Numerical Solving Two-dimensional Variable-order Fractional Advection-dispersion Equation

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Abstract: In this paper, a two-dimensional variable-order fractional advection-dispersion equation with variable coefficient is considered. The numerical method with first order temporal accuracy and first order spatial accuracy is proposed. The convergence and stability of the numerical method are analyzed by using energy method. Finally, the results of a numerical example supports the theoretical analysis.

Key–Words: two-dimensional variable-order fractional advection-dispersion equation, finite difference methods, stability, convergence

1 Introduction

Fractional diffusion equations as the important mathematical models are used widely in physics [1], finance [2-4] and hydrology [5-9]. In the last decades, for more accurately describe the behaviors of some complex systems, the concept of variable-order fractional equations have been introduced by many authors [10-18] and have received tremendous success. Since the kernel of the variable-order fractional operator has a variable exponent, it is more difficult to obtain analytical solutions of fractional differential equations with variable order. Hence, to find the solution of these equations, the numerical approximation method play an important role.

Up to now, the theoretical study of variable-order fractional partial differential equations is quite limited and the numerical process still remains in primary stage. In 2005, to solve the variable-order differential equations, Soon et al. [19] gave a second-order Runge-Kutta method consisting of an explicit Euler predictor step followed by an implicit Euler corrector step. In 2009, Sun et al. [20] used the Crank-Nicolson scheme to get the diffusion curve of the variable-order fractional models. In the same year, Zhuang et al. [21] presented explicit and implicit Euler approximations for the variable-order fractional advection-diffusion equation with a nonlinear source term, and also gave the stability and convergence results of these methods. In 2010, Chen et al. [22] proposed two numerical schemes for a variable-order anomalous sub-diffusion equation: one with first order temporal accuracy and fourth order spatial accuracy, and the other with second order temporal accuracy and fourth spacial accuracy. In 2012, Shen et al. [23] presented numerical techniques for the variable-order time fractional diffusion equation and discussed the stability and convergence. Yang et al. [24] also proposed a finite difference scheme for solving time-variable-order time-space fractional reaction-diffusion equation using the finite difference scheme. Chen et al. [25] investigated the stability and convergence of finite difference-approximation for two dimensional variable-order anomalous sub-diffusion fractional equation.

In this paper, we will structure a numerical scheme for solving more general variable-order fractional advection-dispersion equation and study the stability and convergence of this scheme.

For simplify we introduce the following notations

$$\Omega = \{(x, y, t) \mid 0 \le x \le X, 0 \le y \le Y, 0 \le t \le T\},$$

$$U(\Omega) = \left\{u(x, y, t) \;\Big|\; \frac{\partial u}{\partial t}, \frac{\partial^3 u}{\partial x^3}, \frac{\partial^3 u}{\partial y^3} \in C(\Omega)\right\}.$$

Consider the two-dimensional variable-order fractional advection-dispersion equation

$$\begin{aligned} \frac{\partial u(x, y, t)}{\partial t} \quad = \quad & c(x, y, t)\frac{\partial^{\alpha(x,y,t)} u(x, y, t)}{\partial x^{\alpha(x,y,t)}} \\ & + d(x, y, t)\frac{\partial^{1+\alpha(x,y,t)} u(x, y, t)}{\partial x^{1+\alpha(x,y,t)}} \\ & + e(x, y, t)\frac{\partial^{\beta(x,y,t)} u(x, y, t)}{\partial y^{\beta(x,y,t)}} \\ & + g(x, y, t)\frac{\partial^{1+\beta(x,y,t)} u(x, y, t)}{\partial y^{1+\beta(x,y,t)}}, \end{aligned}$$

$$(x, y, t) \in \Omega \tag{1}$$

with initial condition

$$u(x, y, 0) = \phi(x, y), \quad 0 \le x \le X, 0 \le y \le Y, \tag{2}$$

and boundary conditions

$$\begin{cases} u(0, y, t) = u(x, 0, t) = 0, \\ u(X, y, t) = \psi_1(y, t), \\ u(x, Y, t) = \psi_2(x, t), \\ 0 \le x \le X, \\ 0 \le y \le Y, \\ 0 \le t \le T, \end{cases} \tag{3}$$

