Humbert polynomials and functions in terms of Hermite polynomials towards applications to wave propagation

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Abstract: By starting from the standard definitions of the incomplete two-variable Hermite polynomials, we propose non-trivial generalizations and we show some applications to the Bessel-type functions as the Humbert functions. We also present a generalization of the Laguerre polynomials in the same context of the incomplete-type and we use these to obtain relevant operational techniques for the Humbert-type functions. Final considerations are inserted to include the problem of wave propagation in the present theoretical framework.

Keywords: Hermite Polynomials, Laguerre polynomials, Generating Functions, Orthogonal Polynomials, Humbert polynomials, Humbert Functions, Bessel Functions.

1 Introduction

It is possible to introduce a generalization of the Hermite polynomials which are a vectorial extension of the ordinary Kampé de Feriét one-variable Hermite polynomials [1]. We have indicated this class of the Hermite polynomials, of two-index and two-variable, by the symbol $He_{m,n}(x,y)$, and we stated their definition through the following generating function:

$$e^{\underset{-}{h}^t \hat{M} \underset{-}{z} - \frac{1}{2} \underset{-}{h}^t \hat{M} \underset{-}{h}} = \sum_{m=0}^{+\infty} \sum_{n=0}^{+\infty} \frac{t^m}{m!} \frac{u^n}{n!} He_{m,n}(x,y), \tag{1}$$

where:

$$\underset{-}{z} = \begin{pmatrix} x \\ y \end{pmatrix} \text{ and } \underset{-}{h} = \begin{pmatrix} t \\ u \end{pmatrix}$$

are two vectors of the space $\mathbb{R}^2$ such that: $t \neq u$, $(|t|,|u|) < +\infty$, and the superscript "*t*" denotes transpose.
A different generalization of the Hermite polynomials could be obtained by using the slight similar procedure onto the two-variable generalized Hermite polynomials [1,2]:

$$H_n(x,y) = n! \sum_{r=0}^{\left[\frac{n}{2}\right]} \frac{y^r x^{n-2r}}{r!(n-2r)!}, \tag{2}$$

defined by the generating function of the form:

$$\exp(xt + yt^2) = \sum_{n=0}^{+\infty} \frac{t^n}{n!} H_n(x,y). \tag{3}$$

Let u and v continuous variables, such that $u \neq v$ and $(|u|,|v|) < +\infty$, $\tau \in \mathbb{R}$, we will say incomplete 2-dimensional Hermite polynomials, the polynomials defined by following generating function:

$$\exp(xu + yv + \tau uv) = \sum_{m=0}^{+\infty} \sum_{n=0}^{+\infty} \frac{u^m}{m!} \frac{v^n}{n!} h_{m,n}(x,y \mid \tau). \tag{4}$$

This class of Hermite polynomials has been deeply studied for its importance in applications, as quantum mechanical problems, harmonic oscillator functions and also to investigate the statistical properties of chaotic light [3].
By using the techniques of the generating function method [4,5], it is easy to obtain the explicit form of the above polynomials:

$$h_{m,n}(x,y\,|\,\tau)=m!n!\sum_{r=0}^{[m,n]}\frac{\tau^r x^{m-r}y^{n-r}}{r!(m-r)!(n-r)!}, \tag{5}$$

where $[m,n]=\min(m,n)$.
An interesting particular case of this class of Hermite polynomials is presented when $x=y=1$ and $\tau=x$:

$$h_{m,n}(1,1\,|\,x)=g_{m,n}(x). \tag{6}$$

It is significant to study the polynomials $g_{m,n}(x)$ since they can be used to define other forms of the incomplete 2-dimensional Hermite polynomials of the type $h_{m,n}(.,.\,|\,.)$ themselves and since they often appear in the description of the applications in quantum optics. From the relation (6) and by using the definitions (4) and (5), we can immediately write the following general relation:

