High Accuracy and Multiscale Multigrid Computation for Three Dimensional Biharmonic Equations

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Abstract: The multiscale multigrid method is presented in this article to solve the linear systems arising from a fourth order discretisation. We used a symbolic algebra package *Mathematica* to derive a family of finite difference approximations on a 27 point compact stencil. The unknown solution and its second derivatives are carried as unknowns at selected grid points. A set of test problems are presented to demonstrate the efficiency and accuracy of the fourth order compact scheme.

Key–Words: Boundary value problems; three-dimensional biharmonic equation; fourth order compact scheme; multiscale multigrid method.

1 Introduction

In the field of numerical simulation of three-dimensional partial differential equations, the solutions tends to be computationally intensive due to the requirements on the memory and the CPU time to obtain solutions with desired accuracy , see [18]. For three-dimensional biharmonic equation with Dirichlet boundary conditions of first kind, Altas et al. [1, 2] had presented finite difference approximations of second and fourth order for 2D and 3d biharmonic e-quation; and described the multigrid and Krylov solution of the fourth-order discretisation. Stephenson [18] proposed second-order and fourth-order compact finite difference approximations and solved the resulting linear system via the direct solvers and classical iterative methods. Dehghan et al. [8], had used a *Maple* to obtain compact finite difference approximations on a 27 points. Altas et al. [2] had used a *Mathematica* to obtain compact finite difference approximations on a 25 points. In our previous work [12], we had derived a finite difference approximations for the biharmonic equation on a 18 point compact stencil. Wang et al. [19] had used a 19 point compact stencil.

In the present paper, we considered the three-dimensional biharmonic equation:

$$\begin{aligned}\frac{\partial^4\psi}{\partial x^4}+\frac{\partial^4\psi}{\partial y^4}+\frac{\partial^4\psi}{\partial z^4}+2\frac{\partial^4\psi}{\partial x^2\partial y^2}\\ +2\frac{\partial^4\psi}{\partial x^2\partial z^2}+2\frac{\partial^4\psi}{\partial y^2\partial z^2}=f(x,y,z),\end{aligned}\qquad(1)$$

for a specified forcing function f in a continuous 3D domain Ω, and Dirichlet boundary conditions of second kind:

$$\psi=f_1(x,y,z),\ \frac{\partial^2\psi}{\partial n^2}=f_2(x,y,z),\quad (x,y,z)\in\partial\Omega.$$

The two-dimensional version of Eq.(1) is

$$\frac{\partial^4\psi}{\partial x^4}+\frac{\partial^4\psi}{\partial y^4}+2\frac{\partial^4\psi}{\partial x^2\partial y^2}=f(x,y),\quad (x,y)\in\Omega,\qquad(2)$$
$$\psi=f_1(x,y),\ \frac{\partial^2\psi}{\partial n^2}=f_2(x,y),\quad (x,y)\in\partial\Omega,$$

that has been considered extensively in the literature. As mentioned in [2] a popular approach for solving the three-dimensional biharmonic equations with Dirichlet boundary conditions of first kind is to discrete the equation on a uniform grid using a 25 point computational stencil with truncation error of order h^2. This 25 point approximation connects the values of ψ at grid point (x_i, y_j, z_k) in terms of 24 neighboring values of ψ in a $5\times5\times5$ cube. Also, the value of ψ at the point (x_i, y_j, z_k) is connected to certain neighboring values two grid points away in each direction from the point (x_i, y_j, z_k) and this approximation needs to be modified at grid points near the boundaries see [8]. Altas et al. [2] had presented another plan for solving the biharmonic boundary value problems with Dirichlet boundary conditions of first kind. Their approach involves discreting the biharmonic equation using not just the grid values of the unknown solution ψ but also the values of the gradients ψ_x, ψ_y and ψ_z at selected grid points. This approach introduced extra amount of computation but

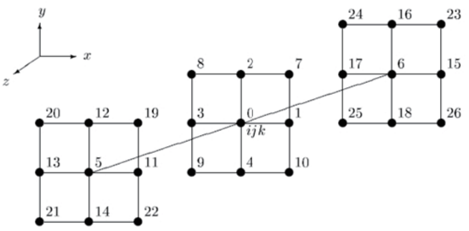

Figure 1: The 27 point stencil of the 3D grid points.

utilising the grid values of the gradients is advantageous.

The objective of this article is to apply the proposed compact finite difference approximations on a 27 point, and the well-known software *Mathematica* to obtain this compact finite difference approximations for three-dimensional biharmonic equation with Dirichlet boundary conditions of second kind. The discretization is carried out on a uniform 3D grid with meshsize h, and we use a local coordinate system where the unit cubic grids are labeled as in Fig.1, see [19].

2 The Compact finite difference approximations

In Appendix A we have written a *Mathematica* code for fourth-order compact finite difference approximations by expanding the *Mathematica* code in [1], by choosing various combinations of the grid values of ψ, ψ_{xx} ,ψ_{yy} and ψ_{zz} to be used in the derivations [2]. Then, the *Mathematica* code produces the following finite-difference approximation:

$$\begin{aligned}
&96\psi_{i,j,k} - 26(\psi_{i,j+1,k} + \psi_{i,j,k-1} + \psi_{i,j-1,k} \\
&+\psi_{i,j,k+1} + \psi_{i+1,j,k} + \psi_{i-1,j,k}) \\
&+5(\psi_{i+1,j,k-1} + \psi_{i,j+1,k-1} + \psi_{i,j-1,k-1} + \psi_{i+1,j-1,k} \\
&+\psi_{i,j-1,k+1} + \psi_{i+1,j,k+1} + \psi_{i+1,j,k} + \psi_{i+1,j+1,k} \\
&+\psi_{i-1,j,k+1} + \psi_{i-1,j,k-1} + \psi_{i-1,j+1,k} + \psi_{i-1,j_1,k}) \\
&+3h^2(\psi_{xxi+1,j,k} + \psi_{xxi-1,j,k} + \psi_{yyi,j+1,k} \\
&+\psi_{yyi,j-1,k} + \psi_{zzi,j,k+1} + \psi_{zzi,j,k-1}) \\
&= \tfrac{5}{2}h^4 f_{i,j,k}.
\end{aligned} \tag{3}$$

The corresponding finite difference approximations for ψ_{xx}, ψ_{yy}, ψ_{zz}, at point (x_i, y_j, z_k) , see [8], are:

$$\begin{aligned}
&-32h^2\psi_{xxi,j,k} - \tfrac{52}{5}(\psi_{i,j+1,k} + \psi_{i,j-1,k} + \psi_{i,j,k+1} \\
&+\psi_{i,j,k-1}) + 2(\psi_{i+1,j,k-1} + \psi_{i,j+1,k-1} + \psi_{i,j-1,k-1} \\
&+\psi_{i+1,j-1,k} + \psi_{i,j-1,k+1} + \psi_{i+1,j,k+1} + \psi_{i,j+1,k+1} \\
&+\psi_{i+1,j+1,k} + \psi_{i-1,j,k+1} + \psi_{i-1,j,k-1} + \psi_{i-1,j+1,k} \\
&+\psi_{i-1,j-1,k}) + \tfrac{44}{5}(\psi_{i+1,j,k} + \psi_{i-1,j,k}) \\
&-\tfrac{2}{5}h^2(\psi_{xxi+1,j,k} + \psi_{xxi-1,j,k}) \\
&+\tfrac{6}{5}h^2(\psi_{yyi,j+1,k} + \psi_{yyi,j-1,k} + \psi_{zzi,j,k+1} + \psi_{zzi,j,k-1}) \\
&= h^4 f_{i,j,k},
\end{aligned} \tag{4}$$

