Robust guaranteed cost control of discrete-time system with mixed time-varying delays and nonlinear perturbations

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Abstract: This paper is concerned with the problem of robust static output feedback guaranteed cost control for a class of nonlinear discrete systems with mixed time-varying delays. The proposed controller uses the output measurement to robustly exponentially stabilize the closed-loop system and guarantee an adequate level of system performance. By constructing a set of improved Lyapunov-Krasovskii functionals, novel criteria for the existence of robust output feedback guaranteed cost controller are established in terms of linear matrix inequality (LMI). Two numerical examples are given to illustrate the effectiveness of the proposed methods.

Key–Words: Guaranteed cost control, Mixed time-varying delays, Discrete-time system, Nonlinear perturbations, Lyapunov-Krasovskii functions, Linear matrix inequality

1 Introduction

Over the past few decades, time-delays have been greatly considered in dynamical systems. Time-delay often appears in many areas such as chemical systems, air-craft stabilization, communication systems, population dynamic models, automatic control systems, neural networks, metallurgical processing systems and so on. It is well known that the existence of time-delays in many cases lead to poor system performance, and even cause system instability. Therefore, stability analysis for time-delay systems have been investigated by many researchers over the past years [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12]. [1, 2, 3, 4, 5, 6] deals with constant delays, and [7, 8, 9, 10, 11, 12] studies the problem of time-varying delays, or mixed time-varying delays, which being more complicated.

On the other hand, the problem of designing controllers for time-delay systems has drawn considerable attention especially designing robust controllers. A great deal of effort has been directed towards finding a controller in order to guarantee robust stability [13, 14, 15, 16, 17]. However, in many practical systems, it is desirable to design control systems which are not only asymptotically or exponentially stable but can also guarantee an adequate level of system performance. One method of dealing with this problem is the guaranteed cost control. Moreover, it has the advantage of providing an upper bound on a given system performance index and thus the system performance degradation incurred by the uncertainties or nonlinearities is guaranteed to be not more than this bound. Bases on this idea, a lot of significant results have been addressed for continuous-time systems in [18, 19, 20], and for discrete-time systems in [21, 22]. When the time-delay is time-varying delays [23] or interval time-varying delays [24, 25], even is mixed time-varying delays [26], the situation turn to more complicated.

In the past studies for guaranteed cost control, almost most of the articles considered linear systems [16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27]. However, in majority dynamic systems, the nonlinear perturbations appears more and more frequently. Therefore, we not only deal with the time-varying delays, uncertainties, but also deal with the nonlinearities, difficulties then arise when on attempts to derive exponential stabilization conditions in order to find controller parameters. And in the past studies, the delay appears in nonlinear perturbations always keep consistent with delay in state vector, however in actual system, there exists inconsistent phenomenon. Hence in this case, the methods in linear systems [21, 22, 26, 27] can not directed applied to nonlinear systems. This calls for a fresh look at the problem with a improved Lyapunov-Krasovskii functionals and a new set of LMI conditions.

In this paper, we aim to design robust static output feedback guaranteed cost controller for a class of uncertain discrete systems with mixed time-varying delays and nonlinear perturbations, where the state and output are subjected to interval time-varying delays. By constructing a set of improved Lyapunov-Krasovskii functionals, novel criteria for the existence of robust static output feedback guaranteed cost controllers are established in terms of linear matrix inequality (LMI). Two numerical examples are provided to show the effectiveness of the proposed method.

Notations: Throughout this paper, R^n denotes the n-dimensional Euclidian space, $R^{n\times r}$ is the set of $n \times r$ real matrices, I represents the identity matrix. A^Tdenotes the transpose of A, a matrix A is symmetric if $A = A^T$, $\lambda_{max}(A)$ (respectively, $\lambda_{min}(A)$) denotes the maximum (respectively,minimum) value of the real parts of eigenvalues of A. The symmetric terms in a matrix are denoted by $*$. $X > 0$ (respectively $X \geq 0$), for $X \in R^{n\times n}$ means that the matrix is real symmetric positive definite (respectively,positive semi definite). N^+ denotes the set of all real non-negative integers, the scalar product of two vectors x, y is denoted by $x^T y$ and x_k denotes the segment of the solution $x(k)$ on $[-\tau, -\tau + 1, \ldots, 0]$ with the norm $\|x_k\| = \max\{\|x(k-\tau)\|, \ldots, \|x(k\|\}$. Matrices, if not explicitly specified, are assumed to have compatible dimensions.

2 Preliminaries

Consider the following uncertain discrete-time system with mixed time-varying delays and nonlinear perturbations:

$$\begin{cases} x(k+1) = [A + \Delta A]x(k) + [A_1 + \Delta A_1]x(k \\ \quad - d(k)) + [B + \Delta B]u(k) + [F + \Delta F]f(x(k)) \\ \quad + [G + \Delta G]g(x(k - h(k))), \\ y(k) = Cx(k) + C_1x(k - d(k)) \\ \quad + C_2x(k - h(k)), \;\; k \in N^+, \\ x(k) = \varphi_k, \;\; k = -\sigma, -\sigma + 1, \ldots, 0, \end{cases} \tag{1}$$

where $x(k) \in R^n$ is the state vector, $y(k) \in R^r$ is the observation output, $u(k) \in R^m$ is the control input. A, A_1, B, F, G, C, C_1, C_2 are given constant matrices with appropriate dimensions. ΔA, ΔA_1, ΔB, ΔF, ΔG are the time-varying parameter uncertainties that are assumed to satisfy the following admissible condition:

$$[\Delta A \Delta A_1 \Delta B \Delta F \Delta G] = MH(k)[N_1 N_2 N_3 N_4 N_5], \tag{2}$$

$$H^T(k)H(k) \leqslant I, \;\; \forall k \in N^+, \tag{3}$$

where M, N_1, N_2, N_3, N_4, N_5 are constant matrices with appropriate dimensions. The positive integers $d(k)$ and $h(k)$ are time-varying delays satisfying:

$$0 < d_1 \leq d(k) \leq d_2, \tag{4}$$

$$0 < h_1 \leq h(k) \leq h_2, \tag{5}$$

where d_1, d_2, h_1, h_2 are known positive integers. $f(x(k)) \in R^n$, and $g(x(k - h(k))) \in R^n$ are unknown nonlinear perturbations with respect to $x(k)$ and $x(k - h(k))$, respectively, assumed as

$$f^T(x(k))f(x(k)) \leqslant \beta_1^2 x^T(k)x(k), \tag{6}$$

$$\begin{aligned} &g^T(x(k-h(k)))g(x(k-h(k))) \\ &\leqslant \beta_2^2 x^T(k-h(k))x(k-h(k)), \end{aligned} \tag{7}$$

where β_1, β_2 are known positive integers.

The initial condition

$$\varphi = (\varphi_{-\sigma}, \varphi_{-\sigma+1}, \ldots, \varphi_0) \in R^{(\sigma+1)n}$$

with the norm

$$\|\varphi\| = \max\{\|x(-\sigma)\|, \ldots, \|x(0)\|\}, \tag{8}$$

where $\sigma = \max\{h_2, d_2\}$.

Associated with the system (1) is the following cost function:

$$\begin{aligned} J = \sum_{k=0}^{\infty}&(x^T(k)Sx(k) + x^T(k - d(k))S_1x(k \\ &- d(k)) + x^T(k - h(k))S_2x(k - h(k)) \\ &+ u^T(k)Ru(k)), \end{aligned} \tag{9}$$

where S, S_1, S_2, R are given symmetric positive definite matrices with appropriate dimensions.

Suppose the system output is available for feedback, the problem under consideration in this paper is to design an output feedback controller $u(k) = Ky(k)$ for system (1) such that for any admissible uncertain matrix $H(k)$, the resulting closed-loop system is robustly exponentially stable with an upper bound for the cost function (9). The corresponding closed-loop system described as follows:

$$\begin{aligned} x(k+1) = &[\tilde{A} + \tilde{B}KC]x(k) + [\tilde{A}_1 + \tilde{B}KC_1] \\ &\times x(k - d(k)) + \tilde{B}KC_2x(k - h(k)) \\ &+ \tilde{F}f(x(k)) + \tilde{G}g(x(k - h(k))), \end{aligned} \tag{10}$$

where $\tilde{A} = A+\Delta A$, $\tilde{A}_1 = A_1+\Delta A_1$, $\tilde{B} = B+\Delta B$, $\tilde{F} = F + \Delta F$, $\tilde{G} = G + \Delta G$.