$$h_{m,n}(x,y\,|\,\tau)=x^m y^n g_{m,n}\left(\frac{\tau}{xy}\right). \tag{7}$$

The incomplete 2-dimensional Hermite polynomials can be used to obtain different forms of the multi-index Bessel functions, in particular for the case of the Humbert functions. We remind that the ordinary cylindrical Bessel functions [6] are specified by the generating function:

$$\exp\left[\frac{x}{2}\left(t-\frac{1}{t}\right)\right]=\sum_{n=-\infty}^{+\infty}t^n J_n(x), \tag{8}$$

and a generalization of them, it is represented by the case of two-index, one-variable type [7,8]:

$$J_{m,n}(x)=\sum_{s=-\infty}^{+\infty}J_{m-s}(x)J_{n-s}(x)J_s(x), \tag{9}$$

with the following generating function:

$$\exp\left[\frac{x}{2}\left(u-\frac{1}{u}\right)+\left(v-\frac{1}{v}\right)\left(uv-\frac{1}{uv}\right)\right]= \\ =\sum_{m=-\infty}^{+\infty}\sum_{n=-\infty}^{+\infty}u^m v^n J_{m,n}(x), \tag{10}$$

where $x\in\mathbb{R}$ and $u,v\in\mathbb{R}$, such that $0<|u|\neq|v|<+\infty$.
This class of Bessel functions satisfied analogous interesting relations as the ordinary Bessel functions. For instance, by deriving in the equation (10) with respect to *x*, we have:

$$\frac{1}{2}\left(u-\frac{1}{u}\right)+\left(v-\frac{1}{v}\right)\left(uv-\frac{1}{uv}\right)\sum_{m=-\infty}^{+\infty}\sum_{n=-\infty}^{+\infty}u^m v^n J_{n,m}(x)= \\ =\sum_{m=-\infty}^{+\infty}\sum_{n=-\infty}^{+\infty}u^m v^n\frac{d}{dx}J_{m,n}(x), \tag{11}$$

which allows us to state the following recurrence relation:

$$\frac{d}{dx}J_{m,n}(x)=\frac{1}{2}\Big\{\big[J_{m-1,n}(x)-J_{m+1,n}(x)\big]+\big[J_{m,n-1}(x)-J_{m,n+1}(x)\big]+ \\ +\big[J_{m-1,n-1}(x)-J_{m+1,n+1}(x)\big]\Big\}. \tag{12}$$

By using the same procedure, it is easy to obtain the other two recurrence relations for this class of Bessel functions:

$$\frac{2m}{x}J_{m,n}(x)=\big[J_{m,n-1}(x)-J_{m,n+1}(x)\big]+ \\ +\big[J_{m-1,n-1}(x)-J_{m+1,n+1}(x)\big], \tag{13}$$

and:

$$\frac{2n}{x}J_{m,n}(x)=\big[J_{m,n-1}(x)-J_{m,n+1}(x)\big]+ \\ +\big[J_{m-1,n-1}(x)-J_{m+1,n+1}(x)\big]. \tag{14}$$

It is interesting to note that, for $x=0$, from the explicit form of the generalized two-index Bessel function (eq. (9)), we get:

$$J_{m,n}(0)=\sum_{s=-\infty}^{+\infty}J_{m-s}(0)J_{n-s}(0)J_s(0), \tag{15}$$

and, since:

$$J_s(0)\neq 0\text{, when } s=0, \tag{16}$$

we, finally, obtain:

$$J_{m,n}(0)=\delta_{m,0}\delta_{n,0}. \tag{17}$$

As a particular case of the two-index, one-variable Bessel functions, we can introduce the Humbert functions [9], by setting:

$$b_{m,n}(x,y\,|\,\tau)=\sum_{r=0}^{+\infty}\frac{\tau^r x^{m+r}y^{n+r}}{r!(m+r)!(n+r)!}, \tag{18}$$

defined through the following generating function:

$$\exp(xu + yv + \tau uv) = \sum_{m=0}^{+\infty}\sum_{n=0}^{+\infty} u^m v^n b_{m,n}(x, y \mid \tau) . \quad (19)$$

It is evident the similar structure between these functions and the incomplete 2-dimensional Hermite polynomials presented previously (see eqs. (4,5)). For this reason, the Humbert functions are usual exploited in connection with the Hermite polynomials of the type $h_{m,n}(x, y \mid \tau)$.