$$\begin{aligned}
&-32h^2\psi_{yyi,j,k} + \tfrac{52}{5}(\psi_{i,j,k+1} + \psi_{i,j,k-1} + \psi_{i+1,j,k} \\
&+\psi_{i-1,j,k}) + 2(\psi_{i+1,j,k-1} + \psi_{i,j+1,k-1} + \psi_{i,j-1,k-1} \\
&+\psi_{i+1,j-1,k} + \psi_{i,j-1,k+1} + \psi_{i+1,j,k+1} + \psi_{i,j+1,k+1} \\
&+\psi_{i+1,j+1,k} + \psi_{i-1,j,k+1} + \psi_{i-1,j,k-1} + \psi_{i-1,j+1,k} \\
&+\psi_{i-1,j-1,k}) + \tfrac{44}{5}(\psi_{i,j+1,k} + \psi_{i,j-1,k}) \\
&-\tfrac{2}{5}h^2(\psi_{yyi,j+1,k} + \psi_{yyi,j-1,k}) \\
&+\tfrac{6}{5}h^2(\psi_{xxi+1,j,k} + \psi_{xxi-1,j,k} + \psi_{zzi,j,k-1} + \psi_{zzi,j,k+1}) \\
&= h^4 f_{i,j,k},
\end{aligned} \tag{5}$$

$$\begin{aligned}
&-32h^2\psi_{zzi,j,k} + \tfrac{52}{5}(\psi_{i,j+1,k} + \psi_{i,j-1,k} + \psi_{i+1,j,k} \\
&+\psi_{i-1,j,k}) + 2(\psi_{i+1,j,k-1} + \psi_{i,j+1,k-1} + \psi_{i,j-1,k-1} \\
&+\psi_{i+1,j-1,k} + \psi_{i,j-1,k+1} + \psi_{i+1,j,k+1} + \psi_{i,j+1,k+1} \\
&+\psi_{i+1,j+1,k} + \psi_{i-1,j,k+1} + \psi_{i-1,j,k-1} + \psi_{i-1,j+1,k} \\
&+\psi_{i-1,j-1,k}) + \tfrac{44}{5}(\psi_{i,j,k-1} + \psi_{i,j,k+1}) \\
&-\tfrac{2}{5}h^2(\psi_{zzi,j,k+1} + \psi_{zzi,j+1,k}) \\
&+\tfrac{6}{5}h^2(\psi_{xxi+1,j,k} + \psi_{xxi-1,j,k} + \psi_{yyi,j+1,k} + \psi_{yyi,j-1,k}) \\
&= h^4 f_{i,j,k}.
\end{aligned} \tag{6}$$

The compact finite difference approximation of order h^4 by choosing values of ψ at 26 neighboring points of (x_i, y_j, z_k) and 10 values of ψ_{xx}, ψ_{yy} and ψ_{zz}, see [8], is

$$\begin{aligned}
&\psi_{i,j,k} - \tfrac{187}{856}(\psi_{i,j+1,k} + \psi_{i,j-1,k} + \psi_{i,j,k+1} + \psi_{i,j,k-1} \\
&+\psi_{i+1,j,k} + \psi_{i-1,j,k}) + \tfrac{21}{172}(\psi_{i+1,j,k-1} + \psi_{i,j+1,k-1} \\
&+\psi_{i,j-1,k-1} + \psi_{i+1,j-1,k} + \psi_{i,j-1,k+1} + \psi_{i+1,j,k+1} \\
&+\psi_{i,j+1,k+1} + \psi_{i-1,j,k+1} + \psi_{i+1,j+1,k} + \psi_{i-1,j,k-1} \\
&+\psi_{i-1,j+1,k} + \psi_{i-1,j-1,k}) + \tfrac{35}{1712}(\psi_{i+1,j-1,k-1} \\
&+\psi_{i-1,j+1,k-1} + \psi_{i-1,j+1,k+1}) \\
&+\tfrac{17}{3424}h^2(\psi_{yyi,j-1,k-1} + \psi_{yyi,j+1,k+1} \\
&+\psi_{i+1,j-1,k+1} + \psi_{i+1,j+1,k+1} + \psi_{i+1,j+1,k-1} \\
&+\psi_{i-1,j-1,k+1} + \psi_{i-1,j-1,k-1} + \psi_{xxi+1,j,k+1} \\
&+\psi_{zzi+1,j,k+1} + \psi_{xxi-1,j,k+1} + \psi_{zzi,j-1,k+1} \\
&+\psi_{zzi,j-1,k-1} + \psi_{yyi+1,j-1,k} + \psi_{yyi-1,j+1,k} \\
&+\psi_{yyi,j-1,k+1} + \psi_{xxi-1,j,k-1} + \psi_{xxi-1,j+1,k} \\
&+\psi_{yyi-1,j-1,k} + \psi_{yyi,j+1,k-1} + \psi_{xxi+1,j,k-1} \\
&+\psi_{zzi,j+1,k-1} + \psi_{zzi-1,j,k-1} + \psi_{xxi+1,j+1,k} \\
&+\psi_{zzi,j+1,k+1} + \tfrac{41}{1712}h^2(\psi_{zzi,j,k+1} \\
&+\psi_{xxi-1,j,k} + \psi_{yyi+1,j_1,k} + \psi_{zzi-1,j,k+1} \\
&+\psi_{zzi+1,j,k-1} + \psi_{xxi-1,j-1,k} + \psi_{xxi+1,j-1,k}) \\
&+\psi_{xxi+1,j,k} + \psi_{zzi,j,k-1} + \psi_{yyi,j-1,k} + \psi_{yyi,j+1,k}) \\
&= h^4(\tfrac{55}{3429}f_{i,j,k} + \tfrac{55}{10{,}272}(f_{i+1,j,k} + f_{i-1,j,k} \\
&+f_{i,j+1,k} + f_{i,j-1,k} + f_{i,j,k-1} + f_{i,j,k+1})),
\end{aligned} \tag{7}$$