where

$$0 < \alpha(x, y, t) < 1, \quad 0 < \beta(x, y, t) < 1,$$

the functions $c(x, y, t), d(x, y, t), e(x, y, t), g(x, y, t)$ satisfy

$$\begin{array}{ll} c(x, y, t) < 0, & d(x, y, t) > 0, \\ e(x, y, t) < 0, & g(x, y, t) > 0, \end{array} \tag{4}$$

for $(x, y, t) \in \Omega$, the operators

$$\frac{\partial^\alpha}{\partial x^\alpha}, \quad \frac{\partial^{\alpha+1}}{\partial x^{\alpha+1}}, \quad \frac{\partial^\beta}{\partial y^\beta}, \quad \frac{\partial^{\beta+1}}{\partial y^{\beta+1}}$$

denote the Riemann-Liouville fractional derivative.

The α order Riemann-Liouville fractional derivative of a function f is defined as (see [26])

$$\frac{d^\alpha f(x)}{dx^\alpha} = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_0^x \frac{f(s)}{(x-s)^\alpha} ds, \quad x > 0, \quad n - 1 < \alpha \le n, \tag{5}$$

where n is a positive integer.

The structure of this paper is as follow: In Section 2, an implicit finite difference method for Eq.(1) is proposed. The stability and convergence of the method are analyzed in Section 3 and 4, respectively. Finally, a numerical example is given.

2 Numerical scheme

In this section, we will structure a numerical scheme to approximate problem (1)-(3). Therefor, we apply the standard Grünwald formula [26] (for $0 < \alpha \le 1$)

$$\frac{d^\alpha f(x)}{dx^\alpha} = \frac{1}{h^\alpha} \sum_{l=0}^{[x/h]} \omega_l^{(\alpha)} f(x - lh) + R_1 \tag{6}$$

and the shifted Grünwald formula [27] (for $1 < \alpha \le 2$)

$$\frac{d^\alpha f(x)}{dx^\alpha} = \frac{1}{h^\alpha} \sum_{l=0}^{[x/h]+1} \omega_l^{(\alpha)} f(x - (l-1)h) + R_2 \tag{7}$$

to discrete the fractional derivatives in equation (1), where

$$\omega_l^{(\alpha)} = (-1)^l \frac{\alpha(\alpha-1)\cdots(\alpha-l+1)}{l!}, l = 0, 1, \cdots$$

is called Grünwald coefficient and the symbol $[x]$ denotes the integer part of x. The symbols R_1 and R_2 are the remainder terms (also called truncation errors). It is not difficult to see from [27] that, when $f \in C^3[0, L]$, there exists constants C_1 and C_2 such that

$$|R_1| \le C_1 h, \quad |R_2| \le C_2 h \tag{8}$$

uniformly on $[0, L]$ for all $0 < \alpha \le 1$ and $1 < \alpha \le 2$, respectively.

Lemma 1 *The Grünwald coefficient $\omega_l^{(\alpha)}$ has the following properties*

(1) $\omega_0^{(\alpha)} = 1, \omega_1^{(\alpha)} = -\alpha$ *and* $\sum_{l=0}^{\infty} \omega_l^{(\alpha)} = 0$.

(2) when $0 < \alpha < 1$, *we have* $\omega_k^{(\alpha)} < 0$ *for* $k \ge 1$ *and* $\sum_{l=0}^{q} \omega_l^{(\alpha)} > 0$ *for* $q = 1, 2, \cdots$,

when $1 < \alpha < 2$, *we have* $\omega_k^{(\alpha)} > 0$ *for* $k \ge 2$ *and* $\sum_{l=0}^{q} \omega_l^{(\alpha)} < 0$ *for* $q = 1, 2, \cdots$.

(3) $\omega_l^{(\alpha)} = (1 - \frac{\alpha+1}{l}) \omega_{l-1}^{(\alpha)}, l = 1, 2, \cdots$.

Through the simple deriving, we can get the following lemma.

