The Solution of a Parabolic Equation with Unbounded Flux Term

HUASHUI ZHAN
School of Applied Mathematics
Xiamen University of Technology
Xiamen
CHINA
2012111007@xmut.edu.cn

ZHENG ZHOU
School of Applied Mathematics
Xiamen University of Technology
Xiamen
CHINA
2010111004@xmut.edu.cn

XIN SI
School of Applied Mathematics
Xiamen University of Technology
Xiamen
CHINA
xsi@xmut.edu.cn

Abstract: Consider Cauchy problem of the degenerate parabolic equation

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x_i}\left(a^{ij}(u)\frac{\partial u}{\partial x_j}\right) + div(uE).$$

A new kind of entropy solution is introduced, which is stronger than the general one. Supposing that $u_0 \in L^\infty(\mathbf{R}^N)$, $E = \{E_i\}$, $E_i \in E^2$, by a modified regularization method, the problem is translated into a approximate Cauchy problem. By choosing suitable testing functions, the BV estimates of the solutions of the approximate Cauchy problem are obtained. According to Kolomogroff's Theorem, a convergent subsequence can be extracted, then the existence of the entropy solution of the original Cauchy problem is obtained. At last, by Kruzkov bi-variables method, the stability of the entropy solutions is obtained, provided that $E_i x_i \geq 0$.

Key–Words: Cauchy problem, degenerate parabolic equation, existence, unbounded flux term

1 Introduction

This paper studies the existence and uniqueness of BV-solutions of Cauchy problem

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x_i}\left(a^{ij}(u)\frac{\partial u}{\partial x_j}\right) + div(uE),\ (x,t) \in Q_T \tag{1}$$

with initial

$$u(x,0) = u_0(x),\ x \in \mathbf{R}^N \tag{2}$$

where $Q_T = \mathbf{R}^N \times (0,T)$, $u_0(x) \in L^\infty(\mathbf{R}^N)$, $E = \{E_i\}$, $E_i \in E^2$ with the definition of that

$$E^2 = \{f \in C^1(Q_T) : f \in L^2(Q_T), divE \in L^2(Q_T)\}.$$

As usual, the pairs of equal indices imply the summation from 1 up to N. We say equation (1) is weakly degenerate, if there is not interior point of the set $\{s$: the determinant $|a^{ij}(s)| = 0\}$. Otherwise, if there are interior points of the set $\{s$: the determinant $|a^{ij}(s)| = 0\}$, then we say equation (1) is strongly degenerate.

If $(a^{ij}(s)) = a(s)I$, I is the unit $N \times N$ matrix, $a(s) \geq \alpha > 0$, some applicative models related to equation (1) were studied in [1]. In this case, the existence and the non-existence of weak solutions of the first initial-boundary value problem of (1) were obtained in [2], provided that $u_0 \in L^1(\Omega)$, and $E = \{E_i\} \in (L^2(\Omega \times (0,T))^N$, where $\Omega \in \mathbf{R}^N$ is a bounded domain.

If (a^{ij}) is only a semidefinite positive matrix , i.e.

$$a^{ij}\xi_i\xi_j \geq 0,\ \forall \xi \in \mathbf{R}^N, \tag{3}$$

then equation (1) is a degenerate parabolic equation and the corresponding problem seems more difficult and few reference could be found.

If the unbounded flux term $div(uE)$ is substituted by a general convection term $div(b(u))$, where $b(u) = \{b^i(u)\}$, $b^i(u)$ generally is a bounded nonlinear function when u is bounded, the following degenerate parabolic-hyperbolic equation

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x_j}\left(a^{ij}(u)\frac{\partial u}{\partial x_j}\right) + div(b(u)),\ in\ Q_T \tag{4}$$

had been studied widely. Equation (4) arises in many applications, e.g., heat flow in materials with temperature dependent conductivity, flow in a porous medium, and the boundary layer theory (see [3], [4] et al.). A.I. Vol′pert and S.I. Hudjaev [5] had firstly got the solvability of equation (4). After that, many mathematicians (e.g. Bénilan, Brezis, DiBenedetto, Carrillo, Gagneux, Madaune-Tort, Wittbold, and Wu-Zhao et al.)(see [4]-[18] et al.) continued to study its solvability, and got many excellent results. The first author of the paper also has studied the posedness of Cauchy problem of equation (4) for a long time (see [10-11] and [23-26]). However, all these results are based on the boundedness and the nonlinearity of $b^i(u)$. In general, one may expect that the linearity of the flux term $div(uE)$ can make the problem easier. If E is bounded, it is indeed likely the corresponding problem can be solved by an easier way. But if $E = \{E_i(x,t)\}$ becomes unbounded, the method used in [4-18] seems impossible to be generalized to deal with Cauchy problem of equation (1), the corresponding problem becomes much more difficult.

In some details, the unbounded property of E makes the general maximum principle ineffective, makes the general parabolically regularized method invalid. To overcome these difficulties, by using some ideas in [10-11], we put forward to a new definition of BV-entropy solution for system (1)-(2), this kind of entropy solution is stronger than the general one. By modifying the general parabolically regularized method, and choosing the suitable testing functions, we are able to get the needed BV estimates. By these estimates, using Kolomogroff's Theorem, we can select the convergent subsequence from the approximate solutions $\{u_K\}$, to get the existence of the entropy solution. The estimating method used here is completely different from that used in [5]-[7], [10]-[12] et al., but we use some inspiring techniques in [19],[23].

At the same time, we shall use Kruzkov's bi-variables method to get the stability of the entropy solutions as we had done on equation (4) in [10]. However, we have to add an auxiliary condition, $E_i x_i \geq 0$, to guarantee that Kruzkov's bi-variables method is still valid.

2 Definition and Main results

Following reference [20], $f \in BV(Q_T)$ if and only if that the generalized derivatives of f are regular measures on Q_T, i.e.

$$\int\int_{Q_T} |\frac{\partial f}{\partial t}| < \infty, \int\int_{Q_T} |\frac{\partial f}{\partial x_i}| < \infty, i = 1, 2, \cdots, N.$$

A basic property of BV function is that ([21]): if $f \in BV(Q_T)$, then there exists a sequence $\{f_n\} \subset C^\infty(Q_T)$ such that

$$\lim_{n\to\infty} \int\int_{Q_T} | f_n - f | \, dxdt = 0,$$

$$\lim_{n\to\infty} \int\int_{Q_T} | \nabla f_n | \, dxdt = \int\int_{Q_T} | \nabla f | \,.$$

So, we can define the trace of the functions in BV space as in Sobolev space. Moreover, the BV functions are the weakest functions that we can define the traces.