Before ending this section, we introduce the following definitions and lemmas:

Definition 1 *Given $\alpha > 0$, the closed-loop system (10) is said to be robustly exponentially stable with a decay rate α, if there exists scalars $\mu > 0$ such that for every solution $x(k, \phi)$ of the system the following inequality holds:*

$$\|x(k, \phi)\| \leq \mu e^{-\alpha k}\|\phi\|, \;\; \forall k \in N^+.$$

Definition 2 *For system (1) and cost function (9), if there exist a output feedback control law $u^*(k) = Ky(k)$ and a positive constant J^* such that the closed-loop system (10) is robustly exponentially stable with a decay rate α and the value (9) satisfies $J \leq J^*$, then the value J^* is said to be a guaranteed cost constant, $u^*(k)$ is said to be a robustly output feedback guaranteed cost control law of the system and its corresponding cost function.*

Lemma 3 *For any $x, y \in R^n$ and positive symmetric definite matrix $W \in R^{n\times n}$, we have*

$$\pm 2x^T y \leq x^T W x + y^T W^{-1} y.$$

Lemma 4 *[17] Let A, E, H and F be real matrices of appropriate dimensions and $F^T F \leq I$, then we have:*
(a) For any scalar $\rho > 0$: $EFH + H^T F^T E^T \leq \rho E E^T + \rho^{-1} H^T H$.
(b) For any matrix $P > 0$ and scalar $\epsilon > 0$ satisfying $P - \epsilon E E^T > 0$:

$$\begin{aligned}(A + EFH)^T P^{-1}(A + EFH) \\ \leq A^T (P - \epsilon E E^T)^{-1} A + \epsilon^{-1} H^T H.\end{aligned}$$

Lemma 5 *(Schur Complement[11]) Given constant matrices S_1, S_2, S_3 with appropriate dimensions, where $S_1 = S_1^T$, $S_2 = S_2^T$, then $S_1 + S_3^T S_2^{-1} S_3 < 0$ if and only if*

$$\begin{pmatrix} S_1 & S_3^T \\ S_3 & -S_2 \end{pmatrix} < 0 \ \ or \ \ \begin{pmatrix} -S_2 & S_3 \\ S_3^T & S_1 \end{pmatrix} < 0.$$

3 Main Results

In this section, we will investigate a sufficient condition for the existence of robust output feedback guaranteed cost control for system (1) by the Lyapunov function method. Before introducing main result, the notations of several matrix variables are defined for simplicity.

Let us denote

$$\Pi_1 = \begin{pmatrix} \Theta_{11} & \Theta_{12} & 0 \\ * & \Theta_{22} & \Theta_{23} \\ * & * & -Q_1 \end{pmatrix},$$

$$\Pi_{14} = \begin{pmatrix} \Delta^T & 0 & 0 \end{pmatrix}^T,$$

$$\Pi_2 = diag\{-Q_2, -\varepsilon_1 I, -\varepsilon_2 I\},$$

$$\Pi_3 = -(W_1 + 3R),$$

$$\Pi_{13} = \begin{pmatrix} W_2 C & 0 & 0 \end{pmatrix}^T,$$

$$\Pi_{15} = \begin{pmatrix} 0 & 0 & 0 \\ e^{\alpha} NB & e^{\alpha d_2} NB & e^{\alpha h_2} NB \\ 0 & 0 & 0 \end{pmatrix},$$

$$\Pi_{16} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ NF & NG & NM & NM & NM \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\begin{aligned}
\Pi_5 &= diag\{-(W_1 - \epsilon N_3^T N_3), -(W_1 - \epsilon N_3^T N_3), \\
&\quad - (W_1 - \epsilon N_3^T N_3)\}, \\
\Pi_4 &= diag\{-(d_2 - d_1 + 1)(W_1 + 3R), \\
&\quad - (h_2 - h_1 + 1)(W_1 + 3R)\}, \\
\Pi_6 &= diag\{-(I - \epsilon N_4^T N_4), -(I - \epsilon N_5^T N_5), \\
&\quad - \epsilon/(e^{2\alpha d_2} + e^{2\alpha} + e^{2\alpha h_2} + 2), -0.5 a_1 e^{-\alpha}, \\
&\quad - 0.5 a_2 e^{-\alpha(d_2+1)}\}, \\
\Theta_{11} &= -P + (d_2 - d_1 + 1)(Q_1 + e^{2\alpha d_2} S_1) + (h_2 - h_1 + 1) \\
&\quad [Q_2 + e^{2\alpha h_2} S_2 + \beta_2^2 (e^{2\alpha(1+h_2)} + \varepsilon_2 e^{2\alpha h_2}) I] \\
&\quad + \beta_1^2 (e^{2\alpha} + \varepsilon_1) I + S + 2 a_1 e^{\alpha} N_1^T N_1 \\
&\quad + 2(d_2 - d_1 + 1) e^{\alpha(d_2+1)} a_2 N_2^T N_2, \\
\Theta_{22} &= P - (N + N^T), \ \ \Theta_{12} = 0.5 e^{\alpha}(NA + A^T N^T), \\
\Theta_{23} &= 0.5 e^{\alpha(d_2+1)}(N A_1 + A_1^T N^T), \\
\Delta &= (\Delta_1 \ \ \Delta_2), \ \ \Delta_1 = (d_2 - d_1 + 1) e^{\alpha d_2} C_1^T W_2^T, \\
\Delta_2 &= (h_2 - h_1 + 1) e^{\alpha h_2} C_2^T W_2^T, \\
R_1 &= e^{2\alpha} C_1^T K^T W_1 K C_1 + 3 e^{2\alpha d_2} C_1^T K^T R K C_1 \\
&\quad + e^{2\alpha d_2} S_1 + 2 e^{\alpha(1+d_2)} a_2 N_2^T N_2, \\
R_2 &= \beta_2^2 e^{2\alpha(1+h_2)} I + e^{2\alpha h_2} S_2 + e^{2\alpha} C_2^T K^T W_1 K C_2 \\
&\quad + \varepsilon_2 \beta_2^2 e^{2\alpha h_2} I + 3 e^{2\alpha h_2} C_2^T K^T R K C_2, \\
\lambda_1 &= \lambda_{min}(P), \\
\lambda_2 &= \lambda_{max}(P) + (d_2 + d_2^2)[\lambda_{max}(Q_1) + \lambda_{max}(R_1)] \\
&\quad + (h_2 + h_2^2)[\lambda_{max}(Q_2) + \lambda_{max}(R_2)].
\end{aligned}$$

Theorem 6 *For given scalar $\alpha > 0$, the control $u(k) = Ky(k)$ is a robustly output feedback guaranteed cost controller for nonlinear uncertain system (1), if there exist symmetric positive definite matrices P, Q_i, $i = 1, 2$, W_1, W_2, an arbitrary matrix N and scalars $\epsilon > 0$, $a_1 > 0$, $a_2 > 0$, $\varepsilon_1 \geq 0$, $\varepsilon_2 \geq 0$ such that the following LMI holds:*

$$\begin{pmatrix} \Pi_1 & 0 & \Pi_{13} & \Pi_{14} & \Pi_{15} & \Pi_{16} \\ * & \Pi_2 & 0 & 0 & 0 & 0 \\ * & * & \Pi_3 & 0 & 0 & 0 \\ * & * & * & \Pi_4 & 0 & 0 \\ * & * & * & * & \Pi_5 & 0 \\ * & * & * & * & * & \Pi_6 \end{pmatrix} < 0, \quad (11)$$

and the guaranteed cost value is given by $J^* = \lambda_2\|\phi\|^2$. *Moreover, the controller parameter* K *is designed by* $K = (W_1 + 3R)^{-T}W_2$.

