Applications of Borel Distribution for a New Family of Bi-Univalent Functions Defined by Horadam Polynomials

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Abstract: - In this paper, by making use of Borel distribution we introduce a new family $\mathcal{G}_\Sigma(\delta,\gamma,\lambda,\tau,r)$ of normalized analytic and bi-univalent functions in the open unit disk U, which are associated with Horadam polynomials. We establish upper bounds for the initial Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$ of functions belonging to the analytic and bi-univalent function family which we have introduced here. Furthermore, we establish the Fekete-Szegö problem of functions in this new family.

Key-Words: - Bi-univalent function, Bazilevič function, λ-Pseudo-starlike function, Borel distribution, Horadam polynomials, Upper bounds, Fekete-Szegö problem.

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1 Introduction

Indicate by $\mathcal{A}$, the collection of analytic functions in the open unit disk $U=\{z\in\mathbb{C}:|z|<1\}$ that have the form:

$$f(z)=z+\sum_{n=2}^{\infty}a_nz^n. \tag{1}$$

.

Further, assume that S stands for the subcollection of the set $\mathcal{A}$ containing of functions in U satisfying (1) which are univalent in U.

A function $f\in\mathcal{A}$ is called Bazilevič function in U if (see [26])

$$Re\left\{\frac{z^{1-\gamma}f'(z)}{(f(z))^{1-\gamma}}\right\}>0,\quad(z\in U