}(s,\alpha_{i+1})h_{k-1}(t,\beta_{j+1})$$

$$f(\sigma(s),\sigma(t))\Delta_2 t\Delta_1 s. \quad (4)$$

On the other hand,

$$\sum_{i=1}^{n-1}\sum_{j=0}^{n-1}(h_k(x_i,\alpha_i)-h_k(x_i,\alpha_{i+1}))h_k(y_{j+1},\beta_{j+1})$$

$$f(x_i,y_{j+1})+\sum_{j=0}^{n-1}h_k(x_n,\alpha_n)h_k(y_{j+1},\beta_{j+1})f(x_n,y_{j+1})$$

$$-\sum_{j=0}^{n-1}h_k(x_0,\alpha_1)h_k(y_{j+1},\beta_{j+1})f(x_0,y_{j+1})$$

$$-\sum_{i=1}^{n-1}\sum_{j=0}^{n-1}(h_k(x_i,\alpha_i)-h_k(x_i,\alpha_{i+1}))h_k(y_j,\beta_{j+1})$$

$$f(x_i,y_j)-\sum_{j=0}^{n-1}h_k(x_n,\alpha_n)h_k(y_j,\beta_{j+1})f(x_n,y_j)$$

$$+\sum_{j=0}^{n-1}h_k(x_0,\alpha_1)h_k(y_j,\beta_{j+1})f(x_0,y_j)$$

$$=\sum_{i=1}^{n-1}\sum_{j=1}^{n-1}(h_k(x_i,\alpha_i)-h_k(x_i,\alpha_{i+1}))(h_k(y_j,\beta_j)-h_k(y_j,\beta_{j+1}))$$

$$f(x_i,y_j)+\sum_{i=1}^{n-1}(h_k(x_i,\alpha_i)-h_k(x_i,\alpha_{i+1}))h_k(y_n,\beta_n)f(x_i,y_n)$$

$$-\sum_{i=1}^{n-1}(h_k(x_i,\alpha_i)-h_k(x_i,\alpha_{i+1}))h_k(y_0,\beta_1)f(x_i,y_0)$$

$$+\sum_{j=1}^{n-1}h_k(x_n,\alpha_n)(h_k(y_j,\beta_j)-h_k(y_j,\beta_{j+1}))f(x_n,y_j)$$

$$+h_k(x_n,\alpha_n)h_k(y_n,\beta_n)f(x_n,y_n)$$

$$-\sum_{j=1}^{n-1}h_k(x_0,\alpha_1)(h_k(y_j,\beta_j)-h_k(y_j,\beta_{j+1}))f(x_0,y_j)$$

$$-h_k(x_0,\alpha_1)h_k(y_n,\beta_n)f(x_0,y_n)$$

$$-h_k(x_n,\alpha_n)h_k(y_0,\beta_1)f(x_n,y_0)$$

$$+h_k(x_0,\alpha_1)h_k(y_0,\beta_1)f(x_0,y_0). \quad (5)$$

Combining (4) and (5) we obtain the desired result. □

Proof of Theorem 2. We deduce

$$\int_a^b\int_c^d|H(s,t,I_n,J_n)|\Delta_2t\Delta_1s$$

$$=\sum_{i=0}^{n-1}\sum_{j=0}^{n-1}\int_{x_i}^{x_{i+1}}\int_{y_j}^{y_{j+1}}|h_k(s,\alpha_{i+1})h_k(t,\beta_{j+1})|\Delta_2t\Delta_1s$$

$$=[\sum_{i=0}^{n-1}\int_{x_i}^{x_{i+1}}|h_k(s,\alpha_{i+1})|\Delta_1s][\sum_{j=0}^{n-1}\int_{y_j}^{y_{j+1}}|h_k(t,\beta_{j+1})|\Delta_2t]$$

$$=\{\sum_{i=0}^{n-1}[\int_{x_i}^{\alpha_{i+1}}(-1)^kh_k(s,\alpha_{i+1})\Delta_1s+\int_{\alpha_{i+1}}^{x_{i+1}}h_k(s,\alpha_{i+1})\Delta_1s]\}$$

$$\times\{\sum_{j=0}^{n-1}[\int_{y_j}^{\beta_{j+1}}(-1)^kh_k(t,\beta_{j+1})\Delta_2t+\int_{\beta_{j+1}}^{y_{j+1}}h_k(t,\beta_{j+1})\Delta_2t]\}$$

$$=\{\sum_{i=0}^{n-1}[\int_{\alpha_{i+1}}^{x_i}(-1)^{k+1}h_k(s,\alpha_{i+1})\Delta_1s+\int_{\alpha_{i+1}}^{x_{i+1}}h_k(s,\alpha_{i+1})\Delta_1s]\}$$

$$\times\{\sum_{j=0}^{n-1}[\int_{\beta_{j+1}}^{y_j}(-1)^{k+1}h_k(t,\beta_{j+1})\Delta_2t+\int_{\beta_{j+1}}^{y_{j+1}}h_k(t,\beta_{j+1})\Delta_2t]\}$$

$$=\{\sum_{i=0}^{n-1}[(-1)^{k+1}h_{k+1}(x_i,\alpha_{i+1})+h_{k+1}(x_{i+1},\alpha_{i+1})]\}\times$$

$$\{\sum_{j=0}^{n-1}[(-1)^{k+1}h_{k+1}(y_j,\beta_{j+1})+h_{k+1}(y_{j+1},\beta_{j+1})]\}. \quad (6)$$

From (6) and Lemma 3 we can obtain the desired inequality (1).