We can immediately note, for instance, that the Humbert functions could be expressed in terms of the incomplete Hermite polynomials. By rewriting, in fact, the expression in equation (19), we have:

$$\exp\left[xu + yv + \tau uv - \tau\left(uv - \frac{1}{uv}\right)\right] = \sum_{m=0}^{+\infty}\sum_{n=0}^{+\infty} u^m v^n b_{m,n}(x, y \mid \tau) , (20)$$

and, from the generating function of the ordinary Bessel function (eq. (8)), we find:

$$b_{m,n}(x, y \mid \tau) = \sum_{r=0}^{[m,n]} \frac{(-1)^r h_{m-r,n-r}(x, y \mid \tau) J_r(2\tau)}{r!(m-r)!(n-r)!} . \quad (21)$$

In the following, we will indicate with:

$g_{m,n}(x)$ and $b_{m,n}(x)$,

the Humbert polynomials and the Humbert functions respectively. In the next sections we will study the properties of these particular polynomials and functions and we will see some their non trivial generalizations along with the analysis of the related applications to facilitate some operational computation.

2 Relevant properties for Humbert polynomials and functions

In the previous section we have introduced the incomplete 2-dimensional Hermite polynomials through the relations (4,5). By using the equivalences stated in equations (6,7), we can now state the expression of the generating function for the Humbert polynomials $g_{m,n}(x)$. We have:

$$\exp(u + v + xuv) = \sum_{m=0}^{+\infty}\sum_{n=0}^{+\infty} \frac{u^m}{m!}\frac{v^n}{n!} g_{m,n}(x) \quad (22)$$

where, again, $u \neq v$ and $(|u|,|v|) < +\infty$, $\tau \in \mathbb{R}$.

By following the same procedure used to derive the recurrence relations related to the two-index, one-variable Bessel function in the previous section, we can find similar expressions for this class of Humbert polynomials. In fact, by deriving, respectively, with respect to *x*, *u* and *v*, we obtain:

$$\begin{aligned} &\frac{d}{dx} g_{m,n}(x) = mn g_{m-1,n-1}(x), \\ &g_{m+1,n}(x) = g_{m,n}(x) + mx g_{m-1,n}(x), \\ &g_{m,n+1}(x) = g_{m,n}(x) + nx g_{m,n-1}(x). \end{aligned} \quad (23)$$

By using the above relations, it is possible to state the differential equation satisfied by the polynomials $g_{m,n}(x)$. After an easy manipulation of the equations in the (23), we get:

$$m g_{m-1,n}(x) = \left(m - x\frac{d}{dx}\right) g_{m,n}(x) , \quad (24)$$

$$n g_{m,n-1}(x) = \left(n - x\frac{d}{dx}\right) g_{m,n}(x) \quad (25)$$

and also:

$$mn g_{m-1,n-1}(x) = \left(m - x\frac{d}{dx}\right)\left(n - x\frac{d}{dx}\right) g_{m,n}(x) . \quad (26)$$

After equating equation (24) with the first of the relations obtained in (23), we can state the following differential equation solved by the Humbert polynomials:

$$x^2 y'' - \left[(m+n-1)x + 1 \right] y' + mny = 0 . \quad (27)$$

We note that, from the equation (25), it also follows:

$$\begin{aligned} &g_{m,+1n}(x) = g_{m,n}(x) + nx g_{m,n}(x) - x^2 \frac{d}{dx} g_{m,n}(x), \\ &g_{m,n+1}(x) = g_{m,n}(x) + mx g_{m,n}(x) - x^2 \frac{d}{dx} g_{m,n}(x) \end{aligned} \quad (28)$$