Proof: First introduce the new variable $z(k) = e^{\alpha k}x(k)$. The closed-loop system (10) is turned to

$$\begin{aligned} z(k+1) &= [\tilde{A} + \tilde{B}KC]e^{\alpha}z(k) \\ &\quad +[\tilde{A}_1 + \tilde{B}KC_1]e^{\alpha(d(k)+1)}z(k-d(k)) \\ &\quad +\tilde{F}e^{\alpha(k+1)}\bar{f}(z(k)) \\ &\quad +\tilde{B}KC_2e^{\alpha(h(k)+1)}z(k-h(k)) \\ &\quad +\tilde{G}e^{\alpha(k+1)}\bar{g}(z(k-h(k))), \end{aligned}$$

where $\bar{f}(z(k)) = f(\frac{z(k)}{e^{\alpha k}})$ and

$$\bar{g}(z(k-h(k))) = g(\frac{z(k-h(k))}{e^{\alpha(k-h(k))}}).$$

Associated with (2), the above equality is reduced to

$$\begin{aligned} &z(k+1) = [\bar{A} + \bar{B}(k)K\bar{C}(k)]z(k) + [\bar{A}_1(k) \\ &+ \bar{B}(k)K\bar{C}_1(k)]z(k-d(k)) + \bar{F}(k)\bar{f}(z(k)) \\ &+ \bar{B}(k)K\bar{C}_2(k)z(k-h(k)) \\ &+ \bar{G}(k)\bar{g}(z(k-h(k))), \end{aligned} \tag{12}$$

where

$$\begin{aligned} &\bar{A} = e^{\alpha}[A + MH(k)N_1], \\ &\bar{A}_1(k) = e^{\alpha(d(k)+1))}[A_1 + MH(k)N_2], \\ &\bar{B}(k) = e^{\alpha(k+1)}[B + MH(k)N_3], \\ &\bar{C}(k) = e^{-\alpha k}C, \\ &\bar{C}_1(k) = e^{-\alpha(k+d(k))}C_1, \\ &\bar{C}_2(k) = e^{-\alpha(k+h(k))}C_2, \\ &\bar{F}(k) = e^{\alpha(k+1)}[F + MH(k)N_4], \\ &\bar{G}(k) = e^{\alpha(k+1)}[G + MH(k)N_5]. \end{aligned}$$

The inequalities (6), (7) turn to, respectively,

$$\bar{f}^T(z(k))\bar{f}(z(k)) \le \beta_1^2 e^{-2\alpha k}z^T(k)z(k), \tag{13}$$

$$\begin{aligned} &\bar{g}^T(z(k-h(k)))\bar{g}(z(k-h(k))) \\ &\le \beta_2^2 e^{-2\alpha(k-h(k))}z^T(k-h(k))z(k-h(k)). \end{aligned} \tag{14}$$

For system (12), choose the following Lyapunov-Krasovskii functional candidate

$$V(k) = \sum_{i=1}^{5} V_i(k), \tag{15}$$

where

$$\begin{aligned} V_1(k) &= z^T(k)Pz(k), \\ V_2(k) &= \sum_{i=k-d(k)}^{k-1} z^T(i)Q_1z(i) + \sum_{i=k-h(k)}^{k-1} z^T(i)Q_2z(i), \\ V_3(k) &= \sum_{j=-d_2+2}^{-d_1+1}\sum_{i=k+j-1}^{k-1} z^T(i)Q_1z(i) \\ &\quad + \sum_{j=-h_2+2}^{-h_1+1}\sum_{i=k+j-1}^{k-1} z^T(i)Q_2z(i), \\ V_4(k) &= \sum_{i=k-d(k)}^{k-1} z^T(i)R_1z(i) + \sum_{i=k-h(k)}^{k-1} z^T(i)R_2z(i), \\ V_5(k) &= \sum_{j=-d_2+2}^{-d_1+1}\sum_{i=k+j-1}^{k-1} z^T(i)R_1z(i) \\ &\quad + \sum_{j=-h_2+2}^{-h_1+1}\sum_{i=k+j-1}^{k-1} z^T(i)R_2z(i). \end{aligned}$$

Take the difference of $V_1(k)$ and $V_2(k)$ along the solution of the system yields:

$$\Delta V_1(k) = z^T(k+1)Pz(k+1) - z^T(k)Pz(k),$$

$$\begin{aligned} &\Delta V_2(k) \\ &= \sum_{i=k+1-d(k+1)}^{k} z^T(i)Q_1z(i) \\ &- \sum_{i=k-d(k)}^{k-1} z^T(i)Q_1z(i) \\ &+ \sum_{i=k+1-h(k+1)}^{k} z^T(i)Q_2z(i) \\ &- \sum_{i=k-h(k)}^{k-1} z^T(i)Q_2z(i) \\ &= \sum_{i=k+1-d(k+1)}^{k-d_1} z^T(i)Q_1z(i) + z^T(k)Q_1z(k) \\ &-z^T(k-d(k))Q_1z(k-d(k)) \\ &+ \sum_{i=k+1-d_1}^{k-1} z^T(i)Q_1z(i) \\ &- \sum_{i=k+1-d(k)}^{k-1} z^T(i)Q_1z(i) + z^T(k)Q_2z(k) \\ &+ \sum_{i=k+1-h(k+1)}^{k-h_1} z^T(i)Q_2z(i) \end{aligned}$$

$$\begin{aligned}
&+\sum_{i=k+1-h_1}^{k-1} z^T(i)Q_2z(i) \\
&-z^T(k-h(k))Q_2z(k-h(k)) \\
&-\sum_{i=k+1-h(k)}^{k-1} z^T(i)Q_2z(i) \\
\leq\ & z^T(k)(Q_1+Q_2)z(k) \\
&-z^T(k-d(k))Q_1z(k-d(k)) \\
&-z^T(k-h(k))Q_2z(k-h(k)) \\
&+\sum_{i=k+1-d(k+1)}^{k-d_1} z^T(i)Q_1z(i) \\
&+\sum_{i=k+1-h(k+1)}^{k-h_1} z^T(i)Q_2z(i). \qquad (16)
\end{aligned}$$

The difference of $\Delta V_3(k)$ is given by

$$\begin{aligned}
&\Delta V_3(k) \\
&=\sum_{j=-d_2+2}^{-d_1+1}\{\sum_{i=k+j}^{k-1} z^T(i)Q_1z(i)-\sum_{i=k+j}^{k-1} z^T(i)Q_1z(i) \\
&+z^T(k)Q_1z(k)-z^T(k+j-1)Q_1z(k+j-1)\} \\
&+\sum_{j=-h_2+2}^{-h_1+1}\{\sum_{i=k+j}^{k-1} z^T(i)Q_2z(i)+z^T(k)Q_2z(k) \\
&-\sum_{i=k+j}^{k-1} z^T(i)Q_2z(i)-z^T(k+j-1)Q_2z(k+j-1)\} \\
&=(d_2-d_1)z^T(k)Q_1z(k)-\sum_{j=k+1-d_2}^{k-d_1} z^T(j)Q_1z(j) \\
&+(h_2-h_1)z^T(k)Q_2z(k)-\sum_{j=k+1-h_2}^{k-h_1} z^T(j)Q_2z(j). \qquad (17)
\end{aligned}$$

Since

$$\sum_{i=k+1-d(k+!1)}^{k-d_1} z^T(i)Q_1z(i)-\sum_{i=k+1-d_2}^{k-d_1} z^T(i)Q_1z(i)\leq 0,$$

$$\sum_{i=k+1-h(k+1)}^{k-h_1} z^T(i)Q_2z(i)-\sum_{i=k+1-h_2}^{k-h_1} z^T(i)Q_2z(i)\leq 0,$$

we can get the followings inequality from (16) and (17):

$$\begin{aligned}
&\Delta V_2(k)+\Delta V_3(k)\leq(d_2-d_1+1)z^T(k)Q_1z(k) \\
&\quad -z^T(k-d(k))Q_1z(k-d(k)) \\
&\quad +(h_2-h_1+1)z^T(k)Q_2z(k) \\
&\quad -z^T(k-h(k))Q_2z(k-h(k)). \qquad (18)
\end{aligned}$$

Similarly, we can get

$$\begin{aligned}
&\Delta V_4(k)+\Delta V_5(k)\leq(d_2-d_1+1)z^T(k)R_1z(k) \\
&\quad -z^T(k-d(k)))R_1z(k-d(k)) \\
&\quad +(h_2-h_1+1)z^T(k)R_2z(k) \\
&\quad -z^T(k-h(k))R_2z(k-h(k)). \qquad (19)
\end{aligned}$$

From (14)-(19), it yields

$$\begin{aligned}
&\Delta V(k)\leq z^T(k+1)Pz(k+1)+z^T(k)[(d_2-d_1 \\
&\quad +1)(Q_1+R_1)+(h_2-h_1+1)(Q_2+R_2) \\
&\quad -P]z(k)-z^T(k-d(k))(Q_1+R_1)z(k-d(k)) \\
&\quad -z^T(k-h(k))(Q_2+R_2)z(k-h(k)). \qquad (20)
\end{aligned}$$

Multiplying $2z^T(k+1)N$ both sides of the identity (12), we have

$$\begin{aligned}
&-2z^T(k+1)Nz(k+1)+2z^T(k+1)N(\bar{A} \\
&\quad +\bar{B}(k)K\bar{C}(k))z(k))+2z^T(k+1)N(\bar{A}_1(k) \\
&\quad +\bar{B}(k)K\bar{C}_1(k))z(k-d(k))+2z^T(k+1)N \\
&\quad \times\bar{B}(k)K\bar{C}_2(k))z(k-h(k))+2z^T(k+1)N \\
&\quad \times\bar{F}(k)\bar{f}+2z^T(k+1)N\bar{G}(k)\bar{g}=0. \qquad (21)
\end{aligned}$$