Lemma 2 *Suppose* $x, x_1, x_2, \cdots, x_n$ *are given real numbers, then we have*

$$|x + x_1 + x_2 + \cdots + x_n| \ge |x| - (|x_1| + |x_2| + \cdots + |x_n|).$$

In the following, we discretize temporal and spatial variables. Let

$$t_k = k\tau, \quad k = 0, 1, 2, \cdots, N;$$
$$x_i = ih_x, \quad i = 0, 1, 2, \cdots, N_x;$$
$$y_j = jh_y, \quad j = 0, 1, 2, \cdots, N_y,$$

where $\tau = T/N$, $h_x = X/N_x$, $h_y = Y/N_y$ are temporal and spatial step-size, respectively.

In order to describe simple, we define

$$\begin{array}{ll} c_{i,j}^k = c(x_i, y_j, t_k), & d_{i,j}^k = d(x_i, y_j, t_k), \\ e_{i,j}^k = e(x_i, y_j, t_k), & g_{i,j}^k = g(x_i, y_j, t_k), \\ \alpha_{i,j}^k = \alpha(x_i, y_j, t_k), & \beta_{i,j}^k = \beta(x_i, y_j, t_k). \end{array}$$

Therefore, on the grid point (x_i, y_j, t_k), Eq. (1) can be rewritten as follows

$$\frac{\partial u(x_i, y_j, t_k)}{\partial t} = c_{i,j}^k \frac{\partial^{\alpha_{i,j}^k} u(x_i, y_j, t_k)}{\partial x^{\alpha_{i,j}^k}}$$

$$+ \quad d_{i,j}^k \frac{\partial^{1+\alpha_{i,j}^k} u(x_i,y_j,t_k)}{\partial x^{1+\alpha_{i,j}^k}} + e_{i,j}^k \frac{\partial^{\beta_{i,j}^k} u(x_i,y_j,t_k)}{\partial y^{\beta_{i,j}^k}}$$
$$+ \quad g_{i,j}^k \frac{\partial^{1+\beta_{i,j}^k} u(x_i,y_j,t_k)}{\partial y^{1+\beta_{i,j}^k}}. \tag{9}$$

Noting that $u(x, y, t) \in U(\Omega)$, we have

$$\frac{\partial u(x_i, y_j, t_k)}{\partial t} = \frac{u(x_i, y_j, t_k) - u(x_i, y_j, t_{k-1})}{\tau} + R_t, \tag{10}$$

where $|R_t| \le C\tau$ with some constant C.

Applying (6), (7) and (10) to (9), we obtain the following formula

$$\begin{aligned}
&\frac{u(x_i, y_j, t_k) - u(x_i, y_j, t_{k-1})}{\tau} + R_t \\
&= \frac{c_{i,j}^k}{h_x^{\alpha_{i,j}^k}} \sum_{l=0}^{i} \omega_l^{(\alpha_{i,j}^k)} u(x_{i-l}, y_j, t_k) + R_{1,x} \\
&+ \frac{d_{i,j}^k}{h_x^{1+\alpha_{i,j}^k}} \sum_{l=0}^{i+1} \omega_l^{(1+\alpha_{i,j}^k)} u(x_{i+1-l}, y_j, t_k) + R_{2,x} \\
&+ \frac{e_{i,j}^k}{h_y^{\beta_{i,j}^k}} \sum_{l=0}^{j} \omega_l^{(\beta_{i,j}^k)} u(_i, y_{j-l}, t_k) + R_{1,y} \\
&+ \frac{g_{i,j}^k}{h_y^{1+\beta_{i,j}^k}} \sum_{l=0}^{j+1} \omega_l^{(1+\beta_{i,j}^k)} u(x_i, y_{j+1-l}, t_k) + R_{2,y},
\end{aligned}$$

which gives

$$\begin{aligned}
u(x_i, y_j, t_k) \quad & -r_{i,j,k}^{(1)} \sum_{l=0}^{i} \omega_l^{(\alpha_{i,j}^k)} u(x_{i-l}, y_j, t_k) \\
& -r_{i,j,k}^{(2)} \sum_{l=0}^{i+1} \omega_l^{(1+\alpha_{i,j}^k)} u(x_{i+1-l}, y_j, t_k) \\
& -r_{i,j,k}^{(3)} \sum_{l=0}^{j} \omega_l^{(\beta_{i,j}^k)} u(x_i, y_{j-l}, t_k) \\
& -r_{i,j,k}^{(4)} \sum_{l=0}^{j+1} \omega_l^{(1+\beta_{i,j}^k)} u(x_i, y_{j+1-l}, t_k) \\
& = u(x_i, y_j, t_{k-1}) + R_{i,j}^k,
\end{aligned} \tag{11}$$