Which suggest the introduction of the following operators:

$$\hat{S}_m^+ = 1 + \hat{n}\,x - x^2 \frac{d}{dx}, \qquad (29)$$
$$\hat{S}_n^+ = 1 + \hat{m}\,x - x^2 \frac{d}{dx},$$

where we have denoted with the symbols:

$\hat{m}$ and $\hat{n}$

a kind of number operators, in the sense that their action read as following:

$$\hat{m}\,\hat{n}\,g_{s,r}(x) = srg_{s,r}(x)\,.$$

It is now evident, by using the relations stated in the equations (23-28) and by the definition of the operators expressed in equation (29), that the following expressions hold:

$$\hat{S}_m^+\, g_{m,n}(x) = g_{m+1,n}(x), \qquad (30)$$
$$\hat{S}_n^+\, g_{m,n}(x) = g_{m,n+1}(x).$$

The above relations, combined with the first in the equation (23), allow us to state the following relevant differential equation:

$$\frac{d}{dx}\left[1 + (n+1)x - x^2 \frac{d}{dx}\right]\left[1 + mx - x^2 \frac{d}{dx}\right] g_{m,n}(x) = \qquad (31)$$
$$= (m+1)(n+1) g_{m,n}(x).$$

It is possible to derive similar relations regarding the Humbert functions. Before to proceed, we remind that, the function defined by the following generating function:

$$\exp\left(t - \frac{x}{t}\right) = \sum_{n=-\infty}^{+\infty} t^n C_n(x) \qquad (32)$$

is known as the Tricomi function [10], which its explicit form is:

$$C_n(x) = \sum_{r=0}^{+\infty} \frac{(-1)^r x^r}{r!(n+r)!}\,. \qquad (33)$$

It is possible to introduce a generalization of the above function in the sense of the Humbert functions. In fact, from the equation (18) it is immediately recognized that we can call generalized Tricomi function, the function expressed by the following relation:

$$C_{m,n}(x) = b_{m,n}(1,1\,|\,x)\,. \qquad (34)$$

By using the same procedure outlined above, we can derive, for the Humbert functions the analogous recurrence relations stated for the polynomials $g_{m,n}(x)$. In fact, by considering the relation (34), we have:

$$\frac{d}{dx} C_{m,n}(x) = C_{m+1,n+1}(x),$$
$$mC_{m,n}(x) = C_{m-1,n}(x) - xC_{m+1,n+1}(x), \qquad (35)$$
$$nC_{m,n}(x) = C_{m,n-1}(x) - xC_{m+1,n+1}(x).$$

We can combine the above relations, to get:

$$C_{m-1,n}(x) = \left(m + x\frac{d}{dx}\right) C_{m,n}(x), \qquad (36)$$
$$C_{m,n-1}(x) = \left(n + x\frac{d}{dx}\right) C_{m,n}(x).$$

These last relations suggest to introduce similar operators acting on these generalized Tricomi function as well as we have done for the Humbert polynomials. We have indeed:

$$\hat{E}_m^- = \hat{m} + x\frac{d}{dx},$$
$$\hat{E}_n^- = \hat{n} + x\frac{d}{dx}, \qquad (37)$$
$$\hat{E}_{m,n}^{+,+} = \frac{d}{dx}.$$

We have used, again, the same notation as expressed for the operators in equation (29). By following the same procedure used for the Humbert polynomials, we can easily to state the following differential equation:

$$x^2 y''' - (m+n+3)xy'' + (mn+m+n+1)y' = y\,. \qquad (38)$$

3 Further generalizations for Humbert polynomials and functions and incomplete Laguerre polynomials

In the paper [10], we have showed some relations linked the cylindrical Bessel function and the Tricomi function; in particular, we have seen that:

$$x^{-\frac{n}{2}} J_0(2\sqrt{x}) = C_n(x) = \sum_{r=0}^{+\infty} \frac{(-1)^r x^r}{r!(n+r)!} \ . \tag{39}$$

