Solutions of the KdV Equation through Analysis of Regular Symmetries.

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Abstract: - We investigate a case of the generalized Korteweg – De Vries Burgers equation. Our aim is to demonstrate the need for the application of further methods in addition to using Lie Symmetries. The solution is found through differential topological manifolds. We apply Lie's theory to take the PDE to an ODE. However, this ODE is of third order and not easily solvable. It is through differentiable topological manifolds that we are able to arrive at a solution.

Key-Words: - Korteweg-De Vries Equation, Lie Symmetry, Ordinary Differential Equations, Partial Differential Equations, Invariant Solutions, Differential Topological Manifolds.

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1 Introduction

The Korteweg - de Vries equation [1] is a mathematical model of waves on shallow water. It is a prototype example of a nonlinear partial differential equation (PDE) whose solutions can be exactly specified. The equation is given as:

$$u_t - uu_x - u_{xxx} = 0. \tag{1}$$

The most referred solution to (1) is the travelling wave solution. After the development of the KdV equation, it was only in 1960 that new application of the model equation was discovered [2]. Gardner and Morikawa [3] found this new model in the study of collision free hydro-magnetic waves

Since this time, many other applications have been found. Kruskal [4] and Zabusky [5] showed the KdV equation models longitudinal waves propogating one dimensional lattice of equal masses coupled by nonlinear springs, the Fermi-Pasta-Ulam problem. Berezin and Karpman [6] derived applications to plasma physics. Washimi and Taniuti [7] applied the equation in their study of ion-acoustic waves in a cold plasma. Wijngaarden [8] found that the KdV equation modelled the pressure waves in a liquid-gas mixture bubble. Nariboli [9] showed the equation modelled waves in elastic rods. The list goes on.

The KdV equation is used in plasma physics, [10] , the study of waves in shallow water [11] as well as in oceanographic studies [12]. The broad spectrum of applications of KdV equation and its ability to model scenarios spanning various sciences, has kept the KdV equation under the microscope of academic study. Most of the more recent research on the equation is driven by examining the solutions of the equation.

In this study we will go back to finding solutions to the KdV equation. In particular, we will look at the scale invariant solution determined through Lie Symmetries. The aim, with Lie Symmetries, is to convert a PDE to a solveable ODE. This study's objective is to determine exact solutions to (1), something that has never been done.

There have been solutions found to variations of the KdV equation [13] [14] [15], and KdV – like equations [16] [17] [18]. Berjawi, Elarwadi, and Israwi conducted a study of an extended KdV equation [19]. However, with regards to (1), and using Lie Symmetries, the scale invariant solution is incomplete. Olver [20], uses a transform developed by Miura [2]. This only leads to an equation that can be integrated once to then yield a second Painlevé transcendent. Thus we still don't have a solution.

A solution to (1) was determined in 1895 by Diederik Korteweg and Gustav de Vries [1] is given as:

$$u = \frac{c}{2}\operatorname{sech}^2\left[\frac{\sqrt{c}}{2}(x - ct - a)\right] \tag{2}$$

This solution does not include integration constants. This makes the result incomplete, or inappropriate [21].

There have been studies which have appended the traditional Lie approach. Ovsiaanikov [22] proposed Group Invariant solutions which he appended to the traditional Lie Approach. However, he still fell short of a solution. Other attempts can be found by Ibragimov and Gazizov [23], Ali, Qadir and Mahomed [24], as well as Momoniat and Mahomed, [25].

The start of our journey to a solution involves determining the Lie Symmetries [26]. This will be covered in Section 2. We will determine the scale invariant solutions based on the symmetries found. These solutions bring us to polynomials which require the use of manifolds to solve. We lay out the theory of manifolds in Section 3.

In Section 4, we make use of differential manifolds and build the solutions.

2 The Pure Lie Approach

In this section, we determine the symmetries for the modified KdV equation (1), by applying the algorithm developed by Lie [26]. The process is laid out in the following subsection.

2.1 The Infinitesimal Generator and Its Prolongations.

The infinitesimal generator of a first order PDE is given as:

$$X = \xi \frac{\partial}{\partial x} + \tau \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial u}. \tag{3}$$

Where ξ, τ, η are all functions of $x, t,$ and u.