In order to prove the sharpness of (1), we take $n=1,\ k=1,\ \alpha_1=b,\ \beta_1=d,\ f(s,t)=st$. Then the left side of (1) becomes

$$|\int_a^b\int_c^d\sigma(s)\sigma(t)\Delta_2t\Delta_1s-(d-c)\int_a^bc\sigma(s)\Delta_1s$$

$$-(b-a)\int_c^da\sigma(t)\Delta_2t+(d-c)(b-a)ac|$$

$$=|\int_a^b\int_c^d[\sigma(s)\sigma(t)-c\sigma(s)-a\sigma(t)+ac]\Delta_2t\Delta_1s|$$

$$=|\int_a^b\int_c^d[\sigma(s)-a][\sigma(t)-c]\Delta_2t\Delta_1s|$$

$$=|\int_a^b\int_c^d\{[(s-a)^2]_s^\Delta[(t-c)^2]_t^\Delta-[(\sigma(s)-a)(t-c)$$

$$+(\sigma(t)-c)(s-a)+(s-a)(t-c)]\}\Delta_2t\Delta_1s|$$

$$=|\int_a^b\int_c^d\{[(s-a)^2]_s^\Delta[(t-c)^2]_t^\Delta-[[(s-a)^2]_s^\Delta(t-c)$$

$$+[(t-c)^2]_t^\Delta(s-a)-(t-c)(s-a)]\}\Delta_2t\Delta_1s|$$

$$=|(b-a)^2(d-c)^2-(b-a)^2\int_c^d(t-c)\Delta_2t$$

$$-(d-c)^2\int_a^b(s-a)\Delta_1s+\int_a^b\int_c^d(t-c)(s-a)\Delta_2t\Delta_1s|$$

$$=|\int_a^b\int_c^d(t-d)(s-b)|=\int_a^b\int_c^d(d-t)(b-s)\Delta_2t\Delta_1s$$

$$=\int_b^a(s-b)\Delta_1s\int_d^c(t-d)\Delta_2t$$

$$=h_2(a,b)h_2(c,d).$$

On the other hand, since $K=1$, the right side of (1) becomes $h_2(a,b)h_2(c,d)$, which implies (2) holds for equality form, and hence the sharpness of (1) is proved. □

Remark 4 *If we take $n = 2$, $\alpha_1 = a$, $\alpha_2 = b$, $\beta_1 = c$, $\beta_2 = d$, then one can see that Lemma 3 will be reduced to [22, Lemma 2] after some computations and simplification for the latter, which is the foundation of [22, Theorem 3]. So in this point, our Theorem 2 extends the result of [22, Theorem 3].*

From Theorem 2 we can obtain some particular Ostrowski type inequalities on time scales. For example, if we take $n = 1$, $\alpha_1 = b$, $\beta_1 = d$, then we have

$$\begin{aligned}&|\int_a^b\int_c^d h_{k-1}(s,b)h_{k-1}(t,d)f(\sigma(s),\sigma(t))\Delta_2t\Delta_1s\\&+h_k(c,d)\int_a^b h_{k-1}(s,b)f(\sigma(s),c)\Delta_1s\\&+h_k(a,b)\int_c^d h_{k-1}(t,d)f(a,\sigma(t))\Delta_2t\\&+h_k(c,d)h_k(a,b)f(a,c)|\le Kh_{k+1}(a,b)h_{k+1}(c,d).\end{aligned}\tag{7}$$

If we take $n = 1$, $\alpha_1 = a$, $\beta_1 = c$, then we have

$$\begin{aligned}&|\int_a^b\int_c^d h_{k-1}(s,a)h_{k-1}(t,c)f(\sigma(s),\sigma(t))\Delta_2t\Delta_1s\\&-h_k(d,c)\int_a^b h_{k-1}(s,a)f(\sigma(s),d)\Delta_1s\\&-h_k(b,a)\int_c^d h_{k-1}(t,c)f(b,\sigma(t))\Delta_2t\\&+h_k(d,c)h_k(b,a)f(b,d)|\le Kh_{k+1}(b,a)h_{k+1}(d,c).\end{aligned}\tag{8}$$