The relations stated in the previous sections and the structure of the 0^{th} order Tricomi function, allow us to introduce a generalization of the Laguerre polynomials. We will say incomplete 2-dimensional Laguerre polynomials, the polynomials defined by the following generating function:

$$\exp(u+v) C_0(xuv) = \sum_{m=0}^{+\infty} \sum_{n=0}^{+\infty} \frac{u^m}{m!} \frac{v^n}{n!} l_{m,n}(x) \,, \tag{40}$$

where their explicit form reads:

$$l_{m,n}(x) = m! n! \sum_{r=0}^{[m,n]} \frac{(-1)^r x^r}{(r!)^2 (m-r)!(n-r)!} . \tag{41}$$

It is evident the similar structure with the Humbert polynomials discussed in the previous sections.
We remind that the ordinary Laguerre polynomials [10] have the following operational expression:

$$L_n(x) = \left(1 - \hat{D}^{-1}\right)^n , \tag{42}$$

where $\hat{D}_x^{-1}$ denotes the inverse of the derivative operator [11-16], being essentially an integral operator, it will be specified by the operational rule:

$$\hat{D}_x^{-1}(1) = \frac{x^n}{n!} . \tag{43}$$

From the above considerations, we can firstly write the following expression for the 0^{th} Tricomi function:

$$C_0(x) = \sum_{r=0}^{+\infty} \frac{(-1)^r \hat{D}_x^{-r}}{r!} = \exp\left(-\hat{D}_x^{-1}\right) , \tag{44}$$

and then we can state the important equivalence between the Humbert polynomials and the incomplete 2-dimensional Laguerre polynomials, that is:

$$l_{m,n}(x) = g_{m,n}\left(-\hat{D}_x^{-1}\right) . \tag{45}$$

By recalling the explicit forms of the generalized two-variable Laguerre polynomials:

$$L_n(x,y) = \left(y - \hat{D}_x^{-1}\right)^n = n! \sum_{r=0}^{n} \frac{(-1)^r y^{n-r} x^r}{(r!)^2 (n-r)!} \tag{46}$$

and from the particular expression of their generating function, in terms of the Tricomi function:

$$\sum_{n=0}^{+\infty} \frac{t^n}{n!} L_n(x,y) = \exp(yt) C_0(xt) \,, \tag{47}$$

we can finally establish a link between the incomplete 2-dimensional laguerre polynomials and the generalized Laguerre of the form $L_n(x,y)$. We have:

$$l_{m,n}(x) = m! n! \sum_{r=0}^{[m,n]} \frac{g_{m-r,n-r}(-y) L_r(x,y)}{r!(m-r)!(n-r)!} . \tag{48}$$

The considerations and the following results obtained to define the incomplete 2-dimensional polynomials, can be used to introduce a similar generalization for the Humbert functions.
By considering indeed the following generating function:

$$\exp(u+v) C_0\left(\frac{x}{uv}\right) = \sum_{m=0}^{+\infty} \sum_{n=0}^{+\infty} u^m v^n A_{m,n}(x) \,, \tag{49}$$

we easily obtain the explicit form of the function $A_{m,n}(x)$:

$$A_{m,n}(x) = \sum_{r=0}^{+\infty} \frac{(-1)^r x^r}{(r!)^2 (m+r)!(n+r)!} . \tag{50}$$

It is easy to note the analogy between the above expression and the generalized Tricomi function presented in the previous section. We find in fact:

$$A_{m,n}(x) = C_{m,n}\left(-\hat{D}_x^{-1}\right) . \tag{51}$$

In the same way, it is possible obtain an expression of the functions $A_{m,n}(x)$ involving the generalized two-variable Laguerre polynomials. From the relation stated in equation (48) and form the (50), we have:

$$A_{m,n}(x) = \sum_{r=0}^{+\infty} \frac{g_{m+r,n+r}(-y) L_r(x,y)}{r!(m+r)!(n+r)!} . \tag{52}$$