where

$$R_{i,j}^k = -\tau R_t + \tau(R_{1,x} + R_{1,x} + R_{1,y} + R_{2,y}) \tag{12}$$

and

$$\begin{aligned}
r_{i,j,k}^{(1)} &= \frac{\tau c_{i,j}^k}{h_x^{\alpha_{i,j}^k}}, \qquad r_{i,j,k}^{(2)} = \frac{\tau d_{i,j}^k}{h_x^{1+\alpha_{i,j}^k}}, \\
r_{i,j,k}^{(3)} &= \frac{\tau e_{i,j}^k}{h_y^{\beta_{i,j}^k}}, \qquad r_{i,j,k}^{(4)} = \frac{\tau g_{i,j}^k}{h_y^{1+\beta_{i,j}^k}}.
\end{aligned} \tag{13}$$

It is clear from (4), that

$$\begin{aligned}
r_{i,j,k}^{(1)} < 0, \qquad r_{i,j,k}^{(2)} > 0, \\
r_{i,j,k}^{(3)} < 0, \qquad r_{i,j,k}^{(4)} > 0.
\end{aligned} \tag{14}$$

Let $u_{i,j}^k$ denote the approximation of $u(x_i, y_j, t_k)$. Truncating the remainder term $R_{i,j}^k$ in (11), we get the implicit finite difference scheme for solving Eq.(1) as follows

$$\begin{aligned}
& u_{i,j}^k - r_{i,j,k}^{(1)} \sum_{l=0}^{i} \omega_l^{(\alpha_{i,j}^k)} u_{i-l,j}^k - r_{i,j,k}^{(2)} \sum_{l=0}^{i+1} \omega_l^{(1+\alpha_{i,j}^k)} u_{i+1-l,j}^k \\
& -r_{i,j,k}^{(3)} \sum_{l=0}^{j} \omega_l^{(\beta_{i,j}^k)} u_{i,j-l}^k - r_{i,j,k}^{(4)} \sum_{l=0}^{j+1} \omega_l^{(1+\beta_{i,j}^k)} u_{i,j+1-l}^k \\
& = u_{i,j}^{k-1}, \\
& \qquad i = 1, 2, \cdots, N_x - 1;\ j = 1, 2, \cdots, N_y - 1; \\
& \qquad k = 1, 2, \cdots, N.
\end{aligned} \tag{15}$$

The initial and boundary value conditions can be described as

$$\begin{aligned}
& u_{i,j}^0 = \phi(x_i, y_j), \\
& i = 0, 1, \cdots, N_x; \\
& j = 0, 1, \cdots, N_y.
\end{aligned} \tag{16}$$

and

$$\begin{aligned}
& u_{0,j}^k = u_{i,0}^k = 0, \qquad u_{N_x,j}^k = \psi_1(y_j, t_k), \\
& u_{i,N_y}^k = \psi_2(x_i, t_k), \quad i = 0, 1, \cdots, N_x; \\
& j = 0, 1, \cdots, N_y; \quad k = 1, 2, \cdots, N.
\end{aligned} \tag{17}$$

A combination of formulas (15), (16) and (17) leads to an implicit finite difference method to solve problem (1)-(3).

3 Stability analysis

In the section, we will analyze the stability of the method (15)-(17). Suppose that $u_{i,j}^k$ and $\tilde{u}_{i,j}^k$ are two solutions of (15) with the same boundary value condition (16) and the different initial conditions (17) and

$$\begin{aligned}
& \tilde{u}_{i,j}^0 = \tilde{\phi}(x_i, y_j), \\
& i = 0, 1, \cdots, N_x; \\
& j = 0, 1, \cdots, N_y,
\end{aligned} \tag{18}$$

respectively, where $\tilde{\phi}(x, y)$ is a given initial function.