The prolongations of the generator are given as:

$$X^{(1)} = \xi \frac{\partial}{\partial x} + \tau \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial u}. \tag{4}$$

$$X^{(2)} = X^{(1)} + \eta^{xx} \frac{\partial}{\partial u_{xx}} + \eta^{xt} \frac{\partial}{\partial u_{xt}} + \eta^{tt} \frac{\partial}{\partial u_{tt}}. \tag{5}$$

$$X^{(3)} = X^{(2)} + \eta^{xxx} \frac{\partial}{\partial u_{xxx}} + \eta^{xxt} \frac{\partial}{\partial u_{xxt}} + \eta^{xtt} \frac{\partial}{\partial u_{xtt}} + \eta^{ttt} \frac{\partial}{\partial u_{ttt}}. \tag{6}$$

To obtain the prolongations of a given transformation, we made use of the total derivative operator:

$$D_x = \partial x + u_x \partial u + u_{xx} \partial u_x + u_{xt} \partial u_t + \cdots. \tag{7}$$

$$D_t = \partial t + u_t \partial u + u_{xt} \partial u_x + u_{tt} \partial u_t + \cdots. \tag{8}$$

The prolongations are determined using the following formulae:

$$\eta^{Jx} = D_x \eta^J - u_{Jx} D_x \xi - u_{Jt} D_x \tau. \tag{9}$$

$$\eta^{Jt} = D_t \eta^J - u_{Jt} D_t \xi - u_{Jt} D_t \tau. \tag{10}$$

2.2 Finding the Symmetries

We make use of the third prolongation (6) and expand Lies Invariance condition. Lie's invariance condition is given as:

$$X^{(3)}(u_t - uu_x - u_{xxx}) = 0. \tag{11}$$

When $u_{xxx} = u_t - u_x$.

We expand (11) and arrive at:

$$\eta u_x + u\eta^x - \eta^t + \eta^{xxx} = 0. \tag{12}$$

We apply Lie's invariance criterion, expand the expression and replace all occurrences of u_{xxx} with $u_t - uu_x$. By equating the coefficients of various monomials to zero, we are left with the defining equations for the symmetry group. The coefficients usually yield an over-determined system of equations.

We begin by looking at the following monomials and equating their coefficients to zero:

$$\boldsymbol{u_{xx}u_{xt}}: \quad \tau_u = 0 \tag{13}$$

$$\boldsymbol{u_{xxt}}: \quad \tau_x = 0 \tag{14}$$

$$\boldsymbol{u_{xx}^2}: \quad \xi_u = 0 \tag{15}$$

$$\boldsymbol{u_x u_{xx}}: \quad \eta_{uu} - 3\xi_{xu} = 0 \tag{16}$$

From (13) and (14), we conclude τ is a function of t. From (16), we determine ξ is a function of x and t. We apply (15) to (16), thus simplifying (16) to: $\eta_{uu} = 0$.

We now introduce arbitrary functions $a(t), b(x,t), c(x,t)$ and $d(x,t)$ to write out an outline of what the infinitesimals will look like:

$$\eta = c(x,t)u + d(x,t). \tag{17}$$

$$\xi = b(x,t). \quad (18)$$

$$\tau = a(t). \quad (19)$$

We continue our examination of the monomials and their coefficients, this time using equations (17), (18), and (19) to define the functions a, b, c, d. We will be introducing constants of integrations A_i and B_i.

$$\boldsymbol{u_x^0}: uc_{xxx} + d_{xxx} + u^2 c_x + ud_x - uc_t - d_t. \quad (20)$$

$$\boldsymbol{u_t}: a'(t) - 3b_x = 0. \quad (21)$$

$$\boldsymbol{u_x}: uc(x,t) + d(x,t) + 2ub_x + b_t + 3c_{xx} - b_{xxx} = 0. \quad (22)$$

$$\boldsymbol{u_{xx}}: c_x - b_{xx} = 0. \quad (23)$$

We focus our attention on the coefficients of the powers of u from equation (20):

$$\boldsymbol{u^0}: d_{xxx} - d_t = 0. \quad (24)$$

$$\boldsymbol{u}: c_{xxx} + d_x - c_t = 0. \quad (25)$$

$$\boldsymbol{u_{xx}}: c_x - b_{xx} = 0. \quad (26)$$

From equation (26), we define c further:

$$c = c(t) + A_1. \quad (27)$$

We substitute (27) into (26):

$$b_{xx} = 0. \quad (28)$$

Taking into account (27) and (28), we consider the coefficients of the powers of u from (22):

$$\boldsymbol{u^0}: \quad d + b_t = 0. \quad (29)$$

$$\boldsymbol{u}: \quad c(t) + A_1 + 2b_x = 0. \quad (30)$$

From equation (30):

$$2b_x = -c(t) - A_1, \quad (31)$$

$$b_x = -\tfrac{1}{2}c(t) - \tfrac{1}{2}A_1, \quad (32)$$

$$b_{xt} = -\tfrac{1}{2}c'(t). \quad (33)$$

We substitute (27) into (25):

$$d_x = c'(t). \quad (34)$$

From (29), we determine d_x:

$$d + b_t = 0. \quad (35)$$

$$d = -b_t. \quad (36)$$

$$d_x = -b_{xt}. \quad (37)$$

From (33), (34), and (37):

$$c'(t) = -\frac{1}{2}c'(t). \quad (38)$$

$$c'(t) = 0. \quad (39)$$

$$c(t) = A_2. \quad (40)$$

We can now state

$$c = B_1. \quad (41)$$

We simplify (24) by applying (34) and (39):

$$d_t = 0, \quad (42)$$

$$d = f(x) + A_3. \quad (43)$$

where $f(x)$ is a function of integration.

However, when we substitute (37) into (25):

$$d_x = 0. \quad (44)$$

$$d = h(t) + A_4. \quad (45)$$

We now equate (43) and (45) and simplify:

$$f(x) + A_3 = h(t) + A_4. \quad (46)$$

$$f(x) = h(t) = A_5. \quad (47)$$

$$d = B_2. \quad (48)$$

We begin defining $b(x,t)$ by looking at (36) and (48):

$$b_t = -B_2, \quad (49)$$

$$b = k(x) - B_2 t. \quad (50)$$

for arbitrary function $k(x)$.

However, from (32) and (41) we have:

$$b_x = -\frac{1}{2}B_1, \quad (51)$$

$$b = m(t) - \frac{1}{2}B_1 x. \quad (52)$$

For arbitrary function $m(t)$.

We now are faced with two definitions of $b(x,t)$ which we need to reconcile. We equate (50) and (52):

$$k(x) - B_2 t = m(t) - \frac{1}{2}B_1 x, \quad (53)$$

$$k(x) = -\frac{1}{2}B_1 x, \quad (54)$$

$$m(t) = -B_2 t, \quad (55)$$

$$b = -\frac{1}{2}B_1 x - B_2 t + B_3. \quad (56)$$

To determine $a(t)$ we substitute (56) into (21):

$$a'(t) = \frac{3}{2}B_1, \quad (57)$$

$$a(t) = \frac{3}{2}B_1 t + B_4. \quad (58)$$

We eliminate the fractions associated with B_1 and replace the constants as follows:

$$a(t) = 3C_1 t + C_4. \quad (59)$$

$$b(x,t) = C_1 x - C_2 t + C_3. \quad (60)$$

$$c = -2C_1. \quad (61)$$

$$d = C_2. \quad (62)$$

2.2.1 The Infinitesimals and Symmetries

The infinitesimals have now been fully determined:

$$\eta = -2C_1 u + C_2. \quad (63)$$

$$\xi = C_1 x - C_2 t + C_3. \quad (64)$$

$$\tau = 3C_1 t + C_4. \quad (65)$$

The infinitesimals give us our symmetries:

$$X_1 = x\frac{\partial}{\partial x} + 3t\frac{\partial}{\partial t} - 2u\frac{\partial}{\partial u}. \quad (66)$$

$$X_2 = -t\frac{\partial}{\partial x} + \frac{\partial}{\partial u}. \quad (67)$$

$$X_3 = \frac{\partial}{\partial x}. \quad (68)$$

$$X_4 = \frac{\partial}{\partial t} \quad (69)$$

2.2.2 The Commutator Table

Consider two symmetry generators, X_i and X_j, the commutator of X_i with X_j is defined as:

$$[X_i, X_j] = X_i X_j - X_j X_i. \quad (70)$$

From the definition of the commutator, the following properties are evident:

$$[X_i, X_j] = -[X_j, X_i]. \quad (71)$$

$$[X_i, X_i] = 0. \quad (72)$$

The commutators for our symmetries are listed in the commutator table:

	X_1	X_2	X_3	X_4
X_1	0	$2X_2$	$-X_3$	$-3X_4$
X_2	$-2X_2$	0	0	X_3
X_3	X_3	0	0	0
X_4	$3X_4$	$-X_3$	0	0

Table 1: Commutator Table

The linear space spanned by X_1, X_2, X_3, X_4 is a Lie Algebra with a skew symmetric operator. From Table 1 we can see that X_2, X_3, X_4 commute. For a symmetry $Y \in \mathcal{L}^3 = \{X_2, X_3, X_4\}$, a subalgebra, we have:

$$[X_1, Y] \in \mathcal{L}^3, \quad (73)$$

From this we have:

$$\{X_2\} \subset \{X_2, X_3\} \subset \{X_2, X_3, X_4\} \subset \{X_1, X_2, X_3, X_4\} = \mathcal{L}. \tag{74}$$

This tells us that $\mathcal{L}$ is solvable.

2.3 Invariant Solution Through X_1

In this section we will detail the calculations in determining the invariant solution through (66)

$$X_1 = x\frac{\partial}{\partial x} + 3t\frac{\partial}{\partial t} - 2u\frac{\partial}{\partial u}$$

The characteristic equation we are required to solve is:

$$\frac{dx}{x} = \frac{dt}{3t} = \frac{du}{-2u}. \tag{75}$$

2.3.1 The Invariants

We begin by solving:

$$\frac{dx}{x} = \frac{dt}{3t}, \tag{76}$$

$$\ln x = \tfrac{1}{3}\ln t + \ln C_1, \tag{77}$$

$$C_1 = xt^{-\frac{1}{3}}. \tag{78}$$

This gives us our first invariant:

$$y = xt^{-\frac{1}{3}}. \tag{79}$$

To determine our second invariant, we look at:

$$\frac{dt}{3t} = \frac{du}{-2u}, \tag{80}$$

$$-\tfrac{2}{3}\ln t + \ln C_2 = \ln u, \tag{81}$$

$$C_2 = ut^{\frac{2}{3}}. \tag{82}$$

Our second invariant is given as:

$$v = ut^{\frac{2}{3}}. \tag{83}$$

From (83), we write u in terms of v:

$$u = vt^{-\frac{2}{3}}. \tag{84}$$

From (84), we determine:

$$u_x = \frac{1}{t}v_y, \tag{85}$$

$$u_{xxx} = t^{-\frac{5}{3}}v_{yyy}, \tag{86}$$

$$u_t = -\tfrac{1}{3}t^{-\frac{5}{3}}(2v + yv_y). \tag{87}$$

Now we substitute (85), (86), (87) into (1)

$$-\frac{1}{3}t^{-\frac{5}{3}}(2v + yv_y) - vt^{-\frac{2}{3}}(t^{-1}v_y) - t^{-\frac{5}{3}}v_{yyy} = 0, \tag{88}$$

$$-\tfrac{1}{3}(2v + yv_y) - vv_y - v_{yyy} = 0, \tag{89}$$

$$v_{yyy} + vv_y + \tfrac{1}{3}yv_y + \tfrac{2}{3}v = 0. \tag{90}$$

We have now taken the original PDE and transformed it to an ODE. However, this third order ODE requires more work to solve. This solution is given, in detail, in Section 4. We first lay the ground work for the technique used in Section 4.

3 The Differential Topological Manifolds Basis

Our approach to the solution of (90) is borrowed from the method of variation of parameters. This procedure is often used to solve second order non-homogeneous linear ODEs:

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x). \tag{91}$$

3.1 The Variation of Parameters Method

The usual steps involved in solving (91) requires first setting:

$$f(x) = 0. \tag{92}$$

so that:

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0. \tag{93}$$

The homogeneous version of the ODE, the result to (93) is given as:

$$y_c = C_1 y_1 + C_2 y_2. \tag{94}$$

This is called the complementary solution. The constants C_1 and C_2 are parameters that have to be varied. At some stage we will need to set:

$$v_i = C_i, i = 1,2. \tag{95}$$

These lead to the particular solution:

$$y_p = v_1 y_1 + v_2 y_2 \tag{96}$$

So that the general solution is given as:

$$y = y_c + y_p \tag{97}$$

We take two assumptions to the next subsection and beyond. The assumption giving rise to (93) will be interpreted as describing points within quotient spaces, leading to (127). The second assumption, the one leading to (95), relates this space to the entire differentiable topological manifold, where it is located. It leads to (103) and (104) which generates (134).