Note that for any $\varepsilon_1\geq 0$, $\varepsilon_2\geq 0$, it follows from (13) to (14) that

$$\varepsilon_1[\beta_1^2e^{-2\alpha k}z^T(k)z(k)-\bar{f}^T(z(k))\bar{f}(z(k))]\geq 0, \qquad (22)$$

$$\begin{aligned}
&\varepsilon_2[\beta_2^2e^{-2\alpha(k-h(k))}z^T(k-h(k))z(k-h(k)) \\
&\quad -\bar{g}^T(z(k-h(k)))\bar{g}(z(k-h(k)))]\geq 0. \qquad (23)
\end{aligned}$$

Adding the above relation (21)-(23) into (20), we can get

$$\begin{aligned}
&\Delta V(k) \\
\leq\ & z^T(k+1)Pz(k+1)+z^T(k)[(d_2-d_1+1)(Q_1 \\
&+R_1)+(h_2-h_1+1)(Q_2+R_2)-P]z(k) \\
&-z^T(k-d(k))(Q_1+R_1)z(k-d(k)) \\
&-z^T(k-h(k))(Q_2+R_2)z(k-h(k)) \\
&-2z^T(k+1)Nz(k+1) \\
&+2z^T(k+1)N(\bar{A}+\bar{B}(k)K\bar{C}(k))z(k) \\
&+2z^T(k+1)N(\bar{A}_1(k)+\bar{B}(k)K\bar{C}_1(k))z(k-d(k)) \\
&+2z^T(k+1)N\bar{B}(k)K\bar{C}_2(k))z(k-h(k)) \\
&+2z^T(k+1)N\bar{F}(k)\bar{f}(z(k)) \\
&+2z^T(k+1)N\bar{G}(k)\bar{g}(z(k-h(k))) \\
&+\varepsilon_1[\beta_1^2e^{-2\alpha k}z^T(k)z(k) \\
&-\bar{f}^T(z(k))\bar{f}(z(k))] \\
&+\varepsilon_2[\beta_2^2e^{-2\alpha(k-h(k))}z^T(k-h(k))z(k-h(k)) \\
&-\bar{g}^T(z(k-h(k)))\bar{g}(z(k-h(k)))]. \qquad (24)
\end{aligned}$$

Dealing with partial idem in (24), we have

$$\begin{aligned}&2z^T(k+1)N(\bar{A}+\bar{B}(k)K\bar{C}(k))z(k)\\&=2z^T(k+1)N[e^{\alpha}(A+MH(k)N_1)\\&+e^{\alpha}(B+MH(k)N_3)KC]z(k)\\&=z^T(k+1)e^{\alpha}[NA+A^TN^T+NMH(k)N_1\\&+N_1^TH(k)^TM^TN^T]z(k)\\&+2e^{\alpha}z^T(k+1)N[B+MH(k)N_3]KCz(k).\end{aligned} \tag{25}$$

Applying Lemma 3 and Lemma 4, we get

$$\begin{aligned}&2z^T(k+1)N(\bar{A}+\bar{B}(k)K\bar{C}(k))z(k)\\&\leq z^T(k+1)e^{\alpha}(NA+A^TN^T)z(k)\\&\quad+2e^{\alpha}z^T(k+1)NMH(k)N_1z(k)\\&\quad+z^T(k)C^TK^TW_1KCz(k)\\&\quad+z^T(k+1)e^{2\alpha}N[B+MH(k)N_3]\times\\&\quad W_1^{-1}[B+MH(k)N_3]^TN^Tz(k+1)\\&\leq z^T(k+1)e^{\alpha}(NA+A^TN^T)z(k)\\&\quad+2\rho_1e^{\alpha}z^T(k+1)NMM^TN^Tz(k+1)\\&\quad+2\rho_1^{-1}e^{\alpha}z^T(k)N_1^TN_1z(k)\\&\quad+z^T(k)C^TK^TW_1KCz(k)\\&\quad+z^T(k+1)e^{2\alpha}N[B(W_1-\epsilon N_3^TN_3)^{-1}B^T\\&\quad+\epsilon^{-1}MM^T]N^Tz(k+1).\end{aligned} \tag{26}$$

Similarly, applying Lemma 3 and Lemma 4, we have

$$\begin{aligned}&2z^T(k+1)N(\bar{A}_1(k)+\bar{B}(k)K\bar{C}_1(k))z(k-d(k))\\&=2z^T(k+1)Ne^{\alpha(d(k)+1)}[A_1+MH(k)N_2\\&\quad+(B+MH(k)N_3)KC_1]z(k-d(k))\\&\leq z^T(k+1)e^{\alpha(d_2+1)}[NA_1+A_1^TN^T)z(k-d(k))\\&\quad+2e^{\alpha(d_2+1)}z^T(k+1)NMH(k)N_2z(k-d(k))\\&\quad+2z^T(k+1)e^{\alpha d(k)}N[B+MH(k)N_3]\\&\quad\times e^{\alpha}KC_1z(k-d(k))\\&\leq z^T(k+1)e^{\alpha(d_2+1)}(NA_1+A_1^TN^T)z(k-d(k))\\&\quad+2\rho_2e^{\alpha(d_2+1)}z^T(k+1)NMM^TN^Tz(k+1)\\&\quad+2\rho_2^{-1}e^{\alpha(d_2+1)}z^T(k-d(k))N_2^TN_2z(k-d(k))\\&\quad+z^T(k+1)e^{2\alpha d_2}N[B+MH(k)N_3]W_1^{-1}[B\\&\quad+MH(k)N_3]^TN^Tz(k+1)\\&\quad+e^{2\alpha}z^T(k-d(k))C_1^TK^TW_1KC_1z(k-d(k))\\&\leq z^T(k+1)e^{\alpha(d_2+1)}(NA_1+A_1^TN^T)z(k-d(k))\\&\quad+2e^{\alpha(d_2+1)}[\rho_2z^T(k+1)NMM^TN^Tz(k+1)\\&\quad+\rho_2^{-1}z^T(k-d(k))N_2^TN_2z(k-d(k))]\\&\quad+z^T(k+1)e^{2\alpha d_2}N[B(W_1-\epsilon N_3^TN_3)^{-1}B^T\\&\quad+\epsilon^{-1}MM^T]N^Tz(k+1)+e^{2\alpha}z^T(k\\&\quad-d(k))C_1^TK^TW_1KC_1z(k-d(k)),\end{aligned} \tag{27}$$

and

$$\begin{aligned}&2z^T(k+1)N\bar{B}(k)K\bar{C}_2(k))z(k-h(k))\\&=2z^T(k+1)Ne^{\alpha(h(k)+1)}[B\\&\quad+MH(k)N_3]KC_2z(k-h(k))\\&\leq e^{2\alpha h_2}z^T(k+1)N[B+MH(k)N_3]W_1^{-1}[B\\&\quad+MH(k)N_3]^TN^Tz(k+1)+e^{2\alpha}z^T(k\\&\quad-h(k))C_2^TK^TW_1KC_2z(k-h(k))\\&\leq z^T(k+1)e^{2\alpha h_2}N[B(W_1-\epsilon N_3^TN_3)^{-1}B^T\\&\quad+\epsilon^{-1}MM^T]N^Tz(k+1)+e^{2\alpha}z^T(k\\&\quad-h(k))C_2^TK^TW_1KC_2z(k-h(k)).\end{aligned} \tag{28}$$

We have

$$\begin{aligned}&2z^T(k+1)N\bar{F}(k)\bar{f}(z(k))\\&=2z^T(k+1)N(F+MH(k)N_4)e^{\alpha(k+1)}\bar{f}(z(k))\\&\leq z^T(k+1)N(F+MH(k)N_4)(F\\&\quad+MH(k)N_4)^TN^Tz(k+1)\\&\quad+e^{2\alpha(k+1)}\bar{f}^T(z(k))\bar{f}(z(k))\\&\leq z^T(k+1)N[F(I-\epsilon N_4^TN_4)^{-1}F^T\\&\quad+\epsilon^{-1}MM^T]N^Tz(k+1)+\beta_1^2e^{2\alpha}z^T(k)z(k),\\&2z^T(k+1)N\bar{G}(k)\bar{g}(z(k-h(k)))\\&=2z^T(k+1)N(G+MH(k)N_5)e^{\alpha(k+1)}\bar{g}(z(k\\&\quad-h(k)))\\&\leq z^T(k+1)N(G+MH(k)N_5)(G\\&\quad+MH(k)N_5)^TN^Tz(k+1)\\&\quad+e^{2\alpha(k+1)}\bar{g}^T(z(k-h(k)))\bar{g}(z(k-h(k)))\\&\leq z^T(k+1)N[G(I-\epsilon N_5^TN_5)^{-1}G^T\\&\quad+\epsilon^{-1}MM^T]N^Tz(k+1)\\&\quad+\beta_2^2e^{2\alpha(1+h_M)}z^T(k-h(k))z(k-h(k)).\end{aligned} \tag{29}$$