If we take $n = 1$, $\alpha_1 = \frac{a+b}{2}$, $\beta_1 = \frac{c+d}{2}$, $x_1 = x$, $y_1 = y$, then we have

$$\begin{aligned}&|\int_a^b\int_c^d h_{k-1}(s,\frac{a+b}{2})h_{k-1}(t,\frac{c+d}{2})f(\sigma(s),\sigma(t))\Delta_2t\Delta_1s\\&-\int_a^b h_{k-1}(s,\frac{a+b}{2})[h_k(d,\frac{c+d}{2})f(\sigma(s),d)\\&-h_k(c,\frac{c+d}{2})f(\sigma(s),c)]\Delta_1s\\&-\int_c^d h_{k-1}(t,\frac{c+d}{2})[h_k(b,\frac{a+b}{2})f(b,\sigma(t))\\&-h_k(a,\frac{a+b}{2})f(a,\sigma(t))]\Delta_2t\\&+h_k(b,\frac{a+b}{2})h_k(d,\frac{c+d}{2})f(b,d)\\&-h_k(a,\frac{a+b}{2})h_k(d,\frac{c+d}{2})f(a,d)\\&-h_k(b,\frac{a+b}{2})h_k(c,\frac{c+d}{2})f(b,c)+\\&h_k(a,\frac{a+b}{2})h_k(c,\frac{c+d}{2})f(a,b)]|\\&\le K[(-1)^{k+1}h_{k+1}(a,\frac{a+b}{2})+h_{k+1}(b,\frac{a+b}{2})]\\&[(-1)^{k+1}h_{k+1}h_2(c,\frac{c+d}{2})+h_{k+1}h_2(d,\frac{c+d}{2})].\end{aligned}\tag{9}$$

In Theorem 2, if we take $\mathbb{T}_1$, $\mathbb{T}_2$ for some special time scales, then we immediately obtain the following corollaries.

Corollary 5 *(Continuous case) Let $\mathbb{T}_1 = \mathbb{T}_2 = \mathbb{R}$ in Theorem 2, then $h_k(t,s) = \frac{(t-s)^k}{k!}$, and we obtain*

$$\begin{aligned}&|\sum_{i=1}^{n-1}\sum_{j=1}^{n-1}(\frac{(x_i-\alpha_i)^k}{k!}-\frac{(x_i-\alpha_{i+1})^k}{k!})\\&(\frac{(y_j-\beta_j)^k}{k!}-\frac{(y_j-\beta_{j+1})^k}{k!})f(x_i,y_j)\\&+\sum_{i=1}^{n-1}(\frac{(x_i-\alpha_i)^k}{k!}-\frac{(x_i-\alpha_{i+1})^k}{k!})[\frac{(y_n-\beta_n)^k}{k!}f(x_i,y_n)\\&-\frac{(y_0-\beta_1)^k}{k!}f(x_i,y_0)]+\sum_{j=1}^{n-1}(\frac{(y_j-\beta_j)^k}{k!}-\frac{(y_j-\beta_{j+1})^k}{k!})\\&[\frac{(x_i-\alpha_n)^k}{k!}f(x_n,y_j)-\frac{(x_0-\alpha_1)^k}{k!}f(x_0,y_j)]\\&+\frac{(x_n-\alpha_n)^k}{k!}\frac{(y_n-\beta_n)^k}{k!}f(x_n,y_n)\\&-\frac{(x_0-\alpha_1)^k}{k!}\frac{(y_n-\beta_n)^k}{k!}f(x_0,y_n)\\&-\frac{(x_n-\alpha_n)^k}{k!}\frac{(y_0-\beta_1)^k}{k!}f(x_n,y_0)\\&+\frac{(x_0-\alpha_1)^k}{k!}\frac{(y_0-\beta_1)^k}{k!}f(x_0,y_0)\\&-\sum_{i=0}^{n-1}\int_{x_i}^{x_{i+1}}\frac{(s-\alpha_{i+1})^{k-1}}{(k-1)!}[\frac{(y_n-\beta_n)^k}{k!}f(s,y_n)\\&-\frac{(y_0-\beta_1)^k}{k!}f(s,y_0)]ds\\&-\sum_{i=0}^{n-1}\sum_{j=1}^{n-1}\int_{x_i}^{x_{i+1}}\frac{(s-\alpha_{i+1})^{k-1}}{(k-1)!}(\frac{(y_j-\beta_j)^k}{k!}\\&-\frac{(y_j-\beta_{j+1})^k}{k!})f(s,y_j)ds\end{aligned}$$

$$-\sum_{j=0}^{n-1}\int_{y_j}^{y_{j+1}}\frac{(t-\beta_{j+1})^{k-1}}{(k-1)!}[\frac{(x_n-\alpha_n)^k}{k!}f(x_n,t)$$

$$-\frac{(x_0-\alpha_1)^k}{k!}f(x_0,t)]dt$$

$$-\sum_{i=1}^{n-1}\sum_{j=0}^{n-1}\int_{y_j}^{y_{j+1}}\frac{(t-\beta_{j+1})^{k-1}}{(k-1)!}(\frac{(x_i-\alpha_i)^k}{k!}$$

$$-\frac{(x_i-\alpha_{i+1})^k}{k!})f(x_i,t)dt+\sum_{i=0}^{n-1}\sum_{j=0}^{n-1}$$

$$\int_{x_i}^{x_{i+1}}\int_{y_j}^{y_{j+1}}\frac{(s-\alpha_{i+1})^{k-1}}{(k-1)!}\frac{(t-\beta_{j+1})^{k-1}}{(k-1)!}f(s,t)dtds|$$