Let

$$\begin{aligned}
& \rho_{i,j}^k = \tilde{u}_{i,j}^k - u_{i,j}^k \ \ i = 1, 2, \cdots, N_x - 1; \\
& j = 1, 2, \cdots, N_y - 1;\ k = 0, 1, 2, \cdots, N.
\end{aligned}$$

Then $\rho_{i,j}^k$ satisfies

$$\begin{aligned}
&\rho_{i,j}^k - r_{i,j,k}^{(1)} \sum_{l=0}^{i} \omega_l^{(\alpha_{i,j}^k)} \rho_{i-l,j}^k \\
&-r_{i,j,k}^{(2)} \sum_{l=0}^{i+1} \omega_l^{(1+\alpha_{i,j}^k)} \rho_{i+1-l,j}^k - r_{i,j,k}^{(3)} \sum_{l=0}^{j} \omega_l^{(\beta_{i,j}^k)} \rho_{i,j-l}^k \\
&-r_{i,j,k}^{(4)} \sum_{l=0}^{j+1} \omega_l^{(1+\beta_{i,j}^k)} \rho_{i,j+1-l}^k \\
&= \rho_{i,j}^{k-1}, \\
&k = 1, 2, \cdots, N. \qquad (19)
\end{aligned}$$

For $k = 0, 1, 2, \cdots, N$, we define vector

$$\begin{aligned}
\mathbf{E}^k =&(\rho_{1,1}^k, \rho_{2,1}^k, \cdots, \rho_{N_x-1,1}^k, \cdots, \\
&\rho_{1,N_y-1}^k, \rho_{2,N_y-1}^k, \cdots, \rho_{N_x-1,N_y-1}^k).
\end{aligned}$$

Hence we have

$$\| \mathbf{E}^0 \|_\infty \leq \max_{0\leq x\leq X, 0\leq y\leq Y} |\phi(x,y) - \tilde{\phi}(x,y)|.$$

Theorem 3 *For the implicit finite difference scheme (15)-(17) we have*

$$\| \boldsymbol{E}^k \|_\infty \leq \max_{0\leq x\leq X, 0\leq y\leq Y} |\phi(x,y) - \tilde{\phi}(x,y)|,$$

for $k = 1, 2, \cdots, N$. Therefore the scheme is unconditionally stable.

Proof. For a given $k \in \{1, 2, \cdots, N\}$, choose p, q so that $|\rho_{p,q}^k| = \| \mathbf{E}^k \|_\infty$. According to Lemma 1 and (14), it is seen easily that

$$\begin{aligned}
&r_{i,j,k}^{(1)} \sum_{l=0}^{i} \omega_l^{(\alpha_{i,j}^k)} < 0, \quad r_{i,j,k}^{(2)} \sum_{l=0}^{i+1} \omega_l^{(1+\alpha_{i,j}^k)} < 0, \\
&r_{i,j,k}^{(3)} \sum_{l=0}^{j} \omega_l^{(\beta_{i,j}^k)} < 0, \quad r_{i,j,k}^{(4)} \sum_{l=0}^{j+1} \omega_l^{(1+\beta_{i,j}^k)} < 0. \qquad (20)
\end{aligned}$$