$$
\begin{aligned}
&h^2\psi_{xxi,j,k} - (\frac{356{,}892}{642{,}000}(-\psi_{i,j+1,k} - \psi_{i,j-1,k} - \psi_{i,j,k+1} \\
&-\psi_{i,j,k-1}) + \frac{51{,}954}{642{,}000}(\psi_{i+1,j,k-1} + \psi_{i+1,j-1,k} \\
&+\psi_{i+1,j,k+1} + \psi_{i-1,j,k+1} + \psi_{i+1,j+1,k} + \psi_{i-1,j,k-1} \\
&+\psi_{i-1,j+1,k} + \psi_{i-1,j-1,k}) + \frac{28{,}842}{642{,}000}(\psi_{i,j+1,k-1} \\
&+\psi_{i,j-1,k-1} + \psi_{i,j-1,k+1} + \psi_{i,j+1,k-1}) \\
&+\frac{36{,}284}{642{,}000}(\psi_{i+1,j,k} + \psi_{i-1,j,k}) \\
&+\frac{20{,}250}{642{,}000}(\psi_{i+1,j-1,k-1} + \psi_{i+1,j-1,k+1} \\
&+\psi_{i+1,j+1,k+1} + \psi_{i+1,j+1,k-1} + \psi_{i-1,j-1,k-1} \\
&+\psi_{i-1,j+1,k-1} + \psi_{i-1,j-1,k+1} + \psi_{i-1,j+1,k+1}) \\
&-\frac{1227}{642{,}000}h^2(\psi_{xxi+1,j,k-1} + \psi_{xxi-1,j-1,k+1} \\
&+\psi_{xxi-1,j,k-1} + \psi_{xxi+1,j,k+1} + \psi_{xxi-1,j+1,k} \\
&+\psi_{xxi-1,j+1,k} + \psi_{xxi-1,j-1,k} + \psi_{xxi+1,j-1,k} \\
&+\psi_{xxi+1,j+1,k}) + \frac{6477}{642{,}000}h^2(\psi_{yyi,j+1,k+1} \\
&+\psi_{yyi,j-1,k-1} + \psi_{zzi,j+1,k+1} + \psi_{yyi,j+1,k-1} \\
&+\psi_{zzi,j-1,k+1} + \psi_{yyi,j-1,k+1} + \psi_{zzi,j-1,k-1} \\
&+\psi_{zzi,j+1,k-1}) - \frac{2142}{642{,}000}h^2(\psi_{xxi-1,j,k} \\
&+\psi_{xxi+1,j,k}) + \frac{38{,}946}{642{,}000}h^2(\psi_{zzi,j,k+1} \\
&+\psi_{zzi,j,k-1} + \psi_{yyi,j+1,k} + \psi_{yyi,j-1,k}) \\
&+\frac{2625}{642{,}000}h^2(\psi_{yyi-1,j-1,k} + \psi_{yyi+1,j+1,k} \\
&+\psi_{yyi-1,j+1,k} + \psi_{zzi-1,j,k+1} + \psi_{zzi+1,j,k-1} \\
&+\psi_{zzi+1,j,k+1} + \psi_{zzi-1,j,k-1} + \psi_{yyi+1,j-1,k})) \\
&= -h^4(\frac{73}{1712}f_{i,j,k} + \frac{889}{128{,}400}(f_{i,j+1,k} + f_{i,j-1,k} \\
&+f_{i,j,k-1} + f_{i,j,k+1}) + \frac{247}{128{,}400}(f_{i+1,j,k} + f_{i-1,j,k})),
\end{aligned}
\tag{8}
$$

$$
\begin{aligned}
&h^2\psi_{yyi,j,k} - (\frac{28{,}842}{642{,}000}(\psi_{i+1,j,k+1} + \psi_{i+1,j,k+1} \\
&+\psi_{i-1,j,k+1} + \psi_{i-1,j,k-1}) + \frac{51{,}954}{642{,}000}(\psi_{i,j+1,k-1} \\
&+\psi_{i,j-1,k} + \psi_{i+1,j-1,k} + \psi_{i,j+1,k+1} \\
&+\psi_{i,j-1,k+1} + \psi_{i+1,j+1,k} + \psi_{i-1,j+1,k} + \psi_{i-1,j-1,k}) \\
&+\frac{367{,}284}{642{,}000}(\psi_{i,j+1,k} + \psi_{i,j-1,k} \\
&-\frac{356{,}892}{642{,}000}(\psi_{i,j,k+1} + \psi_{i+1,j,k} + \psi_{i,j,k-1} \\
&+\psi_{i-1,j,k}) + \frac{20{,}250}{642{,}000}(\psi_{i+1,j-1,k+1} + \psi_{i+1,j+1,k+1} \\
&+\psi_{i+1,j+1,k-1} + \psi_{i-1,j-1,k+1} + \psi_{i-1,j-1,k-1} \\
&+\psi_{i-1,j+1,k-1} + \psi_{i-1,j+1,k+1} + \psi_{i+1,j-1,k-1}) \\
&+\frac{6477}{642{,}000}h^2(\psi_{xxi-1,j,k-1} + \psi_{zzi+1,j,k+1} + \psi_{zzi+1,j,k-1} \\
&+\psi_{zzi-1,j,k-1} + \psi_{zzi-1,j,k+1} + \psi_{xxi-1,j,k+1} \\
&+\psi_{xxi+1,j,k+1} + \psi_{xxi+1,j,k-1}) + \frac{38{,}946}{642{,}000}h^2(\psi_{zzi,j,k+1} \\
&+\psi_{zzi,j,k-1} + \psi_{xxi+1,j,k} + \psi_{xxi-1,j,k}) \\
&-\frac{1227}{642{,}000}h^2(\psi_{yyi-1,j+1,k} + \psi_{yyi+1,j+1,k} \\
&+\psi_{yyi,j+1,k-1} + \psi_{yyi,j-1,k+1} + \psi_{yyi,j-1,k-1} \\
&+\psi_{yyi-1,j-1,k} + \psi_{yyi,j+1,k+1} \\
&-\frac{2142}{642{,}000}h^2(\psi_{yyi,j+1,k} + \psi_{yyi,j-1,k}) \\
&+\psi_{yyi+1,j-1,k}) + \frac{2625}{642{,}000}h^2(\psi_{zzi,j-1,k+1} \\
&+\psi_{xxi-1,j-1,k} + \psi_{xxi+1,j-1,k} + \psi_{xxi-1,j+1,k} \\
&+\psi_{zzi,j+1,k-1} + \psi_{zzi,j-1,k-1} + \psi_{xxi+1,j+1,k} \\
&+\psi_{zzi,j+1,k+1})) = -h^4(\frac{73}{1712}f_{i,j,k} \\
&+\frac{889}{128{,}400}(f_{i,j+1,k} + f_{i,j-1,k} + f_{i,j,k-1} + f_{i,j,k+1}) \\
&+\frac{247}{128{,}400}(f_{i+1,j,k} + f_{i-1,j,k})),
\end{aligned}
\tag{9}
$$