$$\leq K\{\sum_{i=0}^{n-1}[(-1)^{k+1}\frac{(x_i-\alpha_{i+1})^{k+1}}{(k+1)!}+\frac{(x_{i+1}-\alpha_{i+1})^{k+1}}{(k+1)!}]\}\times$$

$$\{\sum_{j=0}^{n-1}[(-1)^{k+1}\frac{(y_j-\beta_{j+1})^{k+1}}{(k+1)!}+\frac{(y_{j+1}-\beta_{j+1})^{k+1}}{(k+1)!}]\},\tag{10}$$

where $K=\sup_{a<s<b,\ c<t<d}|\frac{\partial^2 f(s,t)}{\partial s\partial t}|$.

Corollary 6 *(Discrete case) Let* $\mathbb{T}_1=\mathbb{T}_2=\mathbb{Z}$, $a=m_1$, $b=m_2$, $c=n_1$, $d=n_2$ *in Theorem 2.1. Suppose that* $x_i\in[m_1,m_2]_{\mathbb{Z}}$, $y_i\in[n_1,n_2]_{\mathbb{Z}}$, $i=0,1,...,k$. $I_n: m_1=x_0<x_1<...<x_{n-1}<x_n=m_2$ *is a division of* $[m_1,m_2]_{\mathbb{Z}}$, *while* $J_n: n_1=y_0<y_1<...<y_{n-1}<y_n=n_2$ *is a division of* $[n_1,n_2]_{\mathbb{Z}}$. $\alpha_i\in[x_{i-1},x_i]_{\mathbb{Z}}$, $\beta_i\in[y_{i-1},y_i]_{\mathbb{Z}}$, $i=1,2,...,n$. *Then we have*

$$|\sum_{i=1}^{n-1}\sum_{j=1}^{n-1}[\binom{x_i-\alpha_i}{k}-\binom{x_i-\alpha_{i+1}}{k}]$$

$$[\binom{y_j-\beta_j}{k}-\binom{y_j-\beta_{j+1}}{k}]f(x_i,y_j)$$

$$+\sum_{i=1}^{n-1}[\binom{x_i-\alpha_i}{k}-\binom{x_i-\alpha_{i+1}}{k}]$$

$$[\binom{y_n-\beta_n}{k}f(x_i,y_n)-\binom{y_0-\beta_1}{k}f(x_i,y_0)]$$

$$+\sum_{j=1}^{n-1}[\binom{y_j-\beta_j}{k}-\binom{y_j-\beta_{j+1}}{k}]$$

$$[\binom{x_n-\alpha_n}{k}f(x_n,y_j)-\binom{x_0-\alpha_1}{k}f(x_0,y_j)$$

$$+\binom{x_n-\alpha_n}{k}\binom{y_n-\beta_n}{k}f(x_n,y_n)$$

$$-\binom{x_0-\alpha_1}{k}\binom{y_n-\beta_n}{k}f(x_0,y_n)$$

$$-\binom{x_n-\alpha_n}{k}\binom{y_0-\beta_1}{k}f(x_n,y_0)$$

$$+\binom{x_0-\alpha_1}{k}\binom{y_0-\beta_1}{k}f(x_0,y_0)$$

$$-\sum_{i=0}^{n-1}\sum_{s=x_i}^{x_{i+1}-1}\binom{s-\alpha_{i+1}}{k-1}[\binom{y_n-\beta_n}{k}f(s+1,y_n)$$

$$-\binom{y_0-\beta_1}{k}f(s+1,y_0)]$$

$$-\sum_{i=0}^{n-1}\sum_{j=1}^{n-1}\sum_{s=x_i}^{x_{i+1}-1}\binom{s-\alpha_{i+1}}{k-1}[\binom{y_j-\beta_j}{k}$$

$$-\binom{y_j-\beta_{j+1}}{k}]f(s+1,y_j)$$

$$-\sum_{j=0}^{n-1}\sum_{t=y_j}^{y_{j+1}-1}\binom{t-\beta_{j+1}}{k-1}[\binom{x_n-\alpha_n}{k}f(x_n,t+1)$$

$$-\binom{x_0-\alpha_1}{k}f(x_0,t+1)]$$

$$-\sum_{i=1}^{n-1}\sum_{j=0}^{n-1}\sum_{t=y_j}^{y_{j+1}-1}\binom{t-\beta_{j+1}}{k-1}[\binom{x_i-\alpha_i}{k}$$

$$-\binom{x_i-\alpha_{i+1}}{k}]f(x_i,t+1)+\sum_{i=0}^{n-1}\sum_{j=0}^{n-1}\sum_{s=x_i}^{x_{i+1}-1}\sum_{t=y_j}^{y_{j+1}-1}$$