4 Concluding remarks and applications to wave propagation

Before closing the paper, we want just to mention how the concepts and the formalism discussed in the previous sections allows also the generalizations of other simple distribution functions like the Poisson distribution.
By using the Tricomi function of order *m:*

$$C_m(-x)=\sum_{r=0}^{+\infty}\frac{x^r}{r!(m+r)!}, \tag{53}$$

we can indeed define the following two-index distribution:

$$P_n(x;m)=\frac{x^n}{n!(m+n)!C_m(-x)}, \tag{54}$$

where the generating function is given by the relation:

$$\frac{C_m(-xt)}{C_m(-x)}=\sum_{n=0}^{+\infty}\frac{t^n}{n!}P_n(x;m)\,. \tag{55}$$

The evaluation of the associated momenta can be easily simplified with the use of the well known property, satisfied by the Tricomi functions:

$$(-1)^r\left(\frac{d}{dx}\right)^r C_n(x)=C_{n+r}(x)\,. \tag{58}$$

Accordingly, we calculate the following average values:

$$\begin{aligned}\langle n\rangle&=\frac{C_{m+1}(-x)}{C_m(-x)},\\ \langle n^2\rangle&=\frac{C_{m+2}(-x)}{C_m(-x)}+\langle n\rangle.\end{aligned} \tag{59}$$

The higher order moments are also given by similar closed relations.
It is remarkable about this probability distribution that, unlike the Poisson distribution, the variance:

$$\sigma=\sqrt{\bar{n^2}-\bar{n}^2}\text{, where } \bar{a}=\langle a\rangle \tag{60}$$

is smaller than $\sqrt{\bar{n}}$.
This type of distribution can be exploited in quantum mechanics within the context of bunching phenomena and in nonlinear dynamics [17], continuum mechanics [18] and robustness-oriented design [19]. This example show that the use of multi-index polynomials and Bessel-type functions with their associated formalism offers wide possibilities in the applications of pure and applied mathematics [20,21].
One of the possible application is the study of wave propagation in homogeneous medium that is a challenge for both theoretical research and engineering practice [22,23]. With the rapid development in science and technology, wave motion study of the anisotropic medium (atmosphere, ocean, earth-crust, functionally graded materials and cycle grid structure, etc.) becomes much more important. For engineers, physicists, and seismologists, the study of longitudinal or flexural waves always is a great deal of interest. The theoretical studies are helpful in forecasting geophysical parameters at deep depths through signal processing and seismic data analysis. Metallurgists use this for the analysis of rock and material structures through non-destructive testing. The knowledge of seismic waves is helpful in investigating the exploration of oil, underground water, and gas accumulation. In recent years, efforts have been made in using seismic methods to characterize hydrocarbon reservoirs, to monitor reservoir production, and to enhance oil recovery processes. Our globe is a spherical body with finite dimension, and the generated elastic waves must receive the effect of the boundaries. Naturally, this concept leads us to the investigation of boundary waves or surface waves, which are confined to some surface during their propagation. In fact, the study of surface waves in homogenous, heterogeneous, and layered media has not been of central interest to theoretical seismologists until recently.
We consider the flexural wave propagation in a circular cylinder of hexagonal elastic material of inner and outer radii *a* and *b*, respectively. The cylinder was subjected to an axial magnetic field and initial hydrostatic stress. The material of the elastic cylinder is regarded as a perfect conductor and the regions inside and outside are assumed to be a vacuum. The displacement components for the case of plane motions can be written in the cylindrical coordinates (r,θ,t) as:

$$u=u(r,\theta,t)\quad v=v(r,\theta,t),\quad w=0 \tag{61}$$

where u,v and w are the displacement components in the radial, circumferential, and axial directions, respectively. Note that all other quantities involved are only functions of r,θ and t, where t denotes the time. To separate the dilatational and rotational components of strain, it is possible to introduce the displacement potentials Φ and Ψ,

$$u(r,\theta,t)=\frac{\partial\Phi}{\partial r}+\frac{1}{r}\frac{\partial\Psi}{\partial\theta}, \tag{62}$$

$$v(r,\theta,t)=\frac{1}{r}\frac{\partial\Phi}{\partial\theta}-\frac{\partial\Psi}{\partial r}. \tag{63}$$