$$
\begin{aligned}
&h^2\psi_{zzi,j,k} - (\frac{356{,}892}{642{,}000}(-\psi_{i,j+1,k} - \psi_{i,j-1,k} - \psi_{i+1,j,k} \\
&-\psi_{i-1,j,k}) + \frac{51{,}954}{642{,}000}(\psi_{i+1,j,k-1} + \psi_{i,j+1,k-1} \\
&+\psi_{i,j-1,k-1} + \psi_{i,j-1,k+1} + \psi_{i+1,j,k+1} \\
&+\psi_{i,j+1,k+1} + \psi_{i-1,j,k+1} + \psi_{i-1,j,k-1}) \\
&+\frac{367{,}284}{642{,}000}(\psi_{i,j,k-1} + \psi_{i,j,k+1}) \\
&+\frac{28{,}942}{642{,}000}(\psi_{i+1,j-1,k} + \psi_{i-1,j+,k} \\
&+\psi_{i+1,j+1,k} + \psi_{i-1,j-1,k}) \\
&+\frac{20{,}250}{642{,}000}(\psi_{i+1,j-1,k-1} + \psi_{i+1,j-1,k+1} \\
&+\psi_{i+1,j+1,k+1} + \psi_{i+1,j+1,k-1} + \psi_{i-1,j-1,k-1} \\
&+\psi_{i-1,j+1,k-1} + \psi_{i-1,j-1,k+1} + \psi_{i-1,j+1,k+1}) \\
&-\frac{1227}{642{,}000}h^2(\psi_{zzi+1,j,k+1} + \psi_{zzi,j-1,k+1} + \psi_{zzi,j-1,k-1} \\
&+\psi_{zzi,j+1,k-1} + \psi_{zzi-1,j,k-1} + \psi_{zzi,j+1,k+1} \\
&+\psi_{zzi-1,j,k+1} + \psi_{zzi+1,j,k-1}) \\
&+\frac{6477}{642{,}000}h^2(\psi_{yyi+1,j-1,k} + \psi_{yyi-1,j+1,k} + \psi_{yyi-1,j-1,k} \\
&+\psi_{yyi+1,j+1,k} + \psi_{xxi-1,j+1,k} + \psi_{xxi+1,j+1,k} \\
&+\psi_{xxi-1,j-1,k} + \psi_{xxi+1,j-1,k}) \\
&-\frac{2142}{642{,}000}h^2(\psi_{zzi,j,k-1} + \psi_{zzi,j,k+1}) \\
&+\frac{38{,}946}{642{,}000}h^2(\psi_{xxi-1,j,k} + \psi_{xxi+1,j,k} \\
&+\psi_{yyi,j+1,k} + \psi_{yyi,j-1,k}) \\
&+\frac{2625}{642{,}000}h^2(\psi_{yyi-1,j-1,k-1} + \psi_{yyi,j+1,k-1} \\
&+\psi_{yyi,j+1,k+1} + \psi_{yyi,j-1,k+1} + \psi_{xxi+1,j,k-1} \\
&+\psi_{xxi+1,j,k+1} + \psi_{xxi-1,j,k-1} + \psi_{xxi-1,j,k+1})) \\
&= -h^4(\frac{73}{1712}f_{i,j,k} + \frac{889}{128{,}400}(f_{i+1,j,k} \\
&+f_{i-1,j,k} + f_{i,j+1,k} + f_{i,j-1,k}) \\
&+\frac{247}{128{,}400}(f_{i,j,k-1} + f_{i,j,k+1})).
\end{aligned}
\tag{10}
$$

We discuss the solution of linear systems associated with the above finite difference approximations in the next section.

3 Solution of linear systems

As mentioned in [2] by writing equations (7)–(10) at every interior grid points one obtains a system of linear algebraic equations for equation (1). We obtain a system of equations with a block coefficient matrix such that each entry is a 4×4 matrix, see [8]. Since the resulted linear system is very huge especially for small values of mesh sizes, applying direct solvers is difficult and in some senses is impossible, so we used the iterative methods for calculating all interior grid points complete one step of iteration method.

3.1 Multigrid method

Since the pioneering work of Brandt [3] in the early 1970s, multi-grid methods have been widely applied to the numerical solution of differential equations. A good introductory text on multi-grid is the book by Briggs [5], more advanced treatment is given by Brandt in [4]. We present a brief description of how multi-grid works. While iterative processes are sometimes slow to solve differential equations, they tend to make good smoothers. That is, analyzing Fourier components of the error, an iterative solver will typically sharply reduce the oscillatory components, while leaving the smooth components virtually unchanged. These smooth components can be solved for on a coarser grid by computing the residual of the equation, restricting it to the coarse grid, and solving. This is more efficient, both due to the smaller number of coarse grid points and to the fact that smooth fine grid components become oscillatory on the coarse grid (moothness being measured in grid points per wavelength), thus, are efficiently solved by the iterative method. Components that are still slow to converge on the coarse grid are transferred to a yet coarser grid, and so on, until a grid is reached where all components can be efficiently resolved. The error components solved for on the coarse grid are added to the fine grid solution, using interpolation to determine the correction values at fine grid points. A multigrid cycle starts with a number (ν) of relaxations of the iterative scheme, transfers the (now smoothed) error to a coarser grid where a number (γ) of multigrid cycles are performed before the solution is interpolated back to the fine grid, and some (ν) more relaxations performed. Setting $\gamma = 1$ results in what is called a Vcycle, while $\gamma = 2$ gives a W cycle. A good initial guess for the multigrid cycle may be obtained cheaply by solving a coarsened version of the problem and interpolating it to a finer grid. The FMG (Full Multigrid) algorithm uses this idea recursively, starting at a relatively coarse grid and going to progressively finer grids. This minimizes the work done on fine grids starting out with the interpolated coarse grid solution.

3.1.1 Transfer operators

1. **Interpolation Operators:** For transfer operators, we use two kinds for prolongation. *The first* is inside the V-cycle and is the linear interpolation. That is values of fine grid points that are common with coarse grid points, are directly transferred and the values of other fine grid points are obtained by averaging nearest values of either two or four or eight points on the coarse grid. Whilst *the second* is outside the V-cycle and is called the fourth order interpolation. It produces new intermediate unknown-functions values on a finer grid. This scheme derived from Taylor series expansion about the point to be interpolated, by be written as,

$$\psi_{interp} = \frac{1}{32}\left\{6\sum_{1}^{4}\psi_{nnipi} + \sum_{1}^{8}\psi_{nnipp} - \frac{1}{2}h^2 f\right\}.$$

Where **nnipi** is the nearest neighbours in plane of interest, **nnipp** is the nearest neighbours in adjacent parallel planes, h is the fine-grid interval and $f$is the fine-grid $r.h.s.$ see for more details Holter [13].

2. **Restriction operators:**
The residual restriction we first employ the 0 residual method that Altas and Erhel and Gupta have been given in [2]. That is after calculating the residuals from the corresponding scheme, we transfer three quarters of the residual of Eq. (5) and only one-quarter of the corresponding residuals from second derivatives Eqs. (4)-(6) to the coarser grid [2]. Then we employ the full weighting operator for residual restriction, see [8]. The full weighting operator calculates residuals at coarse grid points by weighted average of residuals of the fine grid points at 27 points in a cubic around coarse grid points and is as follows:

$$(\Re\psi)_{i,j,k} = \frac{1}{64}\left\{8\psi_{2i,2j,2k} + 4S_1 + 2S_2 + S_3\right\},$$

where

$$\begin{aligned}
S_1 &= \psi_{2i+1,2j,2k} + \psi_{2i-1,2j,2k} + \psi_{2i,2j+1,2k}\\
&\quad+\psi_{2i,2j-1,2k} + \psi_{2i,2j,2k-1} + \psi_{2i,2j,2k+1},\\
S_2 &= \psi_{2i+1,2j+1,2k} + \psi_{2i-1,2j+1,2k}\\
&\quad+\psi_{2i-1,2j-1,2k} + \psi_{2i+1,2j-1,2k}\\
&\quad+\psi_{2i+1,2j,2k+1} + \psi_{2i,2j+1,2k+1}\\
&\quad+\psi_{2i,2j-1,2k+1} + \psi_{2i-1,2j,2k+1}\\
&\quad+\psi_{2i+1,2j,2k-1} + \psi_{2i-1,2j,2k-1}\\
&\quad+\psi_{2i,2j+1,2k-1} + \psi_{2i,2j-1,2k-1},\\
S_3 &= \psi_{2i+1,2j+1,2k+1} + \psi_{2i-1,2j+1,2k+1}\\
&\quad+\psi_{2i+1,2j-1,2k+1} + \psi_{2i-1,2j-1,2k+1}\\
&\quad+\psi_{2i+1,2j+1,2k-1} + \psi_{2i-1,2j+1,2k-1}\\
&\quad+\psi_{2i+1,2j-1,2k-1} + \psi_{2i-1,2j-1,2k-1}.
\end{aligned}$$