$$\binom{s-\alpha_{i+1}}{k-1}\binom{t-\beta_{j+1}}{k-1}f(s+1,t+1)|$$

$$\leq K\{\sum_{i=0}^{n-1}[(-1)^{k+1}\binom{x_i-\alpha_{i+1}}{k+1}+\binom{x_{i+1}-\alpha_{i+1}}{k+1}]\}$$

$$\times\{\sum_{j=0}^{n-1}[(-1)^{k+1}\binom{y_j-\beta_{j+1}}{k+1}+\binom{y_{j+1}-\beta_{j+1}}{k+1}]\},\tag{11}$$

where K *denotes the maximum value of the absolute value of the difference* $\Delta_1\Delta_2 f$ *over* $[m_1,m_2-1]_{\mathbb{Z}}\times[n_1,n_2-1]_{\mathbb{Z}}$.

Notice that $h_k(t,s)=\binom{t-s}{k}$ for $\forall t,s\in\mathbb{Z}$, we can easily get the desired result above.

Corollary 7 *(Quantum calculus case) Let* $\mathbb{T}_1 = q_1^{\mathbb{N}_0}$, $\mathbb{T}_2 = q_2^{\mathbb{N}_0}$, $a = q_1^{m_1}$, $b = q_1^{m_2}$, $c = q_2^{n_1}$, $d = q_2^{n_2}$ *in Theorem 2.1, where* $m_1,\ m_2,\ n_1,\ n_2 \in \mathbb{N}_0$ *and* $q_i > 1$, $i = 1, 2$. *Suppose that* $x_i \in [q_1^{m_1}, q_1^{m_2}]_{q_1^{\mathbb{N}_0}}$, $y_i \in [q_2^{n_1}, q_2^{n_2}]_{q_2^{\mathbb{N}_0}}$, $i = 0, 1, ..., n$. $I_n : q_1^{m_1} = x_0 < x_1 < ... < x_{n-1} < x_n = q_1^{m_2}$ *is a division of* $[q_1^{m_1}, q_1^{m_2}]_{q_1^{\mathbb{N}_0}}$, *while* $J_n : q_2^{n_1} = y_0 < y_1 < ... < y_{n-1} < y_n = q_2^{n_2}$ *is a division of* $[q_2^{n_1}, q_2^{n_2}]_{q_2^{\mathbb{N}_0}}$. $\alpha_i \in [x_{i-1}, x_i]_{q_1^{\mathbb{N}_0}}$, $\beta_i \in [y_{i-1}, y_i]_{q_2^{\mathbb{N}_0}}$, $i = 1, 2, ..., n$. *Then we have*