Let Γ_u be the set of all jump points of $u \in BV(Q_T)$. It is well-known that the normal vector of Γ_u exists almost everywhere in the sense of N-dimensional Hausdorff measure [4]. Let v be the normal of Γ_u at $X = (x,t)$, $u^+(X)$ and $u^-(X)$ be the approximate limits of u at $X \in \Gamma_u$ with respect to $(v, Y - X) > 0$ and $(v, Y - X) < 0$ respectively. For continuous function $p(u, x, t)$ and $u \in BV(Q_T)$, as usual, we can define

$$\widehat{p}(u, x, t) = \int_0^1 p\left(\tau u^+ + (1-\tau)u^-, x, t\right) d\tau,$$

$$\overline{u} = \frac{1}{2}(u^+ + u^-).$$

For a given t, we denote $\Gamma_u^t,\ H^t, (v_1^t,\ \cdots, v_N^t)$ and $u_\pm^t$ as all jump points of $u(\cdot,t)$, Hausdorff measure of Γ_u^t, the unit normal vector of Γ_u^t, and the asymptotic limit of $u(\cdot,t)$ respectively. By [20], if $f(s) \in C^1(\mathbf{R}),\ u \in BV(Q_T)$, then $f(u) \in BV(Q_T)$ and

$$\frac{\partial f(u)}{\partial x_i} = \widehat{f'}(u)\frac{\partial u}{\partial x_i},\ i = 1, 2, \cdots, N.$$

For small $\eta > 0$, we set $S_\eta(s) = \int_0^s h_\eta(\tau)d\tau$, where $h_\eta(s) = \frac{2}{\eta}(1 - \frac{|s|}{\eta})_+,\ s \in \mathbf{R}$. Obviously $h_\eta(s) \in C(\mathbf{R})$, and satisfies

$$h_\eta(s) \geq 0,\ \ | sh_\eta(s) | \leq 1,\ | S_\eta(s) | \leq 1;$$

$$\lim_{\eta\to 0} S_\eta(s) = \mathrm{sgn}(s),\ \lim_{\eta\to 0} sS'_\eta(s) = 0.$$

Definition 1 *A function u is said to be a weak solution of Cauchy problem (1)-(2), if*

1. $u \in BV(Q_T) \cap L^\infty(Q_T)$, and there are functions $g^i \in L^2(0,T; L^2_{loc}(\mathbf{R}^N))$ such that for $\forall\varphi(x,t) \in C_0(Q_T)$,

$$\int\int_{Q_T} g^i(x,t)\varphi(x,t)dxdt = \int\int_{Q_T} \varphi(x,t)\hat{r}^{ij}(u)\frac{\partial u}{\partial x_j}dxdt, \qquad (5)$$

where (r^{ij}) is the square root of (a^{ij}), and $i = 1, 2, \cdots, N$.

2. For any $\varphi \in C_0^2(Q_T),\ \varphi \geq 0,\ k \in \mathbf{R},\ \eta > 0$, u satisfies

$$\int\int_{Q_T} [I_\eta(u-k)\varphi_t - E_i I_\eta(u-k)\varphi_{x_i} + A_\eta^{ij}(u,k)\varphi_{x_i x_j}]dxdt - \int\int_{Q_T} [S'_\eta(u-k)\sum_{j=1}^N | g^j |^2 \varphi - \int_k^u sS'_\eta(s-k)dsE_{ix_i}\varphi]dxdt \geq 0, \qquad (6)$$

where $I_\eta(u-k)=\int_0^{u-k}S_\eta(s)ds$, *and*

$$A_\eta^{ij}(u,k)=\int_k^u a^{ij}(s)S_\eta(s-k)ds. \tag{7}$$

3. The initial value is satisfied in the sense that

$$\lim_{t\to0}\int_{B_R}\mid u(x,t)-u_0(x)\mid dx=0,\ \forall R>0, \tag{8}$$

where $B_R=\{x\in\mathbf{R}^N:|x|<R\}$.

To explain the reasonableness of Definition 1, it is supposed that equation (1) has a classical solution u. Let $\varphi\in C_0^2(Q_T),\ \varphi\geq0,\ k\in\mathbf{R},\ \eta>0$. Multiplying equation (1) by $\varphi S_\eta(u-k)$ and integrating over Q_T, we have

$$\begin{aligned}&\int\int_{Q_T}\frac{\partial u}{\partial t}\varphi S_\eta(u-k)dxdt\\&=\int\int_{Q_T}\frac{\partial}{\partial x_i}\left(a^{ij}(u)\frac{\partial u}{\partial x_j}\right)\varphi S_\eta(u-k)dxdt\\&+\int\int_{Q_T}div(uE)\varphi S_\eta(u-k)dxdt.\end{aligned} \tag{9}$$

$$\begin{aligned}&\int\int_{Q_T}\frac{\partial u}{\partial t}\varphi S_\eta(u-k)dxdt\\&=\int\int_{Q_T}\frac{\partial I_\eta(u-k)}{\partial t}\varphi dxdt\\&=-\int\int_{Q_T}I_\eta(u-k)\frac{\partial\varphi}{\partial t}dxdt.\end{aligned} \tag{10}$$

For the first term of the right hand side of (9), we have

$$\begin{aligned}&\int\int_{Q_T}\frac{\partial}{\partial x_i}\left(a^{ij}(u)\frac{\partial u}{\partial x_j}\right)\varphi S_\eta(u-k)dxdt\\&=-\int_0^T dt\int_{\mathbf{R}^N}a^{ij}(u)\frac{\partial u}{\partial x_j}\\&\cdot\left[S'_\eta(u-k)\frac{\partial u}{\partial x_i}\varphi+S_\eta(u-k)\varphi_{x_i}\right]dx,\end{aligned} \tag{11}$$

where

$$\begin{aligned}&\int_0^T dt\int_{\mathbf{R}^N}a^{ij}(u)\frac{\partial u}{\partial x_j}S_\eta(u-k)\varphi_{x_i}dx\\=&\int_0^T dt\int_{\mathbf{R}^N}\frac{\partial A_\eta^{ij}(u,k)}{\partial x_j}\varphi_{x_i}dx\\=&\int_0^T dt\int_{\mathbf{R}^N}\left[\frac{\partial(A_\eta^{ij}(u,k)\varphi_{x_i})}{\partial x_j}-A_\eta^{ij}(u,k)\varphi_{x_ix_j}\right]dx\\=&-\int_0^T dt\int_{\mathbf{R}^N}A_\eta^{ij}(u,k)\varphi_{x_ix_j}dx.\end{aligned} \tag{12}$$

So

$$\begin{aligned}&\int\int_{Q_T}\frac{\partial}{\partial x_i}\left(a^{ij}(u)\frac{\partial u}{\partial x_j}\right)\varphi S_\eta(u-k)dxdt\\&=-\int\int_{Q_T}S'_\eta(u-k)a^{ij}(u)\frac{\partial u}{\partial x_j}\frac{\partial u}{\partial x_i}\varphi\\&+\int\int_{Q_T}A_\eta^{ij}(u,k)\varphi_{x_ix_j}dxdt\\&=-\int\int_{Q_T}S'_\eta(u-k)\sum_{j=1}^N|g^j|^2\varphi dxdt\\&+\int\int_{Q_T}A_\eta^{ij}(u,k)\varphi_{x_ix_j}dxdt\end{aligned} \tag{13}$$

and

$$\begin{aligned}&\int\int_{Q_T}div(uE)S_\eta(u-k)\varphi dxdt\\&=\int\int_{Q_T}\left(\frac{\partial u}{\partial x_i}E_i+u\frac{\partial E_i}{\partial x_i}\right)S_\eta(u-k)\varphi dxdt\\&=\int\int_{Q_T}\frac{\partial I_\eta(u-k)}{\partial x_i}E_i\varphi dxdt\\&+\int\int_{Q_T}u\frac{\partial E_i}{\partial x_i}S_\eta(u-k)\varphi dxdt\\&=-\int\int_{Q_T}I_\eta(u-k)E_i\varphi_{x_i}dxdt\\&-\int\int_{Q_T}I_\eta(u-k)E_{ix_i}\varphi dxdt\\&+\int\int_{Q_T}u\frac{\partial E_i}{\partial x_i}S_\eta(u-k)\varphi dxdt\\&=-\int\int_{Q_T}I_\eta(u-k)E_i\varphi_{x_i}dxdt\\&+\int\int_{Q_T}\int_k^u sS'_\eta(s-k)dsE_{ix_i}\varphi dxdt.\end{aligned} \tag{14}$$