4 Multiscale Multigrid Method

We proposed a multiscale multigrid method in [19] to solve the 3D Biharmonic equation, which computes the fourth order solutions on both the fine and coarse grids. Wang et al. [19], their multiscale multigrid method had based on the standard multigrid V-Cycle. It is similar to the full multigrid method, but they do not start from the coarsest grid level, see Fig.2. We describe it as below:

1. Run the multigrid V-Cycle on 4h grid for one or two cycles to get an approximate solution ψ_{4h}.

2. Use high order interpolation scheme to interpolate ψ_{4h} to 2h grid as the initial guess.

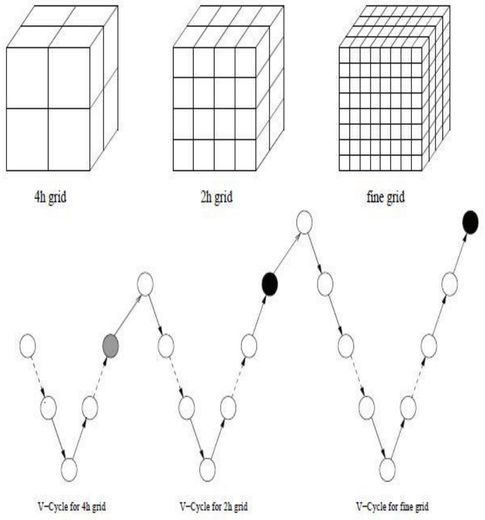

Figure 2: Representation of multiscale multigrid method for 3D biharmonic equation.

3. Run the multigrid V-Cycle on 2h grid until it converges to get the *mth* order solution ψ_{2h}.
4. Use high order interpolation scheme to interpolate ψ_{2h} to h grid as the initial guess.
5. Run the multigrid V-Cycle on h grid until it converges to get the mth order solution ψ_h.

In this work we have observed that w-cycle has better results, so for relaxation sweeps we choose a full multi-grid method V(ν_1, ν_2) cycle, where ν_1 , ν_2 are the relaxation sweep after and before coarse grid correction. For producing better results, we take ν_1 = ν_2 = 4. Also as a smoothing scheme, we use alternating $(x-y)$ line Gauss-Seidel relaxation in the multiscale multigrid method for solving the two dimension problems [19, 20] and use alternating plane relaxation smoother for the three dimension problems.

5 Numerical experiments

We tested our fourth order compact scheme (FOC) and compared the results with that done in [2] and [8]. In our numerical experiments, the domain Ω is chosen as the unit cube $(0,1)^3$. We solve the second kind Dirichlet problems for the 3D Bihaemonic equations (1) for a number of test problems; the forcing function $f(x,y,z)$ in Eq. (1) is generated from the knowledge of exact solution $\psi(x,y,z)$. The numerical results presented in the sequel have been obtained with a code written in FORTRAN77 programming language.The computations were run in double precision on Pentium IV, 2.85 GHZ, with 1GB memory.

In this section we present the numerical results of employing compact finite difference approximations (5)-(8) with multigrid method on several test problems. The stopping criterion is that the absolute maximum value of the differences of values of $\psi(x,y,z)$ in two sequential cycles in grid points be less than 10^{-10}. Because the exact solution is known, we compute the absolute error by computing the differences of the approximate and exact values Ψ.

Problem 1: This test problem is introduced in [2] and [8]. We consider the three-dimensional biharmonic boundary value problem (1) on the unit cubic with the exact solution
$\psi(x,y,z)=(1-\cos 2\pi x)(1-\cos 2\pi y)(1-\cos 2\pi z)$,
$f(x,y,z) = -16(\cos(2\pi x)\pi^4(1-\cos(2\pi y))(1-\cos(2\pi z))) - 16\cos(2\pi x)\pi^4(1-\cos(2\pi y)\pi^4)$
$(1-\cos(2\pi z)) - 16(1-\cos 2\pi x)(1-\cos 2\pi y)\cos(2\pi z)\pi^4$.

Problem 2:
We consider the three-dimensional biharmonic boundary value problem (1) on the unit cubic with the exact solution $\psi(x,y,z)=e^{xyz}$,

$$f(x,y,z)=2\left\{(x^2+y^2+z^2)(2+4xyz\right.$$
$$\left.+x^2y^2z^2)+(x^4z^4+x^4y^4+y^4z^4)\right\}e^{xyz}.$$

Problem 3:
We consider the three-dimensional biharmonic boundary value problem (1) on the unit cubic with the exact solution $\psi(x,y,z)=cosh(x)cosh(y)cosh(z)$.
$f(x,y,z)=9cosh(x)cosh(y)cosh(z)$.

Problem 4:
This test problem is introduced in [2] and [8]. We consider the three-dimensional biharmonic boundary value problem (1) on the unit cubic with the exact solution

$$\psi(x,y,z)= \quad (x^2-x)(y^2-y)(z^2-z)$$
$$e^{p[(x-0.5)^2+(y-0.5)^2+(z-q)^2]}.$$

The exact solution of this test problem is strongly peaked for large values of the parameter p. The second parameter q moves the peak along the z-direction. The computational results are presented here for $q=0.2$ and $p=10$.

According to all these tables we observe that grid-independently property of multigrid method completely satisfy (see Tables 2, 3, 4 and 6). Also according to discretisation error of Tables 1-6 we see that errors decay with order h^4 as the mesh size, h, reduces.

Mesh Size	$\|\|\Psi-\psi\|\|_\infty$	W-Cycles [2]	W-Cycles [8]
$8\times8\times8$	6.4×10^{-3}	10	14
$16\times16\times16$	3.8×10^{-4}	22	14
$32\times32\times32$	2.3×10^{-5}	22	14

Table 1: Multigrid results with the residual restriction scheme introduced by Altas, et al.[2] and Mehdi et al.[8] for test problem 1

Mesh Size	$\|\|\Psi-\psi\|\|_\infty$	$\|\|\Psi-\psi\|\|_2$	W-Cycles
$8\times8\times8$	6.4×10^{-3}	2.1×10^{-2}	13
$16\times16\times16$	3.8×10^{-4}	6.1×10^{-3}	14
$32\times32\times32$	2.3×10^{-5}	3.3×10^{-4}	14

Table 2: Multigrid results with the full weighting residual restriction scheme and higher order interpolation for test problem 1

Mesh Size	$\|\|\Psi-\psi\|\|_\infty$	$\|\|\Psi-\psi\|\|_2$	W-Cycles
$8\times8\times8$	4.1×10^{-4}	3.5×10^{-4}	15
$16\times16\times16$	3.7×10^{-5}	4.8×10^{-4}	16
$32\times32\times32$	2.1×10^{-6}	5.5×10^{-5}	16

Table 3: Multigrid results with the full weighting residual restriction scheme and higher order interpolation for test problem 2

Mesh Size	$\|\|\Psi-\psi\|\|_\infty$	$\|\|\Psi-\psi\|\|_2$	W-Cycles
$8\times8\times8$	9.8×10^{-9}	1.7×10^{-8}	13
$16\times16\times16$	5.9×10^{-10}	4.6×10^{-9}	14
$32\times32\times32$	3.4×10^{-11}	5.1×10^{-9}	14

Table 4: Multigrid results with the full weighting residual restriction scheme and higher order interpolation for test problem 3

Mesh Size	$\|\|\Psi-\psi\|\|_\infty$	W-Cycles [2]	W-Cycles [8]
$8\times8\times8$	9.0×10^{0}	10	14
$16\times16\times16$	8.1×10^{-1}	13	15
$32\times32\times32$	5.2×10^{-2}	21	15