$$
\begin{aligned}
&|\sum_{i=1}^{n-1}\sum_{j=1}^{n-1}[\frac{(x_i-\alpha_i)_{q_1}^k}{[k]^1!}-\frac{(x_i-\alpha_{i+1})_{q_1}^k}{[k]^1!}]\\
&[\frac{(y_j-\beta_j)_{q_2}^k}{[k]^2!}-\frac{(y_j-\beta_{j+1})_{q_2}^k}{[k]^2!}]f(x_i,y_j)\\
&+\sum_{i=1}^{n-1}[\frac{(x_i-\alpha_i)_{q_1}^k}{[k]^1!}-\frac{(x_i-\alpha_{i+1})_{q_1}^k}{[k]^1!}]\\
&[\frac{(y_n-\beta_n)_{q_2}^k}{[k]^2!}f(x_i,y_n)-\frac{(y_0-\beta_1)_{q_2}^k}{[k]^2!}f(x_i,y_0)]\\
&+\sum_{j=1}^{n-1}[\frac{(y_j-\beta_j)_{q_2}^k}{[k]^2!}-\frac{(y_j-\beta_{j+1})_{q_2}^k}{[k]^2!}]\\
&[\frac{(x_n-\alpha_n)_{q_1}^k}{[k]^1!}f(x_n,y_j)-\frac{(x_0-\alpha_1)_{q_1}^k}{[k]^1!}f(x_0,y_j)\\
&+\frac{(x_n-\alpha_n)_{q_1}^k}{[k]^1!}\frac{(y_n-\beta_n)_{q_2}^k}{[k]^2!}f(x_n,y_n)\\
&-\frac{(x_0-\alpha_1)_{q_1}^k}{[k]^1!}\frac{(y_n-\beta_n)_{q_2}^k}{[k]^2!}f(x_0,y_n)\\
&-\frac{(x_n-\alpha_n)_{q_1}^k}{[k]^1!}\frac{(y_0-\beta_1)_{q_2}^k}{[k]^2!}f(x_n,y_0)\\
&+\frac{(x_0-\alpha_1)_{q_1}^k}{[k]^1!}\frac{(y_0-\beta_1)_{q_2}^k}{[k]^2!}f(x_0,y_0)\\
&-\sum_{j=1}^{n-1}h_k(x_0,\alpha_1)(h_k(y_j,\beta_j)-h_k(y_j,\beta_{j+1}))f(x_0,y_j)\\
&-h_k(x_0,\alpha_1)h_k(y_n,\beta_n)f(x_0,y_n)\\
&-h_k(x_n,\alpha_n)h_k(y_0,\beta_1)f(x_n,y_0)\\
&+h_k(x_0,\alpha_1)h_k(y_0,\beta_1)f(x_0,y_0)\\
&-\sum_{i=0}^{n-1}\int_{x_i}^{x_{i+1}}[h_k(y_n,\beta_n)h_{k-1}(s,\alpha_{i+1})f(\sigma(s),y_n)\\
&-h_k(y_0,\beta_1)h_{k-1}(s,\alpha_{i+1})f(\sigma(s),y_0)]\Delta_1 s\\
&-\sum_{i=0}^{n-1}\sum_{j=1}^{n-1}\int_{x_i}^{x_{i+1}}[h_k(y_j,\beta_j)h_{k-1}(s,\alpha_{i+1})\\
&-h_k(y_j,\beta_{j+1})h_{k-1}(s,\alpha_{i+1})]f(\sigma(s),y_j)\Delta_1 s\\
&-\sum_{j=0}^{n-1}\int_{y_j}^{y_{j+1}}[h_k(x_n,\alpha_n)h_{k-1}(t,\beta_{j+1})f(x_n,\sigma(t))\\
&-h_k(x_0,\alpha_1)h_{k-1}(t,\beta_{j+1})f(x_0,\sigma(t))]\Delta_2 t\\
&-(q_1-1)\sum_{i=0}^{n-1}x_i\{\sum_{s=0}^{log_{q_1}^{[x_{i+1}/(q_1x_i)]}}q_1^s\frac{(q_1^sx_i-\alpha_{i+1})_{q_1}^{k-1}}{[k-1]^1!}\\
&[\frac{(y_n-\beta_n)_{q_2}^k}{[k]^2!}f(q_1^{s+1}x_i,y_n)-\frac{(y_0-\beta_1)_{q_2}^k}{[k]^2!}f(q_1^{s+1}x_i,y_0)]\}\\
&-(q_1-1)\sum_{i=0}^{n-1}\sum_{j=1}^{n-1}x_i\{\sum_{s=0}^{log_{q_1}^{[x_{i+1}/(q_1x_i)]}}q_1^s\frac{(q_1^sx_i-\alpha_{i+1})_{q_1}^{k-1}}{[k-1]^1!}\\
&[\frac{(y_j-\beta_j)_{q_2}^k}{[k]^2!}-\frac{(y_j-\beta_{j+1})_{q_2}^k}{[k]^2!}]f(q_1^{s+1}x_i,y_j)\}\\
&-(q_2-1)\sum_{j=0}^{n-1}y_j\{\sum_{t=0}^{log_{q_1}^{[y_{j+1}/(q_2y_j)]}}q_2^t\frac{(q_2^ty_j-\beta_{j+1})_{q_2}^{k-1}}{[k-1]^2!}\\
&[\frac{(x_n-\alpha_n)_{q_1}^k}{[k]^1!}f(x_n,q_2^{t+1}y_j)-\frac{(x_0-\alpha_1)_{q_1}^k}{[k]^1!}f(x_0,q_2^{t+1}y_j)]\}\\
&-(q_2-1)\sum_{i=1}^{n-1}\sum_{j=0}^{n-1}y_j\{\sum_{t=0}^{log_{q_1}^{[y_{j+1}/(q_2y_j)]}}q_2^t\frac{(q_2^ty_j-\beta_{j+1})_{q_2}^{k-1}}{[k-1]^2!}\\
&[\frac{(x_i-\alpha_i)_{q_1}^k}{[k]^1!}-\frac{(x_i-\alpha_{i+1})_{q_1}^k}{[k]^1!}]f(x_i,q_2^{t+1}y_j)\}\\
&+(q_1-1)(q_2-1)\sum_{i=0}^{n-1}\sum_{j=0}^{n-1}x_iy_j\{\sum_{s=0}^{log_{q_1}^{[x_{i+1}/(q_1x_i)]}}\sum_{t=0}^{log_{q_1}^{[y_{j+1}/(q_2y_j)]}}\\
&q_1^sq_2^t\frac{(q_1^sx_i-\alpha_{i+1})_{q_1}^{k-1}}{[k-1]^1!}\frac{(q_2^ty_j-\beta_{j+1})_{q_2}^{k-1}}{[k-1]^2!}f(q_1^{s+1}x_i,q_2^{t+1)y_j}|\}\\
&\le K\{\sum_{i=0}^{n-1}[(-1)^{k+1}\frac{(x_i-\alpha_{i+1})_{q_1}^k}{[k]^1!}+\frac{(x_{i+1}-\alpha_{i+1})_{q_1}^k}{[k]^1!}]\}\times\\
&\{\sum_{i=0}^{n-1}[(-1)^{k+1}\frac{(y_j-\beta_{j+1})_{q_2}^k}{[k]^2!}+\frac{(y_{j+1}-\beta_{j+1})_{q_2}^k}{[k]^2!}]\},\quad(12)
\end{aligned}
$$