At last, adding the following relation

$$\begin{aligned}&[z^T(k)\bar{S}z(k)+z^T(k-d(k))\bar{S}_1z(k-d(k))\\&+z^T(k-h(k))\bar{S}_2z(k-h(k))+u^T(k)Ru(k)]\\&-[z^T(k)\bar{S}z(k)+z^T(k-d(k))\bar{S}_1z(k-d(k))\\&+z^T(k-h(k))\bar{S}_2z(k-h(k))+u^T(k)Ru(k)]\\&=0,\end{aligned} \tag{30}$$

with $\bar{S}=e^{-2\alpha k}S$, $\bar{S}_1=e^{-2\alpha(k-d(k))}S_1$, $\bar{S}_2=$

$e^{-2\alpha(k-h(k))}S_2$, into (24), and using

$$\begin{aligned}&u^T(k)Ru(k) \leq 3z^T(k)C^TK^TRKCz(k)\\&+3e^{2\alpha d_2}z^T(k-d(k))C_1^TK^TRKC_1z(k-d(k))\\&+3e^{2\alpha h_2}z^T(k-h(k))C_2^TK^TRKC_2z(k-h(k)),\end{aligned}$$

we can get

$$\begin{aligned}\Delta V(k) \leq\ &\xi^T(k)\Omega\xi(k) - [z^T(k)\bar{S}z(k) + z^T(k\\&-d(k))\bar{S}_1z(k-d(k)) + z^T(k-h(k))\bar{S}_2z(k\\&-h(k)) + u^T(k)Ru(k)],\end{aligned} \tag{31}$$

where $\xi(k) = [z(k), z(k+1), z(k-d(k)), z(k-h(k))\ \ \bar{f}(z(k))\ \ \bar{g}(z(k-h(k)))]^T$, and

$$\Omega = \begin{pmatrix} \Omega_{11} & \Theta_{12} & 0 & 0 & 0 & 0\\ * & \Omega_{22} & \Theta_{23} & 0 & 0 & 0\\ * & * & -Q_1 & 0 & 0 & 0\\ * & * & * & -Q_2 & 0 & 0\\ * & * & * & * & -\varepsilon_1 I & 0\\ * & * & * & * & * & -\varepsilon_2 I\end{pmatrix},$$

$$\begin{aligned}\Omega_{11} =\ &\Theta_{11} + C_1^TK^T[(d_2-d_1+1)e^{2\alpha d_2}(W_1+3R)\\&+(h_2-h_1+1)e^{2\alpha}W_1]KC_1\\&+C^TK^T(W_1+3R)KC\\&+3(h_2-h_1+1)e^{2\alpha h_2}C_2^TK^TRKC_2\\&+(h_2-h_1+1)e^{2\alpha h_2}C_2^TK^TW_1KC_2,\\ \Omega_{22} =\ &\Theta_{22} + e^{2\alpha}NB(W_1-\epsilon N_3^TN_3)^{-1}B^TN^T\\&+e^{2\alpha d_2}NB(W_1-\epsilon N_3^TN_3)^{-1}B^TN^T\\&+e^{2\alpha h_2}NB(W_1-\epsilon N_3^TN_3)^{-1}B^TN^T\\&+NF(I-\epsilon N_4^TN_4)^{-1}F^TN^T\\&+NG(I-\epsilon N_5^TN_5)^{-1}G^TN^T\\&+\epsilon^{-1}(e^{2\alpha d_2}+e^{2\alpha}+e^{2\alpha h_2}+2)NMM^TN^T\\&+2\rho_1e^{\alpha}NMM^TN^T+2\rho_2e^{\alpha(d_2+1)}NMM^TN^T.\end{aligned}$$

Define $a_1 = \rho_1^{-1}$, $a_2 = \rho_2^{-1}$ and inserting the congruent transformation

$$T = diag\{I, I, \Gamma_1, \Gamma_2, I, I\},$$

where

$$\Gamma_1 = (W_1+3R)^{-1},$$

and

$$\Gamma_2 = (W_1+3R)^{-1}diag\{I, I\},$$

into the LMI (11), we can get a matrix inequality. By Lemma 5 (Schur complement lemma), the condition $\Omega < 0$ is equivalent to the above matrix inequality. Therefore, from (31) it follows that

$$\Delta V(k) < 0,$$

which implies that

$$V(k) \leq V(0), \quad \forall \in N^+. \tag{32}$$

We can easily get that

$$\lambda_1\|z(k)\|^2 \leq V(k) \leq \lambda_2\|z_k\|^2, \tag{33}$$

where $\|z_k\| = \max\{\|z(k-\sigma)\|, \ldots, \|z(k)\|\}$.

From (32) and (33), we can get

$$\lambda_1\|z(k)\|^2 \leq \lambda_2\|\bar{\varphi}\|^2,$$

therefore,

$$\|z(k)\| \leq \sqrt{\frac{\lambda_2}{\lambda_1}}\|\bar{\varphi}\|.$$

Using the relation $z(k) = e^{\alpha k}x(k)$, we can get

$$\|x(k)\| \leq \sqrt{\frac{\lambda_2}{\lambda_1}}e^{-\alpha k}\|\varphi\|, \ \ \forall k \in N^+.$$

Therefore, the closed-loop system (10) is exponentially stable.

Next we will find the guaranteed cost value. From (31), we can get

$$\begin{aligned}&\Delta V(k) \leq\\&-[x^T(k)Sx(k) + x^T(k-d(k))S_1x(k-d(k))\\&\quad + x^T(k-h(k))S_2x(k-h(k)) + u^T(k)Ru(k)],\end{aligned}$$

therefore,

$$\begin{aligned}&[x^T(k)Sx(k) + x^T(k-d(k))S_1x(k-d(k))\\&+x^T(k-h(k))S_2x(k-h(k)) + u^T(k)Ru(k)]\\&\leq V(k) - V(k+1).\end{aligned} \tag{34}$$

Summing up both sides of (34) from 0 to $n-1$, we can get

$$\begin{aligned}&\sum_{k=0}^{n-1}[x^T(k)Sx(k) + x^T(k-d(k))S_1x(k-d(k))\\&+x^T(k-h(k))S_2x(k-h(k)) + u^T(k)Ru(k)]\\&\leq V(0) - V(n).\end{aligned}$$

Let $n \to +\infty$, noting that $V(n) \to 0$, we can get

$$\begin{aligned}&\sum_{k=0}^{\infty}[x^T(k)Sx(k) + x^T(k-d(k))S_1x(k-d(k))\\&+x^T(k-h(k))S_2x(k-h(k)) + u^T(k)Ru(k)]\\&\leq V(0),\end{aligned}$$

that is, $J \leq V(0)$. Associated with (33), we have $J \leq \lambda_2\|\varphi\|^2 = J^*$. □

Remark 7 *When the delay in nonlinear perturbations keeps consistent with the delay in the state variable, the system (1) is turned to:*

$$\begin{cases} x(k+1) = \\ [A+\Delta A]x(k) + [A_1 + \Delta A_1]x(k-d(k)) \\ \ + [B+\Delta B]u(k) + [F+\Delta F]f(x(k)) \\ \ + [G+\Delta G]g(x(k-d(k))), \\ y(k) = Cx(k) + (C_1+C_2)x(k-d(k)), \ \ k \in N^+, \\ x(k) = \varphi_k, \ \ k = -d_2, -d_2+1, \dots, 0, \end{cases} \tag{35}$$

and the closed-loop system (10) and the cost function (9) are reduced to

$$\begin{aligned} x(k+1) &= [\tilde{A} + \tilde{B}KC]x(k) + [\tilde{A}_1 + \tilde{B}K(C_1 \\ &\quad + C_2)]x(k-d(k)) + \tilde{F}f(x(k)) \\ &\quad + \tilde{G}g(x(k-d(k))), \\ J &= \sum_{k=0}^{\infty}(x^T(k)Sx(k) + x^T(k-d(k))(S_1 \\ &\quad + S_2)x(k-d(k)) + u^T(k)Ru(k)). \end{aligned}$$