Table 5: Multigrid results with the residual restriction scheme introduced by Altas, et al.[2] and Mehdi et al.[8] for test problem 4

Mesh Size	$\|\|\Psi-\psi\|\|_\infty$	$\|\|\Psi-\psi\|\|_2$	W-Cycles
$8\times8\times8$	4.2×10^{-1}	9.0×10^{-1}	14
$16\times16\times16$	8.1×10^{-1}	6.1×10^{-2}	15
$32\times32\times32$	5.2×10^{-2}	1.2×10^{-3}	15

Table 6: Multigrid results with the full weighting residual restriction scheme and higher order interpolation for test problem 4

6 Conclusion

In our work, we examine a high-accuracy, compact formulation for the three dimensional biharmonic equation with Dirichlet boundary conditions of second kind. We developed a *Mathematica* sympolic computation software package to handle the extensive algebraic manipulation procedure. The finite difference approximation has derived on a 27-point compact stencil using the values of the solution and its second derivatives as the unknowns. The approximations have derived using a symbolic software package. We proposed a multiscale multigrid method in [18] to solve the 3D Biharmonic equation, which computes the fourth order solutions on both the fine and coarse grids. We have solved several test problems to show the efficiency of the techniques. FMG algorithm $W(4,4)$ cycles producing high accurate solutions of the biharmonic equation.

7 Appendix A

According to work of [1] and [8] we assume that we approximate the solution ψ, locally, by the *n*th degree polynomial as

$$\psi(x,y,z)\cong P(x,y,z)=\sum a_{i,j,k}x^iy^jz^k.$$

We select for fourth-order formula the polynomial

$$\psi(x,y,z)=\sum_{i,j,k=0}^{i+j+k\le6} a_{i,j,k}x^iy^jz^k,$$

where the coefficients $a_{0,0,0}$, $a_{2,0,0}$, $a_{0,2,0}$ and $a_{0,0,2}$ are proportional to the values of ψ, ψ_{xx}, ψ_{yy} and ψ_{zz}. Considering the Taylor series expansion of ψ and evaluating the values of P, P_{xx}, P_{yy} and P_{zz} at the center point $(0,0,0)$, we obtain: $\psi_{0,0,0}=a_{0,0,0}$, $\psi_{xx_{0,0,0}}=2a_{2,0,0}$, $\psi_{yy_{0,0,0}}=2a_{0,2,0}$, $\psi_{zz_{0,0,0}}=2a_{0,0,2}$. So it is sufficient to calculate four coefficients of ψ in the above expansion. In our *Mathematica* code, we have used from eliminate command to construct chosen system in terms of $a_{0,0,0}$, $a_{2,0,0}$, $a_{0,2,0}$, $a_{0,0,2}$ and then solved chosen system. The *Mathematica* code follows with some explanations of critical steps. We give the fourth-order code in below:
Clear$[p,q,i,j,k,n,xx,yy,zz,\psi,\psi_{xx},\psi_{yy},\psi_{zz},eq,a,b]$;

$$\begin{array}{lll} xx[0]=0; & yy[0]=0; & zz[0]=0; \\ xx[1]=0; & yy[1]=h; & zz[1]=0; \\ xx[2]=h; & yy[2]=0; & zz[2]=-h; \\ xx[3]=0; & yy[3]=h; & zz[3]=-h; \\ xx[4]=0; & yy[4]=0; & zz[4]=-h; \\ xx[5]=0; & yy[5]=-h; & zz[5]=-h; \\ xx[6]=0; & yy[6]=-h; & zz[6]=0; \end{array}$$

$xx[7] = h;$	$yy[7] = -h;$	$zz[7] = 0;$
$xx[8] = 0;$	$yy[8] = -h;$	$zz[8] = h;$
$xx[9] = h;$	$yy[9] = 0;$	$zz[9] = h;$
$xx[10] = 0;$	$yy[10] = 0;$	$zz[10] = h;$
$xx[11] = h;$	$yy[11] = 0;$	$zz[11] = 0;$
$xx[12] = 0;$	$yy[12] = h;$	$zz[12] = h;$
$xx[13] = h;$	$yy[13] = h;$	$zz[13] = 0;$
$xx[14] = -h;$	$yy[14] = 0;$	$zz[14] = 0;$
$xx[15] = -h;$	$yy[15] = 0;$	$zz[15] = h;$
$xx[16] = -h;$	$yy[16] = 0;$	$zz[16] = -h;$
$xx[17] = -h;$	$yy[17] = h;$	$zz[17] = 0;$
$xx[18] = -h;$	$yy[18] = -h;$	$zz[18] = 0;$
$xx[19] = h;$	$yy[19] = -h;$	$zz[19] = -h;$
$xx[20] = h;$	$yy[20] = -h;$	$zz[20] = h;$
$xx[21] = h;$	$yy[21] = h;$	$zz[21] = h;$
$xx[22] = h;$	$yy[22] = h;$	$zz[22] = -h;$
$xx[23] = -h;$	$yy[23] = -h;$	$zz[23] = h;$
$xx[24] = -h;$	$yy[24] = -h;$	$zz[24] = -h;$
$xx[25] = -h;$	$yy[25] = h;$	$zz[25] = -h;$
$xx[26] = -h;$	$yy[26] = h;$	$zz[26] = h;$

($*$ Form the polynomial p of degree n $*$)
n=6;
P=0;

$$Do[If[i + j + k <= n, p = p + a[i, j, k]x^i y^j z^k,\\ Continue[\ ]], \{i, 0, n\}, \{j, 0, n\}, \{k, 0, n\}];$$

($*$ Differentiate the polynomial P with respect to x , y and z $*$)

$$dpxx = D[p, \{x, 2\}]; dpyy = D[p, \{y, 2\}];\\ dpzz = D[p, \{z, 2\}];$$

Clear$[i, j, k]$;
($*$ Form the equations to define values of $\psi, \psi_{xx}, \psi_{yy}, \psi_{zz}$ at 19 points $*$)

$$Do[eq[i] = u[i] =\\ = p/.\{x- > xx[i], y- > yy[i],\\ z- > zz[i]\}, \{i, 0, 26\}];\\ Do[eq[i + 27] = uxx[i] =\\ = dpxx/.\{x- > xx[i], y- > yy[i],\\ z- > zz[i]\}, \{i, 0, 26\}];\\ Do[eq[i + 54] = uyy[i] =\\ = dpyy/.\{x- > xx[i], y- > yy[i],\\ z- > zz[i]\}, \{i, 0, 26\}];\\ Do[eq[i + 82] = uzz[i] =\\ = dpzz/.\{x- > xx[i], y- > yy[i],\\ z- > zz[i]\}, \{i, 0, 26\}]$$

($*$ Define the differential equation (1)$*$)

$$q = D[p, \{x, 4\}] + D[p, \{y, 4\}] + D[p, \{z, 4\}]\\ +2D[p, \{x, 2\}, \{y, 2\}] + 2D[p, \{x, 2\}, \{z, 2\}]\\ +2D[p, \{z, 2\}, \{y, 2\}];$$