where K *denotes the maximum value of the absolute value of the* q_1q_2*-difference* $D_{q_1q_2}f(t,s)$ *over* $[q_1^{m_1}, q_1^{m_2-1}]_{q_1^{\mathbb{N}_0}}$ $\times[q_2^{n_1}, q_2^{n_2-1}]_{q_2^{\mathbb{N}_0}}$, *and*

$$(t-s)_{q_i}^k := \prod_{j=0}^{k-1}(t-q_i^j s) \ for \ s,t \in q_i^{\mathbb{N}_0}, \ i=1,2$$

and $k \in \mathbb{N}_0$,

$$[k]^i! := \prod_{j=1}^{k}[j]_{q_i} \ for \ k \in \mathbb{N}_0, \ i=1,2,$$

$[k]_{q_i} := \frac{q_i^k-1}{q_i-1}$ *for* $q_i \in \mathbb{R}, \ q_i \neq 1, \ i=1,2$ *and* $k \in \mathbb{N}_0$.

Proof. Considering $h_k(t,s) := \frac{(t-s)_{q_i}^k}{[k]^i!}$ for $s,t \in q_i^{\mathbb{N}_0}, \ i=1,2$ and $k \in \mathbb{N}_0$, after simple calculations we can get the desired result. □

Theorem 8 *Under the conditions of Theorem 2, if there exist constants* $K_1, \ K_2$ *such that* $K_1 \leq \frac{\partial^2 f(s,t)}{\Delta_1 s \Delta_2 t} \leq K_2$ *for* $a<s<b, \ c<t<d$, *then we have the following inequality*

$$\begin{aligned}
&|\sum_{i=1}^{n-1}\sum_{j=1}^{n-1}(h_k(x_i,\alpha_i)-h_k(x_i,\alpha_{i+1}))(h_k(y_j,\beta_j) \\
&\quad -h_k(y_j,\beta_{j+1}))f(x_i,y_j) \\
&+\sum_{i=1}^{n-1}(h_k(x_i,\alpha_i)-h_k(x_i,\alpha_{i+1}))h_k(y_n,\beta_n)f(x_i,y_n) \\
&-\sum_{i=1}^{n-1}(h_k(x_i,\alpha_i)-h_k(x_i,\alpha_{i+1}))h_k(y_0,\beta_1)f(x_i,y_0) \\
&+\sum_{j=1}^{n-1}h_k(x_n,\alpha_n)(h_k(y_j,\beta_j)-h_k(y_j,\beta_{j+1}))f(x_n,y_j) \\
&\quad +h_k(x_n,\alpha_n)h_k(y_n,\beta_n)f(x_n,y_n) \\
&-\sum_{j=1}^{n-1}h_k(x_0,\alpha_1)(h_k(y_j,\beta_j)-h_k(y_j,\beta_{j+1}))f(x_0,y_j) \\
&\quad -h_k(x_0,\alpha_1)h_k(y_n,\beta_n)f(x_0,y_n) \\
&\quad -h_k(x_n,\alpha_n)h_k(y_0,\beta_1)f(x_n,y_0) \\
&\quad +h_k(x_0,\alpha_1)h_k(y_0,\beta_1)f(x_0,y_0) \\
&-\sum_{i=0}^{n-1}\int_{x_i}^{x_{i+1}}[h_k(y_n,\beta_n)h_{k-1}(s,\alpha_{i+1})f(\sigma(s),y_n) \\
&\quad -h_k(y_0,\beta_1)h_{k-1}(s,\alpha_{i+1})f(\sigma(s),y_0)]\Delta_1 s \\
&-\sum_{i=0}^{n-1}\sum_{j=1}^{n-1}\int_{x_i}^{x_{i+1}}[h_k(y_j,\beta_j)h_{k-1}(s,\alpha_{i+1}) \\
&\quad -h_k(y_j,\beta_{j+1})h_{k-1}(s,\alpha_{i+1})]f(\sigma(s),y_j)\Delta_1 s \\
&-\sum_{j=0}^{n-1}\int_{y_j}^{y_{j+1}}[h_k(x_n,\alpha_n)h_{k-1}(t,\beta_{j+1})f(x_n,\sigma(t)) \\
&\quad -h_k(x_0,\alpha_1)h_{k-1}(t,\beta_{j+1})f(x_0,\sigma(t))]\Delta_2 t \\
&-\sum_{i=1}^{n-1}\sum_{j=0}^{n-1}\int_{y_j}^{y_{j+1}}[h_k(x_i,\alpha_i)h_{k-1}(t,\beta_{j+1}) \\
&\quad -h_k(x_i,\alpha_{i+1})h_{k-1}(t,\beta_{j+1})]f(x_i,\sigma(t))\Delta_2 t \\
&+\sum_{i=0}^{n-1}\sum_{j=0}^{n-1}\int_{x_i}^{x_{i+1}}\int_{y_j}^{y_{j+1}}h_{k-1}(s,\alpha_{i+1})h_{k-1}(t,\beta_{j+1}) \\
&\quad f(\sigma(s),\sigma(t))\Delta_2 t\Delta_1 s \\
&-\frac{K_1+K_2}{2}\{\sum_{i=0}^{n-1}[h_{k+1}(x_{i+1},\alpha_{i+1})-h_{k+1}(x_i,\alpha_{i+1})]\} \\
&\quad \times\{\sum_{j=0}^{n-1}[h_{k+1}(y_{j+1},\beta_{j+1})-h_{k+1}(y_j,\beta_{j+1})]\}| \\
&\leq\frac{K_2-K_1}{2}\{\sum_{i=0}^{n-1}[(-1)^{k+1}h_{k+1}(x_i,\alpha_{i+1})+h_{k+1}(x_{i+1},\alpha_{i+1})]\} \\
&\times\{\sum_{j=0}^{n-1}[(-1)^{k+1}h_{k+1}(y_j,\beta_{j+1})+h_{k+1}(y_{j+1},\beta_{j+1})]\}.
\end{aligned} \tag{13}$$