By (9)-(14), if equation (1) has a classical solution u, then

$$\begin{aligned}&\int\int_{Q_T}[I_\eta(u-k)\varphi_t-E_iI_\eta(u-k)\varphi_{x_i}\\&+A_\eta^{ij}(u,k)\varphi_{x_ix_j}\varphi]dxdt\\&-\int\int_{Q_T}[S'_\eta(u-k)\sum_{j=1}^N\mid g^j\mid^2\varphi\\&-\int_k^u sS'_\eta(s-k)dsE_{ix_i}\varphi]dxdt=0.\end{aligned} \tag{15}$$

Clearly

$$\begin{aligned}&\int\int_{Q_T}[I_\eta(u-k)\varphi_t-E_iI_\eta(u-k)\varphi_{x_i}\\&+A_\eta^{ij}(u,k)\varphi_{x_ix_j}+\int_k^u sS'_\eta(s-k)dsE_{ix_i}\varphi]dxdt\\&\geq0.\end{aligned} \tag{16}$$

Let $\eta\to0$ in this inequality. We have

$$\begin{aligned}&\int\int_{Q_T}[|u-k|\varphi_t-E_i|u-k|\varphi_{x_i}\\&+[A^{ij}(u)-A^{ij}(k)]\mathrm{sgn}(u-k)\varphi_{x_ix_j}\end{aligned}$$

$$+ksgn(u-k)E_{ix_i}\varphi]dxdt \geq 0. \quad (6)'$$

where

$$A^{ij}(u) = \int_0^u a^{ij}(s)ds.$$

Clearly, if one defines the weak solutions u_1, u_2, and u_3 of equation (1) in the sense of expressions (6), (16) and (6)' respectively, then u_1 is also a solution in the senses of expressions (16) and (6)', u_2 is also the solution in the sense of expression (6)'. If equation (1) is weakly degenerate, one can define the weak solution of (1) in sense of (6)'. In this case, the term $-S'_\eta(u-k)\sum_{j=1}^N \mid g^j \mid^2 \varphi$ in (15) seems redundant, and can be drawn away. But, if the equation (1) is strongly degenerate, the term $-S'_\eta(u-k)\sum_{j=1}^N \mid g^j \mid^2 \varphi$ implies very important information of the uniqueness, it can not be drawn away. One can refer to references [11-14].

Also, we note that the classical solution u induces an integral equality (15), whereas the weak solution formula defined as expression (6) is an inequality, this is due to the following weak convergence property.

Lemma 2 *Assume that $U \subset \mathbf{R}^N$ is an open bounded set and as $k \to \infty$,*

$$f_k \rightharpoonup f \text{ weakly in } L^q(U), 1 \leq q < \infty,$$

then

$$\lim_{k\to\infty}\inf \parallel f_k \parallel^q_{L^q(U)} \geq \parallel f \parallel^q_{L^q(U)} . \quad (17)$$

Generally, inequality (17) can not be an equality. In what follows, one can see that this is why we can only define the weak solution as expression (6) instead of expression (15).

Base on the about discussion, we shall prove the following Theorems:

Theorem 3 *Suppose that $(a^{ij}(s))$ is a semidefinite positive matrix, every element $a^{ij}(s) \in C^1(\mathbf{R})$; $u_0(x) \in L^\infty(\mathbf{R}^N) \cap L^2(\mathbf{R}^N)$. Suppose that $E_i(x,t) \in C^1(Q_T)$, and $E = \{E_i\}$ is a vector field, such that*

$$E = \{E_i\}, E_i \in E^2, \quad (18)$$

then the problem (1)-(2) has a weak solution in the sense of Definition 1.

Theorem 4 *Let u, v be solutions of (1)-(2) with initial values $u_0(x), v_0(x) \in L^\infty(\mathbf{R}^N) \cap L^2(\mathbf{R}^N)$ respectively. Suppose that*

$$E \cdot x = E_i x_i \geq 0, \quad (19)$$

then

$$\int_{\mathbf{R}^N} \mid u(x,t) - v(x,t) \mid \omega_\lambda(x)dx$$

$$\leq c\int_{\mathbf{R}^N} \mid u_0 - v_0 \mid \omega_\lambda(x)dx, \quad (20)$$

where c, λ are positive constants and

$$\omega_\lambda(x) = \exp\{-\lambda\sqrt{1+ \mid x \mid^2}\}. \quad (21)$$

Remark 5 *Consider the equation*

$$\frac{\partial u}{\partial t} = \frac{\partial^2 A(u)}{\partial x^2} + \frac{\partial B(u)}{\partial x}, (x,t) \in \mathbf{R} \times (0,T). \quad (22)$$

A.I. Vol'pert and S.I. Hudjaev in [5] defined that, $u \in BV(Q_T) \bigcap L^\infty(Q_T)$ is said to be a weak solution of equation (22), if $\frac{\partial A(u)}{\partial x} \in L^1_{loc}(Q_T)$, and for any $0 \leq \varphi \in C_0^\infty(Q_T)$, any $k \in \mathbf{R}$,

$$\int\int_{Q_T} sgn(u-k)\left[(u-k)\frac{\varphi}{\partial t} - \frac{\partial A(u)}{\partial x}\frac{\partial\varphi}{\partial x}\right]dxdt - \int\int_{Q_T} sgn(u-k)\left[(B(u)-B(k))\frac{\partial\varphi}{\partial x}\right]dxdt \geq 0. \quad (23)$$

We know that only under the condition

$$\frac{\partial A(u)}{\partial x} \in L^\infty Q_T) \bigcap BV_x(Q_T)$$

the uniqueness of the solutions in the sense that expression (23) is true. So, an essential improvement of our paper (also [10-14]) is to get the uniqueness of the solutions in the sense of expression (6) without any bounded restrictions in g^i.

Remark 6 *Consider another equation*

$$u_t - div(a(x,t,u)\nabla u) = -div(uE), \quad (24)$$

$$(x,t) \in Q = \Omega \times (0,T).$$

Assuming that $0 < \alpha \leq a(x,t,s) \leq \beta$, L. Boccardo, L. Orsina and A. Porretta [2] defined that $u \in L^\infty(Q) \bigcap L^2(0,T;H_0^1(\Omega))$ is a weak solution of equation (24) in the sense that

$$< u_t, \varphi > + \int\int_Q a(x,t,u)\nabla u \cdot \nabla\varphi dxdt = \int\int_Q uE\nabla\varphi dxdt, \quad (25)$$

for every $\varphi \in L^2(0,T;H_0^1(\Omega))$, where Ω is a bounded domain in $\mathbf{R}^N$, $< .,. >$ denotes the duality product between $L^2(0,T;H_0^1(\Omega))$ and $L^2(0,T;H^{-1}(\Omega))$. Clearly, if $a(x,t,s) \equiv 0$, equation (24) becomes the type of conservation law equation, and it is well known that in this case, if one defines the weak solution as expression (25), then the uniqueness of the *solutions is not true. Also, L. Boccardo, L. Orsina and A. Porretta [2] had quoted the following unbounded entropy solution. A measurable function $u \in L^\infty(0,T;L^1(\Omega))$ is an entropy solution of equation (24) if $T_k(u) \in L^2(0,T;H_0^1(\Omega))$ for every $k>0$ and u satisfies*

$$\begin{aligned}&\int_\Omega \Theta_k(u-\varphi)(t)dx - <\varphi_s, T_k(u-\varphi)> \\ &+\int_0^t\int_\Omega a(x,s,u)\nabla T_k(u-\varphi)dxds \\ &\le \int_0^t\int_\Omega uE\nabla T_k(u-\varphi)dxds \\ &+\int_\Omega \Theta_k(u_0-\varphi(0))(t)dx, \end{aligned} \tag{26}$$

for almost every $t\in(0,T)$, for every

$$\varphi \in L^2(0,T;H_0^1(\Omega))\bigcap L^\infty(Q)$$

such that

$$\varphi_t \in L^2(0,T;H^{-1}(\Omega))+L^1(Q),$$

where

$$\Theta_k(s)=\int_0^s T_k(r)dr.$$

If we check the proof of the theorems in [2], we have found that the condition $0<\alpha\le a(x,t,s)\le\beta$ acts an important role. If this condition is weakened to $0\le\alpha$, to get the same conclusions seems difficult. By the way, though the authors of [2] did not discussed the uniqueness of the solutions, we believe that the uniqueness of the solutions in the sense of expression (26) is true, provided that $0<\alpha\le a(x,t,s)\le\beta$.

Remark 7 *The space*

$$E^2=\{f\in C^1(Q_T): f\in L^2(Q_T), divE\in L^2(Q_T)\},$$

is a Banach space with the norm defined as

$$\|f\|=\|f\|_{L^2(Q_T)}+\|divf\|_{L^2(Q_T)}.$$

It acts an important role in the studying of compressible flow dynamics theory, see [27].

3 The regularized problem

We need the following **Gronwall Lemma**.

Suppose that $a(t)$ and $c(t)$ are the functions defined on $[0,T]$, $c(t)\ge 0$. Suppose that $y(t)\in C^1[0,T]$ such that $y(0)=0$,

$$y'(t)\le c(t)y(t)+c(t)a(t),$$

then

$$y(t)\le \sup_{0\le t\le T}|a(t)|\left[\exp\int_0^T c(\tau)d\tau -1\right].$$

Now, it is supposed that $A^{ij}(s), u_0(x)$ are functions as in Theorem 3, $u_0(x)\in L^\infty(\mathbf{R}^N)\cap L^2(\mathbf{R}^N)$, and $E\in(C^1(Q_T))^N$ is a vector field such that $E=\{E_i\}$,$E_i\in E^2$. For any given large positive number K, let us introduce the following modified regularized problem.

$$\frac{\partial u}{\partial t}=\frac{\partial}{\partial x_j}(a^{ij}(u)\frac{\partial u}{\partial x_j})+\frac{1}{K}\Delta u+div(u\delta_\varepsilon * T_K E), \tag{27}$$

$$u(x,0)=u_{0K}(x), \tag{28}$$

where δ_ε is the mollifier as usual, i.e. if $y=(x_1,\cdots,x_N,t)$, then

$$\delta(y)=\begin{cases}\frac{1}{A}e^{\frac{1}{|y|^2-1}}, & \text{if } |y|<1,\\ 0, & \text{if } |y|\ge 1,\end{cases}$$

where

$$A=\int_{B_1(0)} e^{\frac{1}{|y|^2-1}}dx.$$

For any given $\varepsilon>0$, $\delta_\varepsilon(y)$ is defined as

$$\delta_\varepsilon(y)=\frac{1}{\varepsilon^{N+1}}\delta(\frac{y}{\varepsilon}).$$

Here, we choose $\varepsilon=\frac{1}{K}$ especially, and

$$\delta_\varepsilon * T_K(E)=\{\delta_\varepsilon(E_i)*T_K(E_i)\},$$

$$i=1,2,\cdots,N.$$

$$T_K(s)=\min\{K,\max\{-K,s\}\}.$$

Moreover, we suppose that $u_{0K}\in C_0^\infty(\mathbf{R}^N)$, $suppu_{0K}\subset B_K=\{x\in\mathbf{R}^N:|x|<K\}$, and

$$\lim_{K\to\infty}\|u_{0K}-u_0\|_{L^2(\mathbf{R}^N)}=0,$$

$$\|u_{0K}\|_{L^\infty(\mathbf{R}^N)}\le\|u_0\|_{L^\infty(\mathbf{R}^N)}. \tag{29}$$

It is well-known that there is a classical solution $u_K\in C^{2,1}(Q_T)$ of system (27)-(28), and

$$\|u_K\|_{L^\infty}\le\|u_0\|_{L^\infty}. \tag{30}$$

Let $gradu_K=(u_{Kx_1},u_{Kx_2},\cdots,u_{Kx_N},u_{Kx_{N+1}})$ and $x_{N+1}=t$, $u_{Kx_{N+1}}=u_{Kt}$. For simplicity, we denote u_K as u in the following calculation. Let us differentiate equation (27) with respect to x_s

$(s=1,2,\cdots,N,N+1)$ and sum up for s after multiplying the resulting relation by $u_{x_s}\frac{S_\eta(|\text{grad}u|)}{|\text{grad}u|}\varphi$. Here, $0\le\varphi\in C_0^\infty(Q_T)$. Integrating over $\mathbf{R}^N$ yields

$$\begin{aligned}&\frac{d}{dt}\int_{\mathbf{R}^N} I_\eta(|\text{grad}u|)\varphi dx\\&-\frac{1}{K}\int_{\mathbf{R}^N}\triangle u_{x_s}u_{x_s}\frac{S_\eta(|\text{grad}u|)}{|\text{grad}u|}\varphi dx\\&-\int_{\mathbf{R}^N}\frac{\partial}{\partial x^j}[a_u^{ij}(u)u_{x_s}u_{x_j}+a^{ij}u_{x_sx_j}]\\&\qquad\cdot u_{x_s}\frac{S_\eta(|\text{grad}u|)}{|\text{grad}u|}\varphi dx\\&-\int_{\mathbf{R}^N}\nabla u_{x_s}\cdot\delta_\varepsilon*T_K(E)u_{x_s}\frac{S_\eta(|\text{grad}u|)}{|\text{grad}u|}\varphi dx\\&-\int_{\mathbf{R}^N}div\left(u\frac{\partial\delta_\varepsilon*T_K(E)}{\partial x_s}\right)u_{x_s}\frac{S_\eta(|\text{grad}u|)}{|\text{grad}u|}\varphi dx\\&=0.\end{aligned}\tag{31}$$

Integrating by part, we have

$$\begin{aligned}&\frac{d}{dt}\int_{\mathbf{R}^N} I_\eta(|\text{grad}u|)\varphi dx\\&+\frac{1}{K}\int_{\mathbf{R}^N}u_{x_sx_i}u_{x_px_i}\frac{\partial^2 I_\eta(|\text{grad}u|)}{\partial\xi_s\partial\xi_p}\varphi dx\\&+\frac{1}{K}\int_{\mathbf{R}^N}\frac{S_\eta(|\text{grad}u|)}{|\text{grad}u|}u_{x_sx_i}u_{x_s}\varphi_{x_i}dx\\&+\int_{\mathbf{R}^N}a^{ij}(u)u_{x_sx_i}u_{x_px_j}\frac{\partial^2 I_\eta(|\text{grad}u|)}{\partial\xi_s\partial\xi_p}\varphi dx\\&+\int_{\mathbf{R}^N}a^{ij}(u)\frac{S_\eta(|\text{grad}u|)}{|\text{grad}u|}u_{x_sx_j}u_{x_s}\varphi_{x_i}dx\\&+\int_{R^N}a_u^{ij}(u)u_{x_j}I_\eta(|\text{grad}u|)\varphi_{x_i}dx\\&-\int_{\mathbf{R}^N}a_u^{ij}(u)u_{x_ix_j}[|\text{grad}u|S_\eta(|\text{grad}u|)\\&-I_\eta(|\text{grad}u|)]\varphi dx\\&-\int_{\mathbf{R}^N}\nabla u_{x_s}\cdot\delta_\varepsilon*T_K(E)u_{x_s}\frac{S_\eta(|\text{grad}u|)}{|\text{grad}u|}\varphi dx\\&-\int_{\mathbf{R}^N}div\left(u\frac{\partial\delta_\varepsilon*T_K(E)}{\partial x_s}\right)u_{x_s}\frac{S_\eta(|\text{grad}u|)}{|\text{grad}u|}\varphi dx\\&=0,\end{aligned}\tag{32}$$

where $\xi_s=u_{x_s}, s=1,2,\cdots,N+1$.

For the last term on the left side of (32),

$$\begin{aligned}&\int_{\mathbf{R}^N}div\left(u\frac{\partial\delta_\varepsilon*T_K(E)}{\partial x_s}\right)u_{x_s}\frac{S_\eta(|\text{grad}u|)}{|\text{grad}u|}\varphi dx\\&=\sum_{i=1}^N\int_{\mathbf{R}^N}[u_{x_i}\frac{\partial\left(\delta_\varepsilon*T_K(E_i)\right)}{\partial x_s}\\&+u\frac{\partial^2(\delta_\varepsilon*T_K(E_i))}{\partial x_s\partial x_i}]u_{x_s}\frac{S_\eta(|\text{grad}u|)}{|\text{grad}u|}\varphi dx.\end{aligned}\tag{33}$$

If we notice that $\varepsilon=\frac{1}{K}$, then

$$\begin{aligned}&\frac{\partial\left(\delta_\varepsilon*T_K(E_i)\right)}{\partial x_s}\\&=-\int_{\{y:|K(x-y)|<1\}}\frac{2K^2(x_s-y_s)}{[|K(x-y)|^2-1]^2}\\&\quad\frac{K^{N+1}}{A}\cdot e^{\frac{1}{|K(x-y)|^2-1}}T_K(E_i(y,s))dy,\end{aligned}\tag{34}$$

where $x=(x_1,\cdots,x_N,t)$ as before. By the facts of that

$$\frac{1}{[|K(x-y)|^2-1]^2}e^{\frac{1}{|K(x-y)|^2-1}}\le c,$$

and $|K(x-y)|<1$, $|T_K(E_i))|\le K$, from (34) we can get

$$\begin{aligned}&|\frac{\partial\left(\delta_\varepsilon*T_K(E_i)\right)}{\partial x_s}|\\&\le c\int_{\{y:|K(x-y)|<1\}}\frac{1}{[|K(x-y)|^2-1]^2}\\&\quad\frac{K^{N+3}}{A}e^{\frac{1}{|K(x-y)|^2-1}}dyds\\&\le cK^{N+3}.\end{aligned}$$

Thus, in (33) (also in (32)), if we choose

$$\varphi(x)=\frac{1}{K^{N+4}}\varphi_1(x),\varphi_1\in C_0^\infty(\mathbf{R}^N),$$

then we have

$$\begin{aligned}&\int_{\mathbf{R}^N}[u_{x_i}\frac{\partial(\delta_\varepsilon*T_K(E))}{\partial x_s}u_{x_s}\frac{S_\eta(|\text{grad}u|)}{|\text{grad}u|}\varphi dx\\&\le\frac{c}{K}\int_{\mathbf{R}^N}|\text{grad}u|\varphi_1 dx.\end{aligned}\tag{35}$$

Similarly, we are able to show that

$$|\frac{\partial^2(\delta_\varepsilon*T_K(E))}{\partial x_s\partial x_i}|\le cK^{N+4},$$

then

$$\begin{aligned}&\int_{\mathbf{R}^N}u\frac{\partial^2(\delta_\varepsilon*T_K(E))}{\partial x_s\partial x_i}u_{x_s}\frac{S_\eta(|\text{grad}u|)}{|\text{grad}u|}\varphi dx\\&\le c\int_{\mathbf{R}^N}\varphi_1 dx.\end{aligned}\tag{36}$$

By a process of limit, we can assume that in the formulas (35)-(36) (also in (32)),

$$\varphi_1=\omega_\lambda(x)=\exp(-\lambda\sqrt{1+|x|^2}).$$

Clearly, there exists a positive constant c_λ such that

$$\begin{aligned}&\omega_{\lambda x_i}=\omega_\lambda\frac{-\lambda x_i}{\sqrt{1+|x|^2}},\\&|\nabla\omega_\lambda|\le c_\lambda\omega_\lambda,\ |\omega_{\lambda x_ix_j}|\le c_\lambda\omega_\lambda.\end{aligned}\tag{37}$$

Now, by the following facts

$$\frac{1}{K}\int_{\mathbf{R}^N}\frac{S_\eta(|\mathrm{grad}u|)}{|\mathrm{grad}u|}u_{x_sx_i}u_{x_s}\varphi_{x_i}dx = -\frac{1}{K^{N+5}}\int_{\mathbf{R}_N}I_\eta(|\mathrm{grad}u|)\triangle\varphi_1 dx, \quad (38)$$

$$\int_{\mathbf{R}^N}a^{ij}(u)u_{x_sx_i}u_{x_px_j}\frac{\partial^2 I_\eta(|\mathrm{grad}u|)}{\partial\xi_s\partial\xi_p}\varphi dx \ge 0, \quad (39)$$

$$\int_{\mathbf{R}^N}a^{ij}(u)\frac{S_\eta(|\mathrm{grad}u|)}{|\mathrm{grad}u|}u_{x_sx_i}u_{x_s}\varphi_{x_j}dx + \int_{\mathbf{R}^N}a_u^{ij}(u)u_{x_i}I_\eta(|\mathrm{grad}u|)\varphi_{x_j}dx = -\frac{1}{K^{N+4}}\int_{\mathbf{R}^N}a^{ij}(u)I_\eta(|\mathrm{grad}u|)\varphi_{1x_ix_j}dx, \quad (40)$$

$$|\mathrm{grad}u|S_\eta(|\mathrm{grad}u|) - I_\eta(|\mathrm{grad}u|) = \int_0^{|\mathrm{grad}u|}\tau h_\eta(\tau)d\tau \to 0, \ \ as\ \eta\to 0, \quad (41)$$

if we let $\eta \to 0$ in (32), using (35)-(41), we have

$$\frac{d}{dt}\int_{\mathbf{R}^N}|\mathrm{grad}u|\omega_\lambda dx \le c + c\int_{\mathbf{R}^N}|\mathrm{grad}u|\omega_\lambda dx,$$

where c depends on λ but is independent of K. By Gronwall Lemma, we have

$$\int_{\mathbf{R}^N}|\mathrm{grad}u|\omega_\lambda dx \le c(T,\lambda). \quad (42)$$

If we multiply with $u\omega_\lambda$ on the two sides of equation (27), and integrate over Q_T, by (42), we can show that

$$\sum_{j=1}^{N}\int\int_{Q_T}|r^{ij}u_{x_i}|^2\omega_\lambda dxdt \le c(T,\lambda,\|u_0\|_{L^\infty}), \quad (43)$$

we omit the details here.

By (30), (42) and Kolomogroff's Theorem, there exists a subsequence $\{u_{K_n}\}$ of the family $\{u_K\}$ which are the solutions of regularized problem (27)-(28), and there exists a function $u \in BV(Q_T)\cap L^\infty(Q_T)$, such that $u_{K_n} \to u$ a.e. on Q_T.

4 The Proof of Theorem 3

Now, let u be the limit function of u_{K_n} as $n \to \infty$. We now prove that u is the weak solution of (1)-(2) in the sense of Definition 1. For simplicity, we denote K_n as K in what follows.

Firstly, from (43), there are

$$g^i \in L^2(0,T;L^2_{loc}(\mathbf{R}^N)),\ i=1,2,\cdots,N,$$

such that, as $K\to\infty$

$$r^{ij}\frac{\partial u_K}{\partial x_j} \rightharpoonup g^j, in\ \ L^2(0,T;L^2_{loc}(\mathbf{R}^N)).$$

For any $\varphi \in C_0^2(Q_T)$,

$$\begin{aligned}&\int\int_{Q_T}\varphi g^i dxdt\\ &= \lim_{K\to\infty}\int\int_{Q_T}\varphi r^{ij}\frac{\partial u_K}{\partial x_j}dxdt\\ &= \lim_{K\to\infty}\int\int_{Q_T}\varphi\frac{\partial}{\partial x_j}(\int_0^{u_K}r^{ij}(s)ds)dxdt\\ &= -\int\int_{Q_T}\varphi_{x_j}\int_0^u r^{ij}(s)dsdxdt\\ &= \int\int_{Q_T}\varphi\hat{r}^{ij}\frac{\partial u}{\partial x_j}dxdt. \end{aligned} \quad (44)$$

This implies that u satisfies (1) of Definition 1.

Secondly, let $\varphi \in C_0^2(Q_T),\ \varphi\ge 0,\ k\in\mathbf{R},\ \eta > 0$. Multiplying equation (27) by $\varphi S_\eta(u_K-k)$ and integrating over Q_T, we obtain

$$\begin{aligned}&-\int\int_{Q_T}I_\eta(u_K-k)\varphi_t dxdt\\ &+\frac{1}{K}\int\int_{Q_T}S_\eta(u_K-k)\frac{\partial u_K}{\partial x_i}\varphi_{x_i}dxdt\\ &+\frac{1}{K}\int\int_{Q_T}S'_\eta(u_K-k)\frac{\partial u_K}{\partial x_i}\frac{\partial u_K}{\partial x_i}\varphi dxdt\\ &-\iint_{Q_T}S_\eta(u_K-k)\left[A^{ij}(u_K)-A^{ij}(k)\right]\varphi_{x_ix_j}dxdt\\ &-\iint_{Q_T}S'_\eta(u_K-k)\left[A^{ij}(u_K)-A^{ij}(k)\right]\frac{\partial u_K}{\partial x_i}\varphi_{x_j}dxdt\\ &+\int\int_{Q_T}S'_\eta(u_K-k)a^{ij}(u_K)\frac{\partial u_K}{\partial x_i}\frac{\partial u_K}{\partial x_j}\varphi dxdt\\ &+\int\int_{Q_T}S_\eta(u_K-k)\delta_\varepsilon * T_K(E_i)u_K\varphi_{x_i}dxdt\\ &+\iint_{Q_T}S'_\eta(u_K-k)\delta_\varepsilon * T_K(E_i)u_K\frac{\partial u_K}{\partial x_i}\varphi dxdt\\ &= 0. \end{aligned} \quad (45)$$

Notice that, on the left-hand side of (45), the second term trends to zero as $K\to\infty$, the third term is nonnegative, and by Lemma 2, the sixth term satisfies that

$$\begin{aligned}&\liminf_{K\to\infty}\iint_{Q_T}S'_\eta(u_K-k)a^{ij}(u_K)\frac{\partial u_K}{\partial x_i}\frac{\partial u_K}{\partial x_j}\varphi dxdt\\ &\ge \int\int_{Q_T}S'_\eta(u-k)\sum_{i=1}^{N}|\ g^i\ |^2\ \varphi dxdt. \end{aligned} \quad (46)$$

At the same time, for the other terms on the left-hand side of (45), we can deal with them as follows.

$$\iint_{Q_T}S'_\eta(u_K-k)\Big[A^{ij}(u_K)-A(k)\Big]\frac{\partial u_K}{\partial x_i}\varphi_{x_j}dxdt$$

$$+\iint_{Q_T} S_\eta(u_K-k)\left[A^{ij}(u_K)-A(k)\right]\varphi_{x_ix_j}dxdt$$
$$=-\iint_{Q_T}\int_k^{u_K} S'_\eta(s-k)\Big[A^{ij}(s)-A^{ij}(k\Big]ds\varphi_{x_ix_j}dxdt$$
$$+\iint_{Q_T} S_\eta(u_K-k)\Big[A^{ij}(u_K)-A^{ij}(k)\Big]\varphi_{x_ix_j}dxdt$$
$$=\iint_{Q_T}\int_k^{u_K} S_\eta(s-k)a^{ij}(s)ds\varphi_{x_ix_j}dxdt, \tag{47}$$

$$\int\int_{Q_T} S_\eta(u_K-k)\delta_\varepsilon * T_K(E_i)u_K\varphi_{x_i}dxdt$$
$$+\int\int_{Q_T} S'_\eta(u_K-k)\delta_\varepsilon * T_K(E_i)u_K\frac{\partial u_K}{\partial x_i}\varphi dxdt$$
$$=\int\int_{Q_T}[\int_k^{u_K} d(sS_\eta(s-k))\varphi_{x_i}$$
$$+\frac{\partial}{\partial x_i}\int_k^{u_K} sS'_\eta(s-k)ds\varphi]\delta_\varepsilon * T_K(E_i)dxdt$$
$$=\int\int_{Q_T}\left[\int_k^{u_K} S_\eta(s-k)ds+\int_k^{u_K} sS'_\eta(s-k)ds\right]$$
$$\varphi_{x_i}\delta_\varepsilon * T_K(E_i)dxdt$$
$$-\int\int_{Q_T}\int_k^{u_K} sS'_\eta(s-k)ds(\varphi_{x_i}\delta_\varepsilon * T_K(E_i)$$
$$+\varphi\delta_\varepsilon * T_{Ki}(E_i))dxdt$$
$$=\int\int_{Q_T}[\delta_\varepsilon * T_K(E_i)I_\eta(u_K-k)\varphi_{x_i}$$
$$-\int_k^{u_K} sS'_\eta(s-k)dsT_{Ki}(E_i)\varphi]dxdt. \tag{48}$$

where $T_{Ki}(E_i)=\frac{\partial(\delta_\varepsilon * T_K(E_i(x,t))}{\partial x_i}$.

At last, let $K\to\infty$ in (45). By (46)-(48), noticing that $E=\{E_i\}\in(L^2(Q_T))^N$ and div$E\in L^2(Q_T)$, we can get (6).

The proof of (8) is similar to that in references [10, 11] et al., we omit the details here.

5 Proof of Theorem 4

Using Vol′pert-Hudjaev's inspiring idea in [5], similar to the proof of Lemma 3.1 in [11], we are able to prove the following lemma, and we omit the details here.

Lemma 8 *Let u be a solution of (1)-(2). Then*

$$\int_{u^-}^{u^+} r^{ij}(s)ds\cdot v_i=0,\ j=1,2,\cdots,N;\ a.e.\ on\ \Gamma_u, \tag{49}$$

where (49) is true in the sense of N dimensional Hausdorff measure.

Proof of Theorem 4. Let u,v be two weak solutions of equation (1) with initial values

$$u(x,0)=u_0(x),\ v(x,0)=v_0(x).$$

By Definition 1, for any $\varphi\in C_0^2(Q_T),\ \varphi\ge 0,\ k,l\in\mathbf{R}$, we have

$$\int\int_{Q_T}[I_\eta(u-k)\varphi_t-E_i(x,t)I_\eta(u-k)\varphi_{x_i}$$
$$+A_\eta^{ij}(u,k)\varphi_{x_ix_j}]dxdt$$
$$-\int\int_{Q_T}[S'_\eta(u-k)\sum_{i=1}^N|g^i|^2\varphi$$
$$-\int_k^u sS'_\eta(s-k)dsE_{ix_i}\varphi]dxdt\ge 0, \tag{50}$$

$$\int\int_{Q_T}[I_\eta(v-l)\varphi_t-E_i(y,\tau)I_\eta(v-l)\varphi_{y_i}$$
$$+A_\eta^{ij}(v,l)\varphi_{y_iy_j}]dxdt$$
$$-\int\int_{Q_T}[S'_\eta(v-l)\sum_{i=1}^N|g^i|^2\varphi$$
$$-\int_l^v sS'_\eta(s-k)dsE_{iy_i}\varphi]dyd\tau\ge 0. \tag{51}$$

Let

$$\psi(x,t,y,\tau)\ge 0,\ \psi\in C^2(Q_T\times Q_T).$$

If for given $(\tau,y)\in Q_T$

$$supp\psi(\cdot,\cdot,\tau,y)\subset Q_T,$$

and if for given $(x,t)\in Q_T$,

$$supp\psi(x,t,\cdot,\cdot)\subset Q_T.$$

We choose

$$k=v(y,\tau),\ l=u(x,t),\ \varphi=\psi(x,t,y,\tau)$$

in (50) (51) respectively, integrate over Q_T, then

$$\iint_{Q_T}\iint_{Q_T}\{I_\eta(u-v)(\psi_t+\psi_\tau)-[E_i(x,t)\psi_{x_i}$$
$$+E_i(y,\tau)\psi_{y_i}]I_\eta(u-v)\}dxdtdyd\tau$$
$$+\int\int_{Q_T}\int\int_{Q_T}[A_\eta^{ij}(u,v)\psi_{x_ix_j}$$

$$+A_\eta^{ij}(v,u)\psi_{y_iy_j} - S'_\eta(u-v)$$
$$\sum_{i=1}^{N} | g^i(u) |^2 + \sum_{i=1}^{N} | g^i(v) |^2)\psi]dxdtdyd\tau$$
$$+\int\int_{Q_T}\int\int_{Q_T}[(E_{ix_i} - E_{iy_i})$$
$$\int_v^u sS'_\eta(s-v)ds\psi]dxdtdyd\tau \geq 0. \quad (52)$$

Let $\psi(x,t,y,\tau) = \phi(x,t)j_h(x-y,t-\tau)$. Here, $\phi(x,t) \geq 0$, $\phi(x,t) \in C_0^\infty(Q_T)$, and

$$j_h(x-y,t-\tau) = \omega_h(t-\tau)\Pi_{i=1}^N \omega_h(x_i - y_i),$$

$$\omega_h(s) = \frac{1}{h}\omega(\frac{s}{h})$$

$$\omega(s) \in C_0^\infty(R),\ \omega(s) \geq 0,\ \omega(s) = 0\ if\ \ | s |> 1,$$

$$\int_{-\infty}^{\infty}\omega(s)ds = 1.$$

Clearly,

$$\frac{\partial j_h}{\partial t} + \frac{\partial j_h}{\partial \tau} = 0,\ \ \frac{\partial j_h}{\partial x_i} + \frac{\partial j_h}{\partial y_i} = 0,$$

$$\frac{\partial \psi}{\partial t} + \frac{\partial \psi}{\partial \tau} = \frac{\partial \phi}{\partial t}j_h,\ \ \frac{\partial \psi}{\partial x_i} + \frac{\partial \psi}{\partial y_i} = \frac{\partial \phi}{\partial x_i}j_h.$$

Since $E_i \in L^2(Q_T)$ and $\psi \in C_0^\infty(Q_T \times Q_T)$, by the control convergent theorem, we have

$$\lim_{\eta\to 0}\int\int_{Q_T}\int\int_{Q_T}[E_i(x,t)\psi_{x_i}$$
$$+E_i(y,\tau)\psi_{y_i}]I_\eta(u-v)dxdtdyd\tau$$
$$= \int\int_{Q_T}\int\int_{Q_T}[E_i(x,t)\psi_{x_i}$$
$$+E_i(y,\tau)\psi_{y_i}]|u-v|dxdtdyd\tau$$
$$= \int\int_{Q_T}\int\int_{Q_T}[E_i(x,t)\psi_{x_i}$$
$$-E_i(y,\tau)(\psi_{x_i} - \phi_{x_i}j_h)]|u-v|dxdtdyd\tau.$$

Let $h \to 0$ in the above equality. We have

$$\lim_{h\to 0}\int\int_{Q_T}\int\int_{Q_T}[E_i(x,t)\psi_{x_i}$$
$$+E_i(y,\tau)\psi_{y_i}]|u-v|dxdtdyd\tau$$
$$= \int\int_{Q_T}E_i(x,t)|u-v|\phi_{x_i}dxdt. \quad (53)$$

At the same time, by the fact of that

$$\lim_{\eta\to 0} sS'_\eta(s) = 0,$$

and by the test function $\psi = \phi(x,t)j_h(x-y,t-\tau)$,

$$\lim_{h\to 0}\lim_{\eta\to 0}\int\int_{Q_T}\int\int_{Q_T}[(E_{ix_i} - E_{iy_i})$$
$$\int_v^u sS'_\eta(s-v)ds\psi]dxdtdyd\tau$$
$$= \lim_{h\to 0}\lim_{\eta\to 0}\int\int_{Q_T}\int\int_{Q_T}[(E_{ix_i} - E_{iy_i})$$
$$\int_v^u (s-v)S'_\eta(s-v)ds\psi]dxdtdyd\tau$$
$$+ \lim_{h\to 0}\lim_{\eta\to 0}\int\int_{Q_T}\int\int_{Q_T}(E_{ix_i} - E_{iy_i})$$
$$v\ sgn(u-v)\psi dxdtdyd\tau$$
$$= \lim_{h\to 0}\int\int_{Q_T}\int\int_{Q_T}(E_{ix_i} - E_{iy_i})v\ sgn(u-v)$$
$$\phi(x,t)j_h(x-y,t-\tau)dxdtdyd\tau = 0. \quad (54)$$

For the third term in the left-hand side of (52), we can deal with it as [10, 11], use Lemma 8, and get the following equality.

$$\lim_{\eta\to 0}\left[A_\eta^{ij}(u,v)\phi_{x_j}j_{hx_i} + A_\eta^{ij}(u,v)\phi_{y_j}j_{hy_i}\right] = 0. \quad (55)$$

Combing (52)-(55), and letting $\eta \to 0, h \to 0$ in (52), we get

$$\int\int_{Q_T}\{[u(x,t) - v(x,t)]\phi_t - |u-v|E_i(x,t)\phi_{x_i}$$
$$+sgn(u-v)\left[A^{ij}(u) - A^{ij}(v)\right]\phi_{x_ix_j}\}dxdt \geq 0. \quad (56)$$

Let

$$\eta(t) = \int_{\tau-t}^{s-t}\alpha_\varepsilon(\sigma)d\sigma,\ \ \varepsilon < \min\{\tau, T-s\}.$$

Here $\alpha_\varepsilon(t)$ is the kernel of mollifier with $\alpha_\varepsilon(t) = 0$ for $t \notin (-\varepsilon, \varepsilon)$.

By approximation, we can replace ϕ in (56) by $\phi(x,t) = \omega_\lambda(x)\eta(t)$, where $\omega_\lambda(x)$ is the function of (21), and $\eta(t) \in C_0^1(0,T)$. By the assumption of that

$$E_ix_i \geq 0,$$

we known that the second term of the left-hand side in (56) is non-positively, it can be drawn away, i.e. we have

$$\int\int_{Q_T}\{[u(x,t) - v(x,t)]\phi_t$$
$$+sgn(u-v)\left[A^{ij}(u) - A^{ij}(v)\right]\phi_{x_ix_j}\}dxdt \geq 0. \quad (57)$$

Using the estimate

$$| \omega_{\lambda x_ix_j}(x) |\leq C_\lambda\omega_\lambda(x),$$

we obtain from (57)

$$\int\int_{Q_T}[u(x,t)-v(x,t)]\,\phi_t dxdt$$

$$=\int\int_{Q_T}[u(x,t)-v(x,t)]]\,\omega_\lambda(x)$$

$$[-\alpha_\varepsilon(s-t)+\alpha_\varepsilon(\tau-t)]\,dxdt$$

$$=\int_{\mathbf{R}^N}[u(x,\tau)-v(x,\tau)]\,\omega_\lambda dx$$

$$-\int_{\mathbf{R}^N}[u(x,s)-v(x,s)]\,\omega_\lambda dx$$

$$\int\int_{Q_T}sgn(u-v)\left[(A^{ij}(u)-A^{ij}(v)\right]\phi_{x_ix_j}$$

$$\leq c\int_\tau^s\int_{\mathbf{R}^N}\mid u(x,t)-v(x,t)\mid\omega_\lambda(x)dxdt,$$

so

$$\int_{\mathbf{R}^N}\mid u(x,s)-v(x,s)\mid\omega_\lambda(x)dx$$

$$\leq\int_{\mathbf{R}^N}\mid u(x,\tau)-v(x,\tau)\mid\omega_\lambda(x)dx$$

$$+c\int_\tau^s\int_{\mathbf{R}^N}\mid u(x,t)-v(x,t)\mid\omega_\lambda(x)dxdt.$$

By Gronwall Lemma

$$\int_{\mathbf{R}^N}\mid u(x,s)-v(x,s)\mid\omega_\lambda(x)dx$$

$$\leq c\int_{\mathbf{R}^N}\mid u(x,\tau)-v(x,\tau)\mid\omega_\lambda(x)dx.$$

Let $\tau\to 0$. The proof of Theorem 4 is complete.

Acknowledgements: The paper is supported by NSF (no. 11371297 and no. 11326150) of China, supported by Education Department of Fujian Province (no: JB13154), supported by SF of Xiamen University of Technology, China.

References:

[1] P. Fabrie and T. Gallouet, Modeling wells in porous media flows. *Math. Models Methods Appl. Sci.* 10, 2000, pp. 673–709.

[2] L. Boccardo, L. Orsina and A. Porretta, Some noncoercive parabolic equations with lower order terms, *J. evol.equ.* 3, 2003, pp. 407–418.

[3] O. A. Oleinik and V. N. Samokhin, *Mathematical Models in boundary Layer Theorem*, Chapman and Hall/CRC, 1999.

[4] Z. Wu, J. Zhao, J. Yinand H. Li, *Nonlinear Diffusion Equations*, Word Scientific Publishing, Singapore, 2001.

[5] A. I. Vol′pert and S. I. Hudjaev, Cauchy's problem for degenerate second order quasilinear parabolic equations, *Mat. Sbornik*, (120)78, 1969, pp. 374-396. Engl. Transl., *Math.USSR Sb.*, (3)7, 1969, pp. 365–387.

[6] J. Zhao, Uniqueness of solutions of quasilinear degenerate parabolic equations, *Northeastern Math. J.*, (2)1, 1985, pp. 153–165.

[7] Z. Wu and J. Yin, Some properties of functions in BV_x and their applications to the uniqueness of solutions for degenerate quasilinear parabolic equations, *Northeastern Math. J.*, (4)5, 1989, pp. 395–422.

[8] H. Brezis and M. G. Crandall, Uniqueness of solutions of the initial value problem for $u_t-\Delta\varphi(u)=0$, *J. Math.Pures et Appl.*, 58, 1979, pp. 564–587.

[9] N. Kružkov, First order quasilinear equations in several independent variables, *Math. USSR-Sb.*, 10, 1970, pp. 217–243.

[10] H. Zhan and J. Zhao, Uniqueness and stability of solution for Cauchy problem of degenerate quasilinear parabolic equations in multi-space variables, *Chinese J. of Contemporary Math.*, 26, 2005. pp. 303–312.

[11] J. Zhao and H. Zhan, Uniqueness and stability of solution for Cauchy problem of degenerate quasilinear parabolic equations, *Science in China Ser. A*, 48, 2005, pp. 583–593.

[12] G. Q. Chen and B. Perthame, Well-Posedness for non-isotropic degenerate parabolic-hyperbolic equations, *Ann. I. H. Poincare-AN*, (4)420, 2003, pp. 645–668.

[13] M. Bendahmane and K. H. Karlsen, Reharmonized entropy solutions for quasilinear anisotropic degenerate parabolic equations, *SLAM J. Math. Annal.*, (2)36, 2004, pp. 405–422.

[14] K. H. Karlsen and M. Ohlberger, A note on the uniqueness of entropy solutions of nonlinear degenerate parabolic equations, *J. of Math. Ann. Appl.*, 275,2002, pp. 439–458.

[15] Ph. Bénilan, M. G. Crandall and M. Pierre, Solutions of the porous medium equation under optimal conditions on initial values. *Indiana Univ. Math. J.*, 33, 1994, pp. 51–87.

[16] J. Carrillo, Entropy solutions for nonlinear degenerate problems, *Arch. Rational Mech. Anal.*, 147, 1999, pp. 269–361.

[17] B. Cockburn and G. Gripenberg, Continuous dependence on the nonlinearities of solutions of degenerate parabolic equations, *J. Diff. Equations*, 151, 1999, pp. 231–251.

[18] G. Q. Chen and K. H. Karlsen, Quasilinear anisotropic degenerate parabolic equations with time-space degenerate diffusion coefficients, *Comm. Pure. Appl. Anal.*, (2)4, 2005, 241–267.

[19] Z. Wu and J. Zhao, The first boundary value problem for quasilinear degenerate parabolic equations of second order in several variables, *Chin. Ann. of Math.*, (3) 4B, 1983, pp. 319–358.

[20] A. I. Vol'pert, The spaces BV and quasilinear equations, *Mat.Sb.*, 73, 1967, pp. 255–302.

[21]] G. Anzellotti and M. Giaquinta, *Funzioni BV e trace*, Rend. Sem. Math. Padova 60, 1978, pp. 1–21.

[22] L. C. Evans, Weak convergence methods for nonlinear partial differential equations, *Conference Board of the Mathematical Sciences, Regional Conferences Series in Mathematics*, Number 74, 1998.

[23] H. Zhan, On existence and uniqueness of the Cauchy problem for parabolic equations with unbounded coefficients, *WSEAS Transactions on Mathematics*, 8, 2009, pp. 541–550.

[24] H. Zhan, Quasilinear degenerate parabolic equation from finance, *WSEAS Transactions on Mathematics*, 9, 2010, pp. 861–873.

[25] H. Zhan and L. Li, On the time-periodic solutions of a quasilinear degenerate parabolic equation, *Proceedings of the 7th IASME / WSEAS International Conference on Fluid Mechanics and Aerodynamics, FMA '09*, 2009, pp. 212–219.

[26] L. Li and H. Zhan, The study of micro-fluid boundary layer theory, *WSEAS Transactions on Mathematics*, (12)8, 2009, pp. 699–711.

[27] A.Novotný and I.Straškraba, *Introduction to the mathematical theory of compressible flow*, Oxford lecture series in mathematics and its applications, Oxford Univ. Press, 27, 2004.