Thus we can give a sufficient condition for the existence of robust output feedback guaranteed cost control for system (35) based on Theorem 1. The notations of several matrix variables are also defined for simplicity. Let us denote

$$\begin{aligned} R_1 &= e^{2\alpha}(C_1+C_2)^T K^T W_1 K(C_1+C_2) \\ &\quad + e^{2\alpha d_2}(S_1+S_2) \\ &\quad + 2e^{2\alpha d_2}(C_1+C_2)^T K^T RK(C_1+C_2) \\ &\quad + 2e^{\alpha(1+d_2)}\rho_2^{-1}N_2^T N_2, \\ \lambda_1 &= \lambda_{min}(P), \\ \lambda_2 &= \lambda_{max}(P) + (d_2+d_2^2)[\lambda_{max}(Q_1) + \lambda_{max}(R_1)]. \end{aligned}$$

Corollary 8 *For given scalars $\alpha > 0$, the control $u(k) = Ky(k)$ is a robustly static output feedback guaranteed cost controller for nonlinear uncertain system (35), if there exist symmetric positive definite matrices P, Q_1, W_1, W_2, an arbitrary matrix N and scalars $\epsilon > 0$, $a_1 > 0$, $a_2 > 0$, $\varepsilon_1 \geq 0$, $\varepsilon_2 \geq 0$ such that the following LMI holds:*

$$\begin{pmatrix} \Pi_1 & 0 & \Pi_{13} & \Pi_{14} & \Pi_{15} & \Pi_{16} \\ * & \Pi_2 & 0 & 0 & 0 & 0 \\ * & * & \Pi_3 & 0 & 0 & 0 \\ * & * & * & \Pi_4 & 0 & 0 \\ * & * & * & * & \Pi_5 & 0 \\ * & * & * & * & * & \Pi_6 \end{pmatrix} < 0, \tag{36}$$

where

$$\begin{aligned} \Pi_1 &= \begin{pmatrix} \Theta_{11} & \Theta_{12} & 0 \\ * & \Theta_{22} & \Theta_{23} \\ * & * & -Q_1 \end{pmatrix}, \\ \Pi_2 &= diag\{-\varepsilon_1 I, -\varepsilon_2 I\}, \\ \Pi_{13} &= \begin{pmatrix} W_2 C & 0 & 0 \end{pmatrix}^T, \ \Pi_{14} = \begin{pmatrix} \Delta^T & 0 & 0 \end{pmatrix}^T, \\ \Pi_{15} &= \begin{pmatrix} 0 & 0 \\ e^{\alpha}NB & e^{\alpha d_M}NB \\ 0 & 0 \end{pmatrix}, \\ \Pi_3 &= -(W_1+2R), \end{aligned}$$

$$\begin{aligned} \Pi_{16} &= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ NF & NG & NM & NM & NM \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \\ \Pi_4 &= -(d_2-d_1+1)(W_1+2R), \\ \Pi_5 &= diag\{-(W_1 - \epsilon N_3^T N_3), -(W_1 - \epsilon N_3^T N_3)\}, \\ \Pi_6 &= diag\left(-(I-\epsilon N_4^T N_4), -(I - \epsilon N_5^T N_5), \right. \\ &\quad -2\epsilon/(e^{2\alpha d_2} + e^{2\alpha} + 2), -0.5a_1 e^{-\alpha}, \\ &\quad \left. -0.5a_2 e^{-\alpha(d_2+1)}\right), \\ \Theta_{11} &= -P + (d_2-d_1+1)[Q_1 + e^{2\alpha d_2}(S_1+S_2)] \\ &\quad + \beta_1^2(e^{2\alpha}+\varepsilon_1)I + S + 2a_1 e^{\alpha}N_1^T N_1 \\ &\quad + 2(d_2-d_1+1)e^{\alpha(d_2+1)}a_2 N_2^T N_2, \\ \Theta_{22} &= P - (N+N^T), \\ \Theta_{12} &= 0.5e^{\alpha}(NA + A^T N^T), \\ \Theta_{23} &= 0.5e^{\alpha(d_2+1)}(NA_1 + A_1^T N^T), \\ \Delta &= (d_2-d_1+1)e^{\alpha d_2}(C_1+C_2)^T W_2^T, \end{aligned}$$

and the guaranteed cost value is given by $J^* = \lambda_2\|\phi\|^2$. Moreover, the controller parameter K is designed by $K = (W_1+2R)^{-T}W_2$.

Proof: The corresponding proof is similar to that in Theorem 1, which are omitted. □

Remark 9 *Without uncertainty idems, the system (1) is reduced to*

$$\begin{cases} x(k+1) = Ax(k) + A_1x(k-d(k)) + Bu(k) \\ \qquad\qquad + Ff(x(k)) + Gg(x(k-h(k))), \\ y(k) = Cx(k) + C_1x(k-d(k)) \\ \qquad\qquad + C_2x(k-h(k)), \ \ k \in N^+, \\ x(k) = \varphi_k, \ \ k = -\sigma, -\sigma+1, \dots, 0, \end{cases} \tag{37}$$

and the closed-loop system (10) is reduced to

$$\begin{aligned}x(k+1) =&(A+BKC)x(k)\\&+(A_1+BKC_1)x(k-d(k))\\&+BKC_2x(k-h(k))+Ff(x(k))\\&+Gg(x(k-h(k))).\end{aligned}$$

Next we give a sufficient condition for the existence of robust output feedback guaranteed cost control for system (36) also based on Theorem 1. For simplicity, let us denote

$$\begin{aligned}R_1 =& e^{2\alpha}C_1^TK^TW_1KC_1+e^{2\alpha d_2}S_1\\&+3e^{2\alpha d_2}C_1^TK^TRKC_1,\\R_2 =& e^{2\alpha}C_2^TK^TW_1KC_2+\beta_2^2e^{2\alpha(1+h_2)}I+e^{2\alpha h_2}S_2\\&+\varepsilon_2\beta_2^2e^{2\alpha h_2}I+3e^{2\alpha h_2}C_2^TK^TRKC_2,\\\lambda_1 =& \lambda_{min}(P),\\\lambda_2 =& \lambda_{max}(P)+(d_2+d_2^2)[\lambda_{max}(Q_1)+\lambda_{max}(R_1)]\\&+(h_2+h_2^2)[\lambda_{max}(Q_2)+\lambda_{max}(R_2)].\end{aligned}$$

Corollary 10 *For given scalars $\alpha > 0$, the control $u(k) = Ky(k)$ is a static output feedback guaranteed cost controller for nonlinear system (36), if there exist symmetric positive definite matrices P, $Q_i, i = 1, 2$, W_1, W_2, an arbitrary matrix N and scalars $\epsilon > 0$, $\varepsilon_1 \geq 0$, $\varepsilon_2 \geq 0$ such that the following LMI holds:*

$$\begin{pmatrix}\Pi_1 & 0 & \Pi_{13} & \Pi_{14} & \Pi_{15} & \Pi_{16}\\ * & \Pi_2 & 0 & 0 & 0 & 0\\ * & * & \Pi_3 & 0 & 0 & 0\\ * & * & * & \Pi_4 & 0 & 0\\ * & * & * & * & \Pi_5 & 0\\ * & * & * & * & * & \Pi_6\end{pmatrix} < 0, \quad (38)$$

where

$$\begin{aligned}\Pi_1 &= \begin{pmatrix}\Theta_{11} & \Theta_{12} & 0\\ * & \Theta_{22} & \Theta_{23}\\ * & * & -Q_1\end{pmatrix},\\\Pi_{16} &= \begin{pmatrix}0 & 0\\ NF & NG\\ 0 & 0\end{pmatrix},\\\Pi_{15} &= \begin{pmatrix}0 & 0 & 0\\ e^{\alpha}NB & e^{\alpha d_2}NB & e^{\alpha h_2}NB\\ 0 & 0 & 0\end{pmatrix},\\\Pi_{13} &= \begin{pmatrix}C^TW_2^T & 0 & 0\end{pmatrix}^T, \Pi_{14} = \begin{pmatrix}\Delta^T & 0 & 0\end{pmatrix}^T,\\\Pi_2 &= diag\{-Q_2, -\varepsilon_1 I, -\varepsilon_2 I\},\\\Pi_3 &= -(W_1+3R),\\\Pi_4 &= diag\{-(d_2-d_1+1)(W_1+3R),\\&\quad -(h_2-h_1+1)(W_1+3R)\},\\\Pi_5 &= diag\{-W_1, -W_1, -W_1\},\\\Pi_6 &= diag\{-I, -I\},\\\Theta_{11} &= -P+(d_2-d_1+1)(Q_1+e^{2\alpha d_M}S_1)\\&+(h_2-h_1+1)[Q_2+e^{2\alpha h_2}S_2+\beta_2^2(e^{2\alpha(1+h_2)}\\&+\varepsilon_2e^{2\alpha h_2})I]+\beta_1^2(e^{2\alpha}+\varepsilon_1)I+S,\\\Theta_{22} &= P-(N+N^T),\\\Theta_{12} &= 0.5e^{\alpha}(NA+A^TN^T),\\\Theta_{23} &= 0.5e^{\alpha(d_2+1)}(NA_1+A_1^TN^T),\\\Delta &= \begin{pmatrix}\Delta_1 & \Delta_2\end{pmatrix}, \Delta_1 = (d_2-d_1+1)e^{\alpha d_2}C_1^TW_2^T,\\\Delta_2 &= (h_2-h_1+1)e^{\alpha h_2}C_2^TW_2^T,\end{aligned}$$

and the guaranteed cost value is given by $J^* = \lambda_2\|\phi\|^2$. Moreover, the controller parameter K is designed by $K = (W_1+3R)^{-T}W_2$.