($*$ Define 7 more equations by using f and its derivatives $*$)

$$eq[99] = b[0, 0, 0] =\\ = (q/.\{x \to 0, y \to 0, z \to 0\});\\ eq[100] = b[1, 0, 0] =\\ = (D[q, x]/.\{x \to 0, y \to 0, z \to 0\});\\ eq[101] = b[0, 1, 0] =\\ = (D[q, y]/.\{x \to 0, y \to 0, z \to 0\});\\ eq[102] = b[0, 0, 1] =\\ = (D[q, z]/.\{x \to 0, y \to 0, z \to 0\});\\ eq[103] = b[2, 0, 0] =\\ = (D[q, x, x]/.\{x \to 0, y \to 0, z \to 0\});\\ eq[104] = b[0, 2, 0] =\\ = (D[q, y, y]/.\{x \to 0, y \to 0, z \to 0\});\\ eq[105] = b[0, 0, 2] =\\ = (D[q, z, z]/.\{x \to 0, y \to 0, z \to 0\});$$

($*$ Solve the chosen system for $a_{0,0,0}, a_{2,0,0}, a_{0,2,0}, a_{0,0,2}$ $*$)

Eliminate [{eq[1], eq[2], eq[3], eq[4], eq[5], eq[6], eq[7], eq[8], eq[9], eq[10], eq[11], eq[12], eq[13], eq[14], eq[15], eq[16], eq[17], eq[18], eq[19], eq[20], eq[21], eq[22], eq[23], eq[24], eq[25], eq[26], eq[29], eq[34], eq[36], eq[38], eq[40], eq[41], eq[42], eq[43], eq[44], eq[45], eq[55], eq[57], eq[59], eq[60], eq[61], eq[62], eq[66], eq[67], eq[71], eq[72], eq[84], eq[85], eq[86], eq[87], eq[90], eq[91], eq[92], eq[94], eq[97] , eq[98], eq[99], eq[100], eq[101], eq[102], eq[103], eq[104], eq[105]},
{*a*[0, 0, 1],*a*[0, 0, 3], *a*[0, 0, 4],*a*[0, 1, 1], *a*[0, 1, 2], *a*[0, 1, 0], *a*[0, 2, 1],*a*[0, 2, 2],*a*[0, 3, 0],*a*[0, 4, 0],*a*[1, 0, 1], *a*[1, 0, 2], *a*[1, 1, 0], *a*[1, 1, 1], *a*[1, 2, 0], *a*[1, 0, 0],*a*[2, 0, 1], *a*[2, 0, 2],*a*[2, 1, 0], *a*[2, 2, 0], *a*[3, 0, 0], *a*[4, 0, 0], *a*[0, 0, 6], *a*[0, 0, 5],*a*[0, 1, 4],*a*[0, 1, 3], *a*[0, 2, 4], *a*[0, 2, 3], *a*[0, 3, 2],*a*[0, 3, 1], *a*[0, 4, 2], *a*[0, 4, 1],*a*[0, 5, 0], *a*[0, 6, 0], *a*[1, 0, 3], *a*[1, 0, 4],*a*[1, 1, 2], *a*[1, 1, 4],*a*[1, 1, 3],*a*[1, 2, 2], *a*[1, 2, 1], *a*[1, 3, 0], *a*[1, 2, 3], *a*[1, 3, 1], *a*[4, 2, 0],*a*[1, 4, 0],*a*[2, 0, 3], *a*[5, 0, 0],*a*[2, 0, 4], *a*[2, 1, 2], *a*[2, 1, 1], *a*[2, 2, 1], *a*[2, 3, 0], *a*[2, 2, 2], *a*[2, 4, 0],*a*[3, 0, 1], *a*[6, 0, 0], *a*[3, 0, 2], *a*[3, 1, 0], *a*[3, 2, 0],*a*[3, 1, 1], *a*[4, 0, 2],*a*[4, 0, 1], *a*[4, 1, 0]}]

Simplify[Solve[%,$\{a[2, 0, 0], a[0, 2, 0], a[0, 0, 0], a[0, 0, 2]\}$]]
$a[0, 0, 0]- > 187/856 * u[1] - 21/1712 * u[2] - 21/1712 * u[3] + 187/856 * u[4] - 21/1712 * u[5] + 187/856 * u[6] - 21/1712 * u[7] - 21/1712 * u[8] - 21/1712 * u[9] + 187/856 * u[10] + 187/856 * u[11] -$

$21/1712 * u[12] - 21/1712 * u[13] + 186/856 * u[14] - 21/1712 * u[15] - 21/1712 * u[16] - 21/1712 * u[17] - 21/1712 * u[18] - 35/1712 * u[19] - 35/1712 * u[20] - 35/1712 * u[21] - 35/1712 * u[22] - 35/1712 * u[23] - 35/1712 * u[24] - 35/1712 * u[25] - 35/1712 * u[26] - 41/1712 * h^2 * uzz[10] - 17/3424 * uzz[12] * h^2 - 17/3424 * uxx[17] * h^2 - 17/3424 * uzz[3] * h^2 - 17/3424 * h^2 * uyy[13] - 17/3424 * uyy[12] * h^2 - 17/3424 * uzz[5] * h^2 + 35/10272 * h^6 * b[0,2,0] - 17/3424 * h^2 * uyy[18] - 17/3424 * uxx[13] * h^2 - 17/3424 * uyy[5] * h^2 - 17/3424 * h^2 * uyy[7] - 41/1712 * h^2 * uxx[11] - 41/1712 * h^2 * uxx[14] + 35/10272 * h^6 * b[0,0,2] - 17/3424 * uxx[18] * h^2 + 125/3424 * h^4 * b[0,0,0] - 41/1712 * h^2 * uzz[4] - 17/3424 * uxx[7] * h^2 - 17/3424 * h^2 * uzz[9] - 17/3424 * uzz[8] * h^2 - 17/3424 * uxx[2] * h^2 - 17/3424 * uxx[9] * h^2 - 17/3424 * h^2 * uxx[15] - 17/3424 * h^2 * uyy[8] - 17/3424 * uyy[3] * h^2 + 35/10272 * h^6 * b[2,0,0] - 41/1712 * h^2 * uyy[6] - 41/1712 * h^2 * uyy[1] - 17/3424 * h^2 * uzz[2] - 17/3424 * uxx[16] * h^2 - 17/3424 * h^2 * uyy[17] - 17/3424 * h^2 * uzz[16] - 17/3424 * h^2 * uzz[15];$