Applying (19), (20) and Lemma 2, we have

$$\begin{aligned}
|\rho_{p,q}^k| \leq& |\rho_{p,q}^k| - \Bigg\{ r_{p,q,k}^{(1)} \sum_{l=0}^{p} \omega_l^{(\alpha_{p,q}^k)} \\
&+ r_{p,q,k}^{(2)} \sum_{l=0}^{p+1} \omega_l^{(1+\alpha_{p,q}^k)} + r_{p,q,k}^{(3)} \sum_{l=0}^{q} \omega_l^{(\beta_{p,q}^k)} \\
&+ r_{p,q,k}^{(4)} \sum_{l=0}^{q+1} \omega_l^{(1+\beta_{p,q}^k)} \Bigg\} |\rho_{p,q}^k| \\
\leq& \Big(1 - r_{p,q,k}^{(1)} \omega_0^{(\alpha_{p,q}^k)} - r_{p,q,k}^{(2)} \omega_1^{(1+\alpha_{p,q}^k)} \\
&- r_{p,q,k}^{(3)} \omega_0^{(\beta_{p,q}^k)} - r_{p,q,k}^{(4)} \omega_1^{(1+\beta_{p,q}^k)} \Big) |\rho_{p,q}^k| \\
&- r_{p,q,k}^{(1)} \sum_{l=1}^{p} \omega_l^{(\alpha_{p,q}^k)} |\rho_{p-l,q}^k| \\
&- r_{p,q,k}^{(2)} \sum_{l=0, l\neq 1}^{p+1} \omega_l^{(1+\alpha_{p,q}^k)} |\rho_{p+1-l,q}^k| \\
&- r_{p,q,k}^{(3)} \sum_{l=1}^{q} \omega_l^{(\beta_{p,q}^k)} |\rho_{p,q-l}^k| \\
&- r_{p,q,k}^{(4)} \sum_{l=0, l\neq 1}^{q+1} \omega_l^{(1+\beta_{p,q}^k)} |\rho_{p,q+1-l}^k| \\
\leq& \Bigg| \rho_{p,q}^k - r_{p,q,k}^{(1)} \sum_{l=0}^{p} \omega_l^{(\alpha_{p,q}^k)} \rho_{p-l,q}^k \\
&- r_{p,q,k}^{(2)} \sum_{l=0}^{p+1} \omega_l^{(1+\alpha_{p,q}^k)} \rho_{p+1-l,q}^k \\
&- r_{p,q,k}^{(3)} \sum_{l=0}^{q} \omega_l^{(\beta_{p,q}^k)} \rho_{p,q-l}^k \\
&- r_{p,q,k}^{(4)} \sum_{l=0}^{q+1} \omega_l^{(1+\beta_{p,q}^k)} \rho_{p,q+1-l}^k \Bigg| \\
=& |\rho_{p,q}^{k-1}| \leq \| \mathbf{E}^{k-1} \|_\infty .
\end{aligned}$$

Therefore we can obtain

$$\| \mathbf{E}^k \|_\infty \leq \| \mathbf{E}^{k-1} \|_\infty \leq \cdots \leq \| \mathbf{E}^0 \|_\infty .$$

This completes the proof of Theorem 1.

4 Convergence analysis

In the section, we will consider the convergence of (15)-(17). Suppose that the problem(1)-(3) has a smooth solution $u(x, y, t) \in U(\Omega)$ and $u_{i,j}^k$ be the numerical solution of $u(x_i, y_j, t_k)$. Let

$$\begin{aligned}
\eta_{i,j}^k &= u(x_i, y_j, t_k) - u_{i,j}^k, \ \ i = 1, 2, \cdots, N_x - 1; \\
&\quad j = 1, 2, \cdots, N_y - 1; \quad k = 1, 2, \cdots, N.
\end{aligned}$$

Then $\eta_{i,j}^k$ satisfies

$$\begin{aligned}
&\eta_{i,j}^k - r_{i,j,k}^{(1)} \sum_{l=0}^{i} \omega_l^{(\alpha_{i,j}^k)} \eta_{i-l,j}^k \\
&-r_{i,j,k}^{(2)} \sum_{l=0}^{i+1} \omega_l^{(1+\alpha_{i,j}^k)} \eta_{i+1-l,j}^k - r_{i,j,k}^{(3)} \sum_{l=0}^{j} \omega_l^{(\beta_{i,j}^k)} \eta_{i,j-l}^k
\end{aligned}$$

$$-r_{i,j,k}^{(4)}\sum_{l=0}^{j+1}\omega_l^{(1+\beta_{i,j}^k)}\eta_{i,j+1-l}^k$$
$$=\eta_{i,j}^{k-1}+R_{i,j}^k, \tag{21}$$

where, since $u(x,y,t)\in U(\Omega)$, from (8), for truncation error $R_{i,j}^k$, there exists a positive constant C_3 independent of τ, h_x, h_y, such that

$$|R_{i,j}^k|\le C_3\tau(\tau+h_x+h_y) \tag{22}$$

holds uniformly for $i=1,2,\cdots,N_x$; $j=1,2,\cdots,N_y$; $k=1,2,\cdots,N$.

Let

$$\mathbf{Y}^k=(\eta_{1,1}^k,\eta_{2,1}^k,\cdots,\eta_{N_x-1,1}^k,\cdots,\\ \eta_{1,N_y-1}^k,\cdots,\eta_{N_x-1,N_y-1}^k).$$

Then we have $Y^0=0$ and can prove the following Theorem.