3.2 Differentiable Topological Manifolds

We start with a topological space $M = (X, J_X)$, a Hausdroff topology. That is, a set X with a topology J_X. For it to be a differentiable topological manifold, or simply a differentiable manifold, we require an atlas A as well, then we have $DM = (X, J_x, A)$.

We now consider two points $p \in U_p$ and $q \in U_q$ such that the sets U_p and U_q are elements of the same manifold. We can then build the sub-topologies $(U_p, J_X \mid_{U_P})$ and $(U_q, J_x \mid_{U_q})$. That is, the topology of X restricted to U_p and U_q. A mapping φ_p, if it exists, maps the space $(U_p, J_X \mid_{U_P})$ into the Euclidean space $(R^N, \mathcal{J}_{\mathcal{R}^N}|_{\varphi_p(\mathcal{U}_p)})$. Similarly, φ_q maps $(U_q, J_x \mid_{U_q})$ into the Euclidean space $(\mathbb{R}^N, \mathcal{J}_{\mathcal{R}^N}|_{\varphi_q(\mathcal{U}_q)})$.

If these mappings are homomorphisms, then the set A, with:

$$A = (U_p, \varphi_p), (U_q, \varphi_q) \tag{98}$$

is called an atlas, with φ_p, φ_q called coordinates.

Our interest is in one of the charts mapping equivalence classes. That is:

$$A = ([U_p], [\varphi_p]), (U_q, \varphi_q) \tag{99}$$

Similarly, for manifolds of derivatives of φ, we get the atlases:

$$\mathcal{A}^{(i)} = \left([U_p], \left[\varphi_p^{(i)}\right]\right), \left(U_q, \varphi_q^{(i)}\right) \tag{100}$$

3.2.1 Transmission Mappings

The mapping from $\left(\mathbb{R}, J_{\mathbb{R}}|_{[\varphi([U_p])]}\right)$ to $\left(\mathbb{R}, J_{\mathbb{R}}|_{\varphi([U_q])}\right)$ having stepped down from $\mathbb{R}^N$ to $\mathbb{R}$, is given by:

$$\varphi_p\left(\varphi_q^{-1}\left(\varphi_q([U_p])\right)\right) \tag{101}$$

and it is called a transition mapping. Its inverse is:

$$\varphi_q\left(\varphi_p^{-1}\left(\varphi_p(U_q)\right)\right) \tag{102}$$

We are interested in case(s) where $[U_p]$ and U_q overlap, so that there is a point x in the neighborhood of both p and q such that:

$$[\varphi[x]] = \varphi(x) \tag{103}$$

The transmission mappings in derivative spaces lead to:

$$\frac{d^n[\varphi[x]]}{dx^n} = \frac{d^n\varphi(x)}{dx^n} \tag{104}$$

for $n = 1, 2, 3, \ldots$

3.2.2 Tangent Spaces

As indicated earlier, tangent spaces assist in establishing a function f, that allows for results to be projected onto the metric space.

A tangent space is a set:

$$TP = \{V_{\gamma,P} | \gamma : \mathbb{R} \to X\} \tag{105}$$

Such that:

$$V_{\gamma,P} f = (f \circ \gamma^{-1})[\gamma(\tau_0)] \tag{106}$$

where $f \in C^\infty(X), V_{\gamma,P}: C^\infty(M) \to \mathbb{R}, \gamma(\tau_0) = P$.