$a[2,0,0]- \;>\; (-356892/1284000 * u[1] + 51954/1284000 * u[2] + 28842/1284000 * u[3] - 356892/1284000 * u[4] + 28842/1284000 * u[5] - 356892/1284000 * u[6] + 51954/1284000 * u[7] + 28842/1284000 * u[8] + 51954/1284000 * u[9] - 356892/1284000 * u[10] + 367284/1284000 * u[11] + 28842/1284000 * u[12] + 51954/1284000 * u[13] + 367284/1284000 * u[14] + 51954/1284000 * u[15] + 51954/1284000 * u[16] + 51954/1284000 * u[17] + 51954/1284000 * u[18] + 20250/1284000 * u[19] + 20250/1284000 * u[20] + 20250/1284000 * u[21] + 20250/1284000 * u[22] + 20250/1284000 * u[23] + 20250/1284000 * u[24] + 20250/1284000 * u[25] + 20250/1284000 * u[26] + 38946/1284000 * h^2 * uzz[10] + 6477/1284000 * uzz[12] * h^2 - 1227/1284000 * uxx[17] * h^2 + 6477/1284000 * uzz[3] * h^2 + 2625/1284000 * h^2 * uyy[13] + 6477/1284000 * uyy[12] * h^2 + 6477/1284000 * uzz[5] * h^2 - 4445/1284000 * h^6 * b[0,2,0] + 2625/1284000 * h^2 * uyy[18] - 1227/1284000 * uxx[13] * h^2 + 6477/1284000 * uyy[5] * h^2 + 2625/1284000 * h^2 * uyy[7] - 2142/1284000 * h^2 * uxx[11] - 2142/1284000 * h^2 * uxx[14] - 4445/1284000 * h^6 * b[0,0,2] - 1227/1284000 * uxx[18] * h^2 - 47625/1284000 * h^4 * b[0,0,0] + 38946/1284000 * h^2 * uzz[4] - 1227/1284000 * uxx[7] * h^2 + 2625/1284000 * h^2 * uzz[9] + 6477/1284000 * uzz[8] * h^2 - 1227/1284000 * uxx[2] * h^2 - 1227/1284000 * uxx[9] * h^2 - 1227/1284000 * uxx[15] * h^2 + 6477/1284000 * uyy[8] * h^2 + 6477/1284000 * uyy[3] * h^2 - 1235/1284000 * h^6 * b[2,0,0] + 38946/1284000 * h^2 * uyy[6] + 38946/1284000 * h^2 * uyy[1] + 2625/1284000 * h^2 * uzz[2] - 1227/1284000 * uxx[16] * h^2 + 2625/1284000 * h^2 * uyy[17] + 2625/1284000 * h^2 * uzz[16] + 2625/1284000 * h^2 * uzz[15])/h^2;$

$a[0,2,0]- \;>\; (367284/1284000 * u[1] + 28842/1284000 * u[2] + 51954/1284000 * u[3] - 356892/1284000 * u[4] + 51954/1284000 * u[5] + 367284/1284000 * u[6] + 51954/1284000 * u[7] + 51954/1284000 * u[8] + 28842/1284000 * u[9] - 356892/1284000 * u[10] - 356892/1284000 * u[11] + 51954/1284000 * u[12] + 51954/1284000 * u[13] - 356892/1284000 * u[14] + 28842/1284000 * u[15] + 28842/1284000 * u[16] + 51954/1284000 * u[17] + 51954/1284000 * u[18] + 20250/1284000 * u[19] + 20250/1284000 * u[20] + 20250/1284000 * u[21] + 20250/1284000 * u[22] + 20250/1284000 * u[23] + 20250/1284000 * u[24] + 20250/1284000 * u[25] + 20250/1284000 * u[26] + 38946/1284000 * h^2 * uzz[10] + 2625/1284000 * uzz[12] * h^2 + 2625/1284000 * uxx[17] * h^2 + 2625/1284000 * uzz[3] * h^2 - 1227/1284000 * h^2 * uyy[13] - 1227/1284000 * uyy[12] * h^2 + 2625/1284000 * uzz[5] * h^2 - 1235/1284000 * h^6 * b[0,2,0] - 1227/1284000 * h^2 * uyy[18] + 2625/1284000 * uxx[13] * h^2 - 1227/1284000 * uyy[5] * h^2 - 1227/1284000 * h^2 * uyy[7] + 38946/1284000 * h^2 * uxx[11] + 38946/1284000 * h^2 * uxx[14] - 4445/1284000 * h^6 * b[0,0,2] + 2625/1284000 * uxx[18] * h^2 - 47625/1284000 * h^4 * b[0,0,0] + 38946/1284000 * h^2 * uzz[4] + 2625/1284000 * uxx[7] * h^2 + 6477/1284000 * h^2 * uzz[9] + 2625/1284000 * uzz[8] * h^2 + 6477/1284000 * uxx[2] * h^2 + 6477/1284000 * uxx[9] * h^2 + 6477/1284000 * uxx[15] * h^2 - 1227/1284000 * uyy[8] * h^2 - 1227/1284000 * uyy[3] * h^2 - 4445/1284000 * h^6 * b[2,0,0] - 2142/1284000 * h^2 * uyy[6] - 2142/1284000 * h^2 * uyy[1] + 6477/1284000 * h^2 * uzz[2] + 6477/1284000 * uxx[16] * h^2 - 1227/1284000 * h^2 * uyy[17] + 6477/1284000 * h^2 * uzz[16] - +6477/1284000 * h^2 * uzz[15])/h^2;$

$a[0,0,2]- \;>\; (356892/1284000 * u[1] + 51954/1284000 * u[2] + 51954/1284000 * u[3] + 367284/1284000 * u[4] + 51954/1284000 * u[5] - 356892/1284000 * u[6] + 28842/1284000 * u[7] + 51954/1284000 * u[8] + 51954/1284000 * u[9] + 367284/1284000 * u[10] - 356892/1284000 * u[11] + 51954/1284000 * u[12] + 28842/1284000 * u[13] - 356892/1284000 * u[14] + 51954/1284000 * u[15] + 51954/1284000 * u[16] + 28842/1284000 * u[17] + 28842/1284000 * u[18] + 20250/1284000 * u[19] + 20250/1284000 * u[20] + 20250/1284000 * u[21] + 20250/1284000 * u[22] + 20250/1284000 * u[23] + 20250/1284000 * u[24] + 20250/1284000 *$

$$u[25] + 20250/1284000 \ast u[26] - 2142/1284000 \ast h^2 \ast uzz[10] - 1227/1284000 \ast uzz[12] \ast h^2 + 6477/1284000 \ast uxx[17] \ast h^2 - 1227/1284000 \ast uzz[3] \ast h^2 + 6477/1284000 \ast h^2 \ast uyy[13] + 2625/1284000 \ast uyy[12] \ast h^2 - 1227/1284000 \ast uzz[5] \ast h^2 - 4445/1284000 \ast h^6 \ast b[0,2,0] + 2625/1284000 \ast uyy[5] \ast h^2 + 6477/1284000 \ast h^2 \ast uyy[7] + 38946/1284000 \ast h^2 \ast uxx[11] + 38946/1284000 \ast h^2 \ast uxx[14] - 1235/1284000 \ast h^6 \ast b[0,0,2] + 6477/1284000 \ast uxx[18] \ast h^2 - 47625/1284000 \ast h^4 \ast b[0,0,0] - 2142/1284000 \ast h^2 \ast uzz[4] + 6477/1284000 \ast uxx[7] \ast h^2 - 1227/1284000 \ast h^2 \ast uzz[9] - 1227/1284000 \ast uzz[8] \ast h^2 + 2625/1284000 \ast uxx[2] \ast h^2 + 2625/1284000 \ast uxx[9] \ast h^2 + 2625/1284000 \ast uxx[15] \ast h^2 + 2625/1284000 \ast uyy[8] \ast h^2 + 2625/1284000 \ast uyy[3] \ast h^2 - 4445/1284000 \ast h^6 \ast b[2,0,0] + 38946/1284000 \ast h^2 \ast uyy[6] + 38946/1284000 \ast h^2 \ast uyy[1] - 1227/1284000 \ast h^2 \ast uzz[2] + 6477/1284000 \ast h^2 \ast uyy[18] + 6477/1284000 \ast uxx[13] \ast h^2 + 2625/1284000 \ast uxx[16] \ast h^2 + 6477/1284000 \ast h^2 \ast uyy[17] - 1227/1284000 \ast h^2 \ast uzz[16] - 1227/1284000 \ast h^2 \ast uzz[15])/h^2;$$

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