Theorem 4 *Suppose that the problem (1)-(3) has a smooth solution $u(x,y,t)\in U(\Omega)$, let $u_{i,j}^k$ be the numerical solution computed by using (15)-(17), then there exists a positive constant C independent of i,j,k,τ,h_x,h_y such that*

$$\|\boldsymbol{Y}^k\|_\infty\le Ck\tau(\tau+h_x+h_y),\qquad k=1,2,\cdots,N \tag{23}$$

and since $k\tau\le T$, then we have

$$|u_{i,j}^k-u(x_i,y_j,t_k)|\le\tilde{C}(\tau+h_x+h_y),$$
$$i=1,2,\cdots,N_x;\\ j=1,2,\cdots,N_y;k=1,2,\cdots,N, \tag{24}$$

where $\tilde{C}=CT$.

Proof We prove this Theorem by mathematical induction.

First, for $k=1$, we choose p,q so that

$$|\eta_{p,q}^1|=\|\mathbf{Y}^1\|_\infty .$$

Noting that (20) and $\mathbf{Y}^0=\mathbf{0}$, from (21) and (22) we have

$$\begin{aligned}
&\|\mathbf{Y}^1\|_\infty=|\eta_{p,q}^1|\\
&\le|\eta_{p,q}^1|-r_{p,q,1}^{(1)}\sum_{l=0}^{p}\omega_l^{(\alpha_{p,q}^1)}|\eta_{p-l,q}^1|\\
&-r_{p,q,1}^{(2)}\sum_{l=0}^{p+1}\omega_l^{(1+\alpha_{p,q}^1)}|\eta_{p+1-l,q}^1|-r_{p,q,1}^{(3)}\sum_{l=0}^{q}\omega_l^{(\beta_{p,q}^1)}|\eta_{p,q-l}^1|\\
&-r_{p,q,1}^{(4)}\sum_{l=0}^{q+1}\omega_l^{(1+\beta_{p,q}^1)}|\eta_{p,q+1-l}^1|\\
&\le\Bigg|\eta_{p,q}^1-r_{p,q,1}^{(1)}\sum_{l=0}^{p}\omega_l^{(\alpha_{p,q}^1)}\eta_{p-l,q}^1\\
&\quad-r_{p,q,1}^{(2)}\sum_{l=0}^{p+1}\omega_l^{(1+\alpha_{p,q}^1)}\eta_{p+1-l,q}^1-r_{p,q,1}^{(3)}\sum_{l=0}^{q}\omega_l^{(\beta_{p,q}^1)}\eta_{p,q-l}^1\\
&\quad-r_{p,q,1}^{(4)}\sum_{l=0}^{q+1}\omega_l^{(1+\beta_{p,q}^1)}\eta_{p,q+1-l}^1\Bigg|\\
&=|\eta_{p,q}^0+R_{p,q}^1|\\
&\le\|\mathbf{Y}^0\|_\infty+|R_{p,q}^1|\\
&\le C\tau(\tau+h_x+h_y),
\end{aligned}$$

which shows that (23) holds when $k=1$.

Secondly, we suppose that

$$\|\mathbf{Y}^{k-1}\|_\infty\le C(k-1)\tau(\tau+h_x+h_y),\quad 1<k\le N. \tag{25}$$

Similar to above derived, we can obtain

$$\begin{aligned}
&|\eta_{p,q}^k|\le|\eta_{p,q}^k|-r_{p,q,k}^{(1)}\sum_{l=0}^{p}\omega_l^{(\alpha_{p,q}^k)}|\eta_{p-l,q}^k|\\
&-r_{p,q,k}^{(2)}\sum_{l=0}^{p+1}\omega_l^{(1+\alpha_{p,q}^k)}|\eta_{p+1-l,q}^k|-r_{p,q,k}^{(3)}\sum_{l=0}^{q}\omega_l^{(\beta_{p,q}^k)}|\eta_{p,q-l}^k|\\
&-r_{p,q,k}^{(4)}\sum_{l=0}^{q+1}\omega_l^{(1+\beta_{p,q}^k)}|\eta_{p,q+1-l}^k|\\
&\le\Bigg|\eta_{p,q}^k-r_{p,q,k}^{(1)}\sum_{l=0}^{p}\omega_l^{(\alpha_{p,q}^k)}\eta_{p-l,q}^k\\
&-r_{p,q,k}^{(2)}\sum_{l=0}^{p+1}\omega_l^{(1+\alpha_{p,q}^k)}\eta_{p+1-l,q}^k-r_{p,q,k}^{(3)}\sum_{l=0}^{q}\omega_l^{(\beta_{p,q}^k)}\eta_{p,q-l}^k\\
&-r_{p,q,k}^{(4)}\sum_{l=0}^{q+1}\omega_l^{(1+\beta_{p,q}^k)}\eta_{p,q+1-l}^k\Bigg|\\
&=|\eta_{p,q}^{k-1}+R_{p,q}^k|\\
&\le\|\mathbf{Y}^{k-1}\|_\infty+|R_{p,q}^k|\\
&\le Ck\tau(\tau+h_x+h_y).
\end{aligned}$$