Proof. According to the definition of $H(s,t,I_n,J_n)$ in Lemma 3 we have

$$\begin{aligned}
&\int_a^b\int_c^d H(s,t,I_n,J_n)\Delta_2 t\Delta_1 s \\
&=\sum_{i=0}^{n-1}\sum_{j=0}^{n-1}\int_{x_i}^{x_{i+1}}\int_{y_j}^{y_{j+1}}h_k(s,\alpha_{i+1})h_k(t,\beta_{j+1})\Delta_2 t\Delta_1 s \\
&=[\sum_{i=0}^{n-1}\int_{x_i}^{x_{i+1}}h_k(s,\alpha_{i+1})\Delta_1 s][\sum_{j=0}^{n-1}\int_{y_j}^{y_{j+1}}h_k(t,\beta_{j+1})\Delta_2 t] \\
&=\{\sum_{i=0}^{n-1}[\int_{x_i}^{\alpha_{i+1}}h_k(s,\alpha_{i+1})\Delta_1 s+\int_{\alpha_{i+1}}^{x_{i+1}}h_k(s,\alpha_{i+1})\Delta_1 s]\} \\
&\times\{\sum_{j=0}^{n-1}[\int_{y_j}^{\beta_{j+1}}h_k(t,\beta_{j+1})\Delta_2 t+\int_{\beta_{j+1}}^{y_{j+1}}h_k(t,\beta_{j+1})\Delta_2 t]\} \\
&=\{\sum_{i=0}^{n-1}[-\int_{\alpha_{i+1}}^{x_i}h_k(s,\alpha_{i+1})\Delta_1 s+\int_{\alpha_{i+1}}^{x_{i+1}}h_k(s,\alpha_{i+1})\Delta_1 s]\} \\
&\times\{\sum_{j=0}^{n-1}[-\int_{\beta_{j+1}}^{y_j}h_k(t,\beta_{j+1})\Delta_2 t+\int_{\beta_{j+1}}^{y_{j+1}}h_k(t,\beta_{j+1})\Delta_2 t]\} \\
&=\{\sum_{i=0}^{n-1}[h_{k+1}(x_{i+1},\alpha_{i+1})-h_{k+1}(x_i,\alpha_{i+1})]\}
\end{aligned}$$

$$\times\{\sum_{j=0}^{n-1}[h_{k+1}(y_{j+1},\beta_{j+1})-h_{k+1}(y_j,\beta_{j+1})]\}. \quad (14)$$

We also have $|\frac{\partial^2 f(s,t)}{\Delta_1 s\Delta_2 t}-\frac{K_1+K_2}{2}|\leq\frac{K_2-K_1}{2}$, and

$$|\int_a^b\int_c^d H(s,t,I_n,J_n)(\frac{\partial^2 f(s,t)}{\Delta_1 s\Delta_2 t}-\frac{K_1+K_2}{2})\Delta_2 t\Delta_1 s|$$

$$\leq\frac{K_2-K_1}{2}\int_a^b\int_c^d|H(s,t,I_n,J_n)|\Delta_2 t\Delta_1 s. \quad (15)$$

Then combining (6), (14) and (15) we obtain the desired inequality (13). □

Remark 9 *If we take* $\mathbb{T}_1=\mathbb{T}_2=\mathbb{R},\ n=2,\ k=1,\ \alpha_1=a+\lambda\frac{b-a}{2},\ \alpha_2=b-\lambda\frac{b-a}{2},\ \beta_1=c+\lambda\frac{d-c}{2},\ \beta_2=d-\lambda\frac{d-c}{2},\ x_1=x,\ y_1=y$, *where* $\lambda\in[0,1]$, *then Theorem 8 reduces to [10, Theorem 4].*

Remark 10 *For Theorem 8, we can also obtain similar results as shown in Corollary 5-7, which are omitted here.*

3 Conclusions

In this paper, we establish some new Ostrowski type inequalities on time scales involving functions of two independent variables for k^2 points, which unify continuous and discrete analysis. Some of the established results are sharp, and are further improvements of some known results in the literature.

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