Proof: Construct the Lyapunov-Krasovskii functional (15). The corresponding proof is similar to that in Theorem 1, which are omitted. □

4 Numerical examples

In this section, two numerical examples are given to demonstrate the effectiveness of the proposed methods.

Example 11 *Consider the nonlinear uncertain discrete-time system (1) with the following parameters :*

$$A = \begin{bmatrix}-0.18 & 0.1\\ 0.2 & -0.06\end{bmatrix}, \quad A_1 = \begin{bmatrix}-0.02 & 0.02\\ 0.01 & -0.01\end{bmatrix},$$
$$B = \begin{bmatrix}0.01 & 0.03\\ -0.05 & 0.02\end{bmatrix}, \quad F = \begin{bmatrix}0.01 & -0.03\\ 0.05 & 0.02\end{bmatrix},$$
$$G = \begin{bmatrix}-0.01 & 0.04\\ 0.01 & 0.02\end{bmatrix}, \quad C = \begin{bmatrix}-0.06 & -0.3\\ 0.02 & 0.03\end{bmatrix},$$
$$C_1 = \begin{bmatrix}-0.04 & 0.2\\ -0.05 & 0.07\end{bmatrix}, \quad C_2 = \begin{bmatrix}-0.05 & 0.25\\ -0.04 & 0.03\end{bmatrix},$$
$$M = \begin{bmatrix}-0.1 & 0.2\\ 0.1 & 0.2\end{bmatrix}, \quad N_1 = \begin{bmatrix}0.04 & 0.01\\ 0.02 & -0.03\end{bmatrix},$$
$$N_2 = \begin{bmatrix}0.01 & 0.03\\ 0.02 & 0.02\end{bmatrix}, \quad N_3 = \begin{bmatrix}0.02 & -0.02\\ 0.05 & 0.01\end{bmatrix},$$
$$N_4 = \begin{bmatrix}0.06 & -0.01\\ 0.03 & 0.02\end{bmatrix}, \quad N_5 = \begin{bmatrix}0.02 & 0.01\\ 0.03 & 0.03\end{bmatrix},$$
$$S = \begin{bmatrix}0.2 & 0.15\\ 0.15 & 0.3\end{bmatrix}, \quad S_1 = \begin{bmatrix}0.2 & 0.1\\ 0.1 & 0.4\end{bmatrix},$$
$$S_2 = \begin{bmatrix}0.1 & 0.1\\ 0.1 & 0.3\end{bmatrix}, \quad R = \begin{bmatrix}0.1 & 0.2\\ 0.2 & 0.6\end{bmatrix},$$

and for $k = 0, 1, 2, \ldots$

$$\begin{aligned} &f(x(k-d(k))) = \sin(x(k-d(k))), \\ &g(x(k-h(k))) = 2\sin(x(k-h(k))), \\ &d(k) = 2 + \sin^2\frac{k\pi}{2}, h(k) = 2 + \frac{1+(-1)^k}{2}, \end{aligned}$$

Given $\alpha = 0.1$, $d_1 = 2$, $d_2 = 3$, $h_1 = 2$, $h_2 = 3$, $\beta_1 = 0.2$, $\beta_2 = 0.3$. By using the LMI Toolbox in MTALAB [29], the LMI (11) in Theorem 1 is satisfied with

$$P = \begin{bmatrix} 10.2226 & -1.1379 \\ -1.1379 & 11.6238 \end{bmatrix}, Q_1 = \begin{bmatrix} 1.2429 & -0.3419 \\ -0.3419 & 1.1140 \end{bmatrix},$$
$$Q_2 = \begin{bmatrix} 1.2429 & -0.3419 \\ -0.3419 & 1.1140 \end{bmatrix}, W_1 = \begin{bmatrix} 3.7245 & -0.1804 \\ -0.1804 & 3.1272 \end{bmatrix},$$
$$W_2 = \begin{bmatrix} 2.0821 & -0.2124 \\ -0.2124 & 4.3941 \end{bmatrix}, N = \begin{bmatrix} 8.2587 & -0.2481 \\ -0.5235 & 8.8085 \end{bmatrix},$$
$$\epsilon = 14.0832, a_1 = 8.4647, a_2 = 9.6368,$$
$$\varepsilon_1 = 4.5770, \varepsilon_2 = 3.1914.$$

Moreover, according to the controller design

$$K = (W_1 + 3R)^{-1} W_2,$$

we have

$$K = \begin{bmatrix} 0.5265 & -0.1471 \\ -0.0880 & 0.9043 \end{bmatrix}.$$

Therefore,the guaranteed cost controller is as follows:

$$\begin{aligned} u(k) = &\begin{bmatrix} -0.0345 & -0.1624 \\ 0.0234 & 0.0535 \end{bmatrix} x(k) \\ &+ \begin{bmatrix} -0.0137 & 0.0950 \\ -0.0417 & 0.0457 \end{bmatrix} x(k-d(k)) \\ &+ \begin{bmatrix} -0.0204 & 0.1272 \\ -0.0318 & 0.0051 \end{bmatrix} x(k-h(k)). \end{aligned}$$

Moreover, the solution of the closed-loop system satisfies $\|x(k)\| \le 2.8261e^{-0.1k}\|\phi\|$, and the guaranteed cost control value of the closed-loop system is $J^* = 76.5695\|\phi\|^2$. Given the initial condition of system (11) as follows:

$$x(-3) = \begin{pmatrix} -1 & 0 \end{pmatrix}^T, x(-2) = \begin{pmatrix} -1 & 1 \end{pmatrix}^T,$$
$$x(-1) = \begin{pmatrix} 0 & -0.5 \end{pmatrix}^T, x(0) = \begin{pmatrix} 0.5 & -0.1 \end{pmatrix}^T,$$

then associated with the above values, we can get state trajectories of the closed-loop system as shown in Figure 1.

From the Figure 1, it is easy to see the system is exponentially stable.

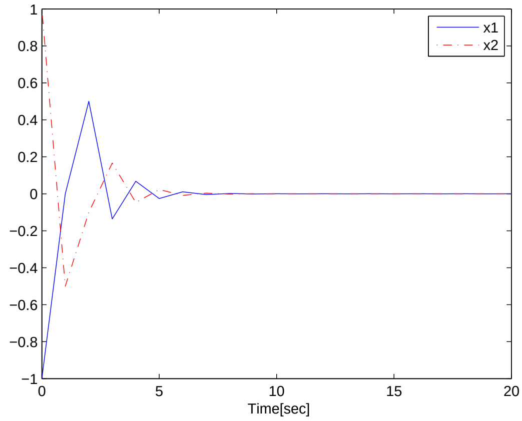

Figure 1: State trajectories of the closed-loop system

Example 12 *Consider the uncertain discrete-time system (35) with the following parameters:*

$$A = \begin{bmatrix} 0.05 & 0 & -0.01 \\ 0.2 & 0.03 & 0 \\ 0 & 0.02 & -0.05 \end{bmatrix}, \quad B = \begin{bmatrix} 0.01 \\ 0.02 \\ 0.03 \end{bmatrix},$$
$$A_1 = \begin{bmatrix} 0.02 & 0.1 & 0.02 \\ 0 & -0.01 & 0.1 \\ 0.01 & 0 & -0.05 \end{bmatrix}, \quad F = G = 0,$$
$$C = \begin{bmatrix} 0.04 & 0.05 & 0 \\ 0.03 & 0.01 & 0.02 \end{bmatrix}, \quad C_1 = C_2 = 0,$$
$$M = \begin{bmatrix} 0.01 \\ 0.03 \\ 0.01 \end{bmatrix}, N_1 = \begin{bmatrix} 0.002 & 0 & 0.01 \\ 0.01 & 0.002 & 0.03 \end{bmatrix},$$
$$N_2 = \begin{bmatrix} 0.01 & 0 & 0.03 \\ 0.03 & 0.01 & 0.02 \end{bmatrix},$$
$$N_3 = \begin{bmatrix} 0.02 & 0.01 & 0.001 \end{bmatrix}, N_4 = N_5 = 0,$$
$$S = \begin{bmatrix} 0.001 & 0 & 0 \\ 0 & 0.002 & 0 \\ 0 & 0 & 0.001 \end{bmatrix}, S_1 = S_2 = 0, R = 2.$$