Therefore, (23) holds for $k = 1, 2, \cdots, N$.

Since $k\tau \leq T$, then form (23) we have

$$|u_{i,j}^k - u(x_i, y_j, t_k)| \leq |\eta_{p,q}^k| \leq \| \mathbf{Y}^k \|_\infty \leq \tilde{C}(\tau + h_x + h_y),$$

where $\tilde{C} = CT$. This completes the proof of Theorem 2.

Remark 5 *The inequality (24) characterize the convergence property of the method (15)-(17) and it also indicates that the numerical method possesses first order temporal accuracy and first order spatial accuracy.*

5 Numerical examples

Consider equation (1) on a finite rectangular domain $0 \leq x, y, t \leq 1$, with the diffusion coefficients

$$\begin{aligned} c(x, y, t) &= -\tfrac{\Gamma(4-\alpha(x,y,t))}{4} x^{\alpha(x,y,t)}, \\ d(x, y, t) &= \tfrac{\Gamma(3-\alpha(x,y,t))}{8} x^{1+\alpha(x,y,t)}, \\ e(x, y, t) &= -\tfrac{\Gamma(4-\beta(x,y,t))}{4} y^{\beta(x,y,t)}, \\ g(x, y, t) &= \tfrac{\Gamma(3-\beta(x,y,t))}{8} y^{1+\beta(x,y,t)}, \end{aligned} \tag{26}$$

and the initial and boundary conditions in the form

$$u(x, y, 0) = x^3 y^3, \tag{27}$$

$$\begin{aligned} &u(0, y, t) = u(x, 0, t) = 0, \quad u(1, y, t) = e^{-t} y^3, \\ &u(x, 1, t) = e^{-t} x^3, \quad t > 0. \end{aligned} \tag{28}$$

This problem posses the exact solution

$$u(x, y, t) = e^{-t} x^3 y^3. \tag{29}$$

Let

$$E_{max} = \max_{1 \leq k \leq N} \left\{ \| u_{i,j}^k - u(x_i, y_j, t_k) \|_\infty \right\}.$$

Table 1 provides the maximum errors of the numerical solution and accuracy solution at $t = 1.0$ for the problem (1), (26)-(28) calculated by (15)-(17), with some $\alpha(x, y, t), \beta(x, y, t)$.

From Table 1, it can be seen that our theoretical analysis results have been verified by the numerical results, and the convergence is one order $O(\tau + h_x + h_y)$.

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Table 1: the maximum error of the numerical solution and accuracy solution at $t = 1.0$

$\alpha(x,y,t)$	xyt	$\sin(\frac{xyt}{2})$	$\frac{e^{xyt}-\sin(xyt)}{10}$	$\frac{e^{xyt}-xyt}{8}$
$\beta(x,y,t)$	xyt	$\cos(\frac{xyt}{2})$	$\frac{e^{xyt}-\cos(xyt)}{10}$	$\frac{e^{xyt}+xyt}{8}$
$\tau = h_x = h_y = \frac{1}{8}$	3.2087e-2	2.9940e-2	3.5430e-2	2.8896e-2
$\tau = h_x = h_y = \frac{1}{16}$	8.2217e-3	7.6840e-3	8.6564e-3	7.0043e-3
$\tau = h_x = h_y = \frac{1}{64}$	2.1454e-3	2.0714e-3	2.2288e-3	1.9361e-3

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