That satisfy the following equations

$$\nabla^2\Phi=\frac{1}{c_1{}^2}\frac{\partial^2\Phi}{\partial t^2}, \quad \nabla^2\Psi=\frac{1}{c_2{}^2}\frac{\partial^2\Psi}{\partial t^2} \tag{64}$$

where

$$\nabla^2=\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}, \tag{65}$$

$$c_1=\sqrt{\frac{c_{11}-p_o}{\rho}+\alpha^2},\ c_2=\sqrt{\frac{c_{66}-p_o}{\rho}} \tag{66}$$

where c_1 and c_2 are velocity of longitudinal and flexural waves, respectively. The coefficients c_{11} and c_{66} come from the constitutive stress-strain relation of the present anisotropic medium, p_o is the hydrostatic tension or compression (tension when $p_o<0$ and compression when $p_o>0$), ρ is the mass density of the material and where α is related to the magnetic field B_0 as follows,

$$\alpha^2=\frac{\mu_o B_o^2}{4\pi\rho}.$$

Now, consider harmonic solutions for $\Phi=\Phi(r,\theta,t)$ and $\Psi=\Psi(r,\theta,t)$ in the form:

$$\Phi(r,\theta,t)=\varphi(r)\cos(n\theta)\exp(i\omega t), \tag{67}$$

$$\Psi(r,\theta,t)=\psi(r)\sin(n\theta)\exp(i\omega t), \tag{68}$$

where ω is the frequency of the vibrations and $n(n=0,1,2,...)$ is an integer indicating the number of circumferential waves. Substituting equations (67-68) into equation. (64), we obtain the well-known Bessel equations for $\varphi(r)$ and $\psi(r)$:

$$r^2\frac{d^2\varphi}{dr^2}+r\frac{d\varphi}{dr}+\left(r^2\gamma_1^2-n^2\right)\varphi=0, \tag{69}$$

$$r^2\frac{d^2\psi}{dr^2}+r\frac{d\psi}{dr}+\left(r^2\gamma_2^2-n^2\right)\psi=0 \tag{70}$$

where

$$\gamma_1^2=\omega^2/c_1^2, \qquad \gamma_2^2=\omega^2/c_2^2. \tag{71}$$

The general solutions of the equations (69) and (70) may take the following form:

$$\varphi(r)=A_1Z_n(\gamma_1 r)+B_1W_n(\gamma_1 r), \tag{72}$$

$$\psi(r)=A_2Z_n(\gamma_2 r)+B_2W_n(\gamma_2 r) \tag{73}$$

where A_1,A_2,B_1 and B_2 are constants of integration and for brevity Z_n denotes the Bessel function J_n or I_n and W_n denotes the Bessel function Y_n or K_n, all of them are of order n, according to the signs of γ_1^2,γ_2^2. We remind, for instance, that one-variable, cylindrical, first type Bessel function is defined by the following relation, involved its generating function:

$$\exp\left[\frac{x}{2}\left(t-\frac{1}{t}\right)\right]=\sum_{m=-\infty}^{+\infty}t^m J_m(x) \tag{74}$$

and, more in general, the two-variable one-parameter cylinder generalized Bessel function (GBF) is represented by the relation [8]:

$$\exp\left[\frac{x}{2}\left(t-\frac{1}{t}\right)+\frac{y}{2}\left(t^2\tau-\frac{1}{t^2\tau}\right)\right]=\sum_{n=-\infty}^{+\infty}t^n J_n(x,y;\tau) \tag{75}$$

Where $x,y\in\mathbb{R}$ and $t,\tau\in\mathbb{R}$, such that $0<|t|\neq|\tau|<+\infty$. In fact, It is immediately recognized that for $y=0$, the function in the previous relation, reduces to the well-known generating function of the one-variable cylinder Bessel function $J_n(x)$ showed in equation (74) .

In a further paper, we will discuss about the applications of the previous results related to the Laguerre polynomials [24] and in some special cases of Chebyshev polynomials [25].

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