Given $\alpha = 0.001$, $d_1 = 1$, $d_2 = 1$, $\beta_1 = 0$, $\beta_2 = 0$. By using the LMI Toolbox in MATLAB, the LMI (36) in Corollary 8 is satisfied with

$$P = \begin{bmatrix} 11.1259 & -0.0238 & 0.0028 \\ -0.0238 & 11.0401 & -0.0046 \\ 0.0028 & -0.0046 & 11.1363 \end{bmatrix},$$
$$Q_1 = \begin{bmatrix} 5.5514 & -0.0151 & -0.0108 \\ -0.0151 & 5.5180 & -0.0049 \\ -0.0108 & -0.0049 & 5.5445 \end{bmatrix},$$

$$W_1 = 7.9057, W_2 = \begin{bmatrix} -8.2895 & 0.0281 \\ 0.0281 & -8.3341 \end{bmatrix},$$

$$N = \begin{bmatrix} 9.7229 & -0.0328 & -0.0023 \\ -0.0299 & 9.6205 & -0.0055 \\ -0.0029 & -0.0087 & 9.7299 \end{bmatrix},$$

$$\epsilon = 9.6416, \ \ a_1 = 10.0245, \ \ a_2 = 10.0193,$$

$$\varepsilon_1 = 8.3573, \ \ \varepsilon_2 = 8.3573.$$

Therefore, according to the controller design

$$K = (W_1 + 2R)^{-1} W_2,$$

we have

$$K = \begin{bmatrix} -0.6963 & 0.0024 \\ 0.0024 & -0.7000 \end{bmatrix}.$$

Furthermore, the solution of the closed-loop system satisfies $\|x(k)\| \leq 1.4231e^{-0.001k}\|\phi\|$, and the guaranteed cost value of the closed-loop system with the initial condition $\|\phi\| = [1, 1, 1]^T$ is $J^* = 22.3472$, while basing on the same values of system vectors in [24], we have the least guaranteed cost value is $J^* = 24.3218$.

5 Conclusion

Throughout this paper, we have studied the problem of robust output feedback guaranteed cost control for a class of nonlinear uncertain discrete system with mixed time-varying delays and nonlinear perturbations. A static output feedback guaranteed cost controller has been designed for all admissible uncertainties such that the resulting closed-loop system is robust exponentially stable and guarantee an adequate level of system performance. Two numerical examples have been provided to illustrate the usefulness of the results we got. The paper mainly discussed the delay in perturbations inconsistent with which in state vector. And in future we will study the system with the random perturbations and stochastic delays.

References:

[1] M. Slemrod, E. F. Infante, Asymptotic stability creteria for linear systems of differential equations of nrutral type and their discrete analogues, *J.Math.Anal. Appl.,* 38, 1972, pp. 399-415.

[2] J. H. Kim, Delay and its time-derivative dependent robust stability of time-delayed linear systems with uncertainty, *IEEE Trans.Autom.Control,* 46, 2001, pp. 789-792.

[3] Q. L. Han, Robust stability of uncertain delay-differential systems of neutral type, *Automatica,* 38, 2002, pp. 719-723.

[4] J. P. Richard, Time-delay systems: an overview of some recent advances and open prolems, *Automa tica,* 39 (10), 2003, pp. 1667-1694.

[5] C. Lin, Q. G. Wang, T. H. Lee, A less conservative robust stability test for linear uncertain time-delay systems, *IEEE Trans. Autom.Control,* 51, 2006, pp. 87-91.

[6] S. Xu, J. Lam, A survey of linear matrix inequality techniques in stability analysis of delay systems, *Int. J. Syt. Sci.,* 39 (12), 2008, pp. 1095-1113.

[7] Y. Dong, J. Liu, S. Mei, M. Li, Stabilization for switched nonlinear time-delay systems, *Nonlinear Analysis: Hybrid Systems,* 5, 2011, pp. 78-88.

[8] D. Zhang, L. Yu, H_∞ filtering for linear neutral systems with mixed time-varying delays and nonlinear perturbations, *J. Franklin Inst.,* 347, 2010, pp. 1374-1390.

[9] Y. Dong, J. Liu, S. Mei, Observer design for a class of nonlinear discrete-time systems with time- delay, *Kybernetika,* 49(2), 2013, pp. 341-358.

[10] Q. Fang, B. T. Cui, J. Yan, Further results on robust stability of neutral system with mixed time-varying delays and nonlinear perturbations, *Nonlinear Anal.RWA.,* 11(2), 2011, pp.895-906.

[11] F. Yang, Y. Dong, global exponential stabilization of uncertain switched nonlinear systems with time-varying delays *WSEAS Transactions on Mathematics,* 12 (3), 2013, pp. 256-265.

[12] X. Wei, Exponential stability of periodic solutions for inertial Cohen-Grossberg-type BAM neural networks with time delays, *WSEAS Transactions on Mathematics,* 12 (2), 2013, pp. 159-169.

[13] Y. Dong, X. Wang, S. Mei, W. Li, Exponential stabilization of nonlinear uncertain systems with time-varying delay, *Journal of Engineering Mathematics,* 77, 2012, pp. 225-237.

[14] Y. Dong, Y. Zhang, X. Zhang, Design of observer-based feedback control for a class of discrete-time nonlinear systems with time-delay, *Applied and Computational Mathematics,* 13(1), 2014, pp. 107-121.

[15] J. H. Park, Robust stabilization for dynamic sysytems with multiple time-varying delays and nonlinear uncertainties, *Journal of Optimization Theory and Applications,* 108, 2001, pp. 155-174.

[16] S. Chang, T. Peng, Adaptive guaranteed cost control of systems with uncertain parameters, *IEEE Trans. Automat.Control,* 17, 1972, pp. 474-483.

[17] Y. Wang, L. Xie, C. E. de Souza, Robust control of a class of uncertain nonlinear systems, *Syst. Control Lett.,* 22, 1992, pp. 139-149.

[18] L. Yu, J. Chu, A LMI approach to guaranteed cost control of linear uncetain time-delay systems, *Automatica,* 35, 1999, pp. 1155-1159.

[19] C. H. Lien, Non-fragile guaranteed cost control for uncertain neutral dynamic systems with time-varying delays in state and control input, *Chaos Solitons Fract.,* 31, 2007, pp. 889-899.

[20] J. Wei, Y, Dong, Guaranteed cost control of uncertain T-S Fuzzy systems via output feedback approach, *WSEAS Transactions on Systems,* 9 (10), 2011, pp. 306-317.

[21] X. Guan, Z. Lin, G. Duan, Robust guatanteed cost control for discrete-time uncertain systems with delay, *IEE Proc.Control Theory Appl.,* 146, 1999, pp. 598-602.

[22] W. H. Chen, Z. H. Guan, X. Lu, Delay-dependent guaranteed cost control for discrete-time uncertain system with delay, *IEE Proc. Control Theory Appl.* 150, 2003, pp. 412-416.

[23] T. Weihua, Z. Huaguang, Optimal guaranteed cost control for fuzzy descriptor systems with time-varying delay, *J. Syst. Eng. Electron.,* 19, 2008, pp. 584-590.

[24] L. Yu, F. Gao, Optimal guaranteed cost control of discrete-time uncertain systems with both state and input delays, *Journal of the Franklin Institute,* 338, 20081, pp. 101-110.

[25] M. V. Thuan, V. N. Phat, H. M. Trinh, Dynamic output feedback guaranteed cost control for linear systems with interval time-varying delays in states and outputs. *J. Comput. Appl. Math.,* 218, 2012, pp. 10697-10707.

[26] Q. X. Chen, L. Yu, W. A. Zhang, Delay-dependent output feedback guaranteed cost control for uncertain discrete-time with multiple time-varying delays. *IET J.Control Theory Appl.,* 1, 2007, pp. 97-103.

[27] J. H. Park, On dynamic output feedback guaranteed cost control for uncertain discrete-time systems: LMI optimization approach, *J.Optimiz.Theory Appl.,* 121, 2004, pp. 147-162.

[28] S. Boyd, L. El. Ghaoui, E. Feron, V. Balakrishan. Linear Matrix Inequalities in System and Control Theory, *SIAM, Philadelphia,* 1994.

[29] P. Gahinet, A. Nemirovskii, A. J. Laub, M. Chilali, LMI control toolbox: for use with MATLAB, *MathWorks, Inc., Natick,* 1995.