The tangent space TP has the basis vectors $\{\partial X^i\}$. Any vector can be represented in terms of it, so that

$$X = \xi^i \frac{\partial}{\partial X^i}|_P \tag{107}$$

That is $X \in T_pX = T_pM$

3.2.3 Cotangent Spaces

A tangent space is a vector space, and where there is one there should also be a co-vector space, hence the cotangent space. It is the set of all maps in the tangent space to $\mathbb{R}$. That is:

$$\omega: T_pX \to \mathbb{R} \tag{108}$$

With ω being an element of the cotangent space. The symbol $(df)_p$ represents a co-vector acting on mapping f at P. A cotangent space, therefore, is:

$$TP^* = \{(df)_p | f \in C^\infty(X)\} \tag{109}$$

and it is a vector space and is the dual of TP.

The basis vectors of a cotangent space requires that:

$$(d\omega^j)_p \left(\frac{\partial}{\partial x^i}\right)|_P = \delta_i^j \tag{110}$$

so that:

$$(\mathcal{TP}^*) = \left\{\frac{\partial}{\partial x^i}\right\}|_p \tag{111}$$

Therefore, an element ω of TP^* can be written:

$$\omega = \omega_i(dx^i)|_p \tag{112}$$

3.3 Quotient Spaces

Consider the general ordinary differential equation:

$$f(x, \varphi, \varphi', \varphi'', \varphi^{(3)} \dots) \tag{113}$$

with

$$\varphi: X \to Y \tag{114}$$

A set:

$$S = \{x_0, x_1, x_2, \dots\} \subset X \tag{115}$$

such that:

$$x_i = P(x_j) = x_j + 2\pi k_s \tag{116}$$

where k_s is an integer, is called an equivalence class. This leads to a Quotient Space $\mathbb{R}/\sim$. It is the set of all equivalent classes in $\mathbb{R}$, and is given by:

$$\mathbb{R}/\sim = \{[x_0], [x_1], [x_2], \dots\} \tag{117}$$

It is a differentiable topological space. In our study, the image of this set, is also an equivalence class:

$$S = \{[\varphi(x_0)], [\varphi(x_1)], [\varphi(x_2)], \dots\} \tag{118}$$

as such there is a homomorphism, and it extends to the derivative spaces:

$$T = \{[\varphi^{(i)}(x_0)], [\varphi^{(i)}(x_1)], [\varphi^{(i)}(x_2)], \dots\} \tag{119}$$

for $i = 1, 2, 3, \dots$

4 The Solution through X_1

We isolate $v(y)$ from (90):

$$v(y) = \frac{-yv' - 3v^{(3)}}{2 + 3v'}. \tag{120}$$

We differentiate (90):

$$v^{(4)}y + vv'' + \frac{1}{3}yv'' + (v')^2 + v' = 0. \tag{121}$$

We isolate v'' from (121):

$$v''(y) = \frac{3(2 + 3v')(v' + (v')^2 + v^{(4)})}{2y - 9v^{(3)}} \tag{122}$$

We differentiate (121):

$$v^{(5)} + vv^{(3)} + \frac{1}{3}yv^{(3)} + 3v'v'' + \frac{4}{3}v'' = 0 \tag{123}$$

To use the method of differential manifolds we require generating equations of the form:

$$[q''] = \omega^2[q]. \tag{124}$$

We are looking for:

$$[q^{(n+2)}] = [q^n] = 0. \tag{125}$$

We substitute (120) and (122) into (123) and determine the equivalence class generators to be:

$$[q'] = v' + \frac{2}{3}. \tag{126}$$

Where $v' = -\frac{2}{3}$

From (126) we have $q^{(3)} = 0$
The condition laid out in (125) has now been satisfied.

From L'Hopitals Rule,

$$\frac{[q^{(3)}]}{[q']} = \frac{[q^{(4)}]}{[q'']} \tag{127}$$

From (127)

$$\frac{[q^{(3)}]}{[q']} = \frac{[q^{(4)}]}{[q'']} \tag{128}$$

$$\frac{[q^{(3)}]}{[q^{(4)}]} = \frac{[q']}{[q'']} \tag{129}$$

$$\frac{d}{dy} ln[q^{(3)}] = \frac{d}{dy} ln[q'] \tag{130}$$

$$ln[q^{(3)}] = ln[q'] + ln\, k \tag{131}$$

$$[q^{(3)}] = k[q'] \tag{132}$$

We let $k = \omega^2$ so (132)can be written as:

$$[q^{(3)}] = \omega^2 [q'] \tag{133}$$

From (133), we have:

$$[q'] = \frac{a}{i\omega} sin(i\omega y + \alpha) \tag{134}$$

Given the condition, (126), we have:

$$\frac{a}{i\omega} sin(i\omega y + \alpha) = -\frac{2}{3}, \tag{135}$$

$$sin(i\omega y + \alpha) = \frac{-2}{3}\left(\frac{i\omega}{a}\right). \tag{136}$$

From (136) we calculate:

$$cos(i\omega y + \alpha) = \sqrt{1 - \left(\frac{-2}{3}\left(\frac{i\omega}{a}\right)\right)^2} \tag{137}$$

We take:

$$v = \frac{a}{i\omega} sin(i\omega y + \alpha) + \frac{2}{3}, \tag{138}$$

In terms of elements in our equivalence class conditions, (138) is written as:

$$[v] = 0. \tag{139}$$

We differentiate (138)

$$v' = \mathrm{a}\, cos(i\omega y + \alpha)\,, \tag{140}$$

The equivalence class element is:

$$[v'] = \mathrm{a}\sqrt{1 - \left(\frac{-2}{3}\left(\frac{i\omega}{a}\right)\right)^2}. \tag{141}$$

We differentiate (140) :

$$v'' = -a(i\omega)\, sin(i\omega y + \alpha) \tag{142}$$

The equivalence class element is:

$$[v''] = \frac{2}{3}(i\omega)^2 \tag{143}$$

We differentiate(142):

$$v^{(3)} = -a\,(i\omega)^2 cos(i\omega y + \alpha) \tag{144}$$

The equivalence class element is:

$$[v^{(3)}] = -a(i\omega)^2\sqrt{1 - \left(\frac{-2}{3}\left(\frac{i\omega}{a}\right)\right)^2} \tag{145}$$

We differentiate (144):

$$v^{(4)} = a(i\omega)^3\, sin(i\omega y + \alpha) \tag{146}$$

The equivalence class element is given as:

$$[v^{(4)}] = \frac{-2}{3}(i\omega)^4 \tag{147}$$

We now substitute (138), (140) and (144) into (90) and integrate once, and then substitute (136) and (137). To make the result easier to read, we make the substitutions:

$$\lambda = a\sqrt{\frac{9a^2-4(i\omega)^2}{a^2}}, \tag{148}$$

$$\omega^2 = \lambda^2 - 9a^2 \tag{149}$$

The numerator yields:

$$\lambda^2(-27a^2 + 36C_1 + 8y - 8) - 324a^2C_1 - 72a^2y + \frac{243a^4}{2} + 63a^2 + \frac{3\lambda^4}{2} - 4 \tag{150}$$

We integrate a second time, making the same substitutions and get:

$$-162a^2C_1y - 162a^2C_2 - 36a^2y^2 + \lambda(-54a^2 - 2y - 3) \tag{151}$$

$$+\lambda^2(18C_1y + 18C_2 + 4y^2) + 6\lambda^3$$

We solve for a^2 by equating (151) to 0:

$$a^2 = \frac{18C_2\lambda^2 + 18C_1\lambda^2y + 6\lambda^3 - 3\lambda + 4\lambda^2y^2 - 2\lambda y}{18(9C_1y + 9C_2 + 3\lambda + 2y^2)} \tag{152}$$

We now can solve for λ, α and ω:

$$\lambda = -\frac{2}{9}(6 + 2y^2 + 9yC_1 + 9C_2) - \tag{153}$$

$$(1 + i\sqrt{3})\left(\frac{f_1}{216 \times 2^{\frac{2}{3}} f_3^{\frac{1}{3}}}\right) + \frac{1}{432 \times 2^{\frac{1}{3}}}(1 - i\sqrt{3})f_3^{\frac{1}{3}}.$$

Where:

$$f_1 = -20736C_1y^3 - 46656C_1^2y^2 \tag{154}$$
$$-20736C_2y^2 - 62208C_1y - 93312C_1C_2y$$
$$-46656C_2^2 - 279936C_1 + 124416C_2$$
$$-2304y^4 - 16416y^2 - 33696y - 34344$$

$$f_2 = 5785344 + 29113344y \tag{155}$$
$$+23701248y^2 - 1492992y^3$$
$$-1244160y^4 - 221184y^6$$
$$+241864704C_1 + 96577920C_1y$$
$$-47029248C_1y^2 - 14556672C_1y^3$$
$$-2985984C_1y^5 - 181398528C_1^2y$$
$$-40310784C_1^2y^2 - 13436928C_1^2y^4$$
$$-20155392C_1^3y^3 - 64665216C_2$$
$$-6718464C_2y + 12317184C_2y^2$$
$$-2985984C_2y^4 - 181398528C_1C_2$$
$$+40310784C_1C_2y - 26873856C_1C_2y^3$$
$$-60466176C_1^2C_2y^2 + 80621568C_2^2$$

$$-13436928C_2^2y^2 - 60466176C_1C_2^2y$$

$$-20155392C_2^3 +$$

$$f_3 = f_2 + \sqrt{4f_1^3 + f_2^2} \quad (156)$$

We can solve for a using (152) and (153).
We can solve for ω using (149) and (153).
We integrate a third time, now substituting our solutions for λ, a and ω.

We let $C_1 = 0, C_2 = -1, C_3 = 1$, to get a solution for $v(y)$. We can now substitute our solution for $v(y)$ into (84) to obtain a solution to (1). This solution is illustrated as a 2D plot in
Figure 1, and as a 3D plot in
Figure 2.
The presence of the constants lifts our solution over those of the founding fathers. The original solution did not have integration constants, which, in the theory of differential equations, is an anomaly. [21]

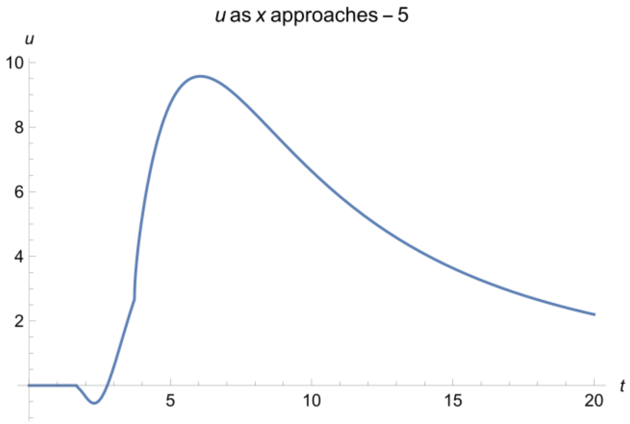

Figure 1: Plot of the solution from (83)

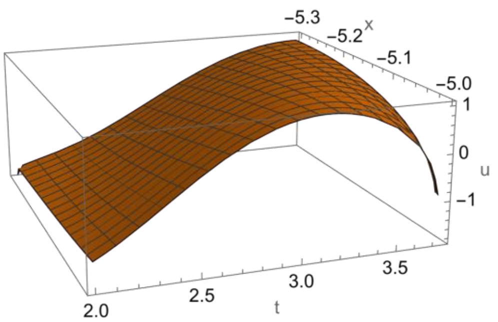

Figure 2: 3D Plot of the solution from (83)

5 Conclusion

We began with finding the Lie Symmetries for (1). Our intention was to find solutions through (66). The invariant solution yielded an ODE, (90), that was not simple to solve. Though Lie Symmetries help us determine solutions, it is not a one size fits all algorithm. We have demonstrated here, that the there are times where we need to apply aggressive techniques to determine solutions. Through differential manifolds we have established solutions to (90) which we used to determine solutions to (1). The solutions are numerous, this even after simplifying our case. The plot of our solution,
Figure 1, gives us a visual that is far more pleasing to the eye than the lengthy equation it illustrates. The 3D plot, shown in
Figure 2, provides a clear visual of the rise and fall of the wave form.
Through the application of manifolds, we have determined solutions where the traditional Lie approach could not. We have also determined an exact solution to (1), which includes integration constants, making our solution mathematically sound.
The results illustrated here, open the door to us examining previously solved equations. It gives us a new set of tools, a new algorithm to apply. There is the exciting possibility of finding new/ additional solutions. One such extension to consider is to consider the KdV equation where all spatial variables are considered. That is the case where $u = u(t, x, y, z)$. This would provide a connection to uses of the equation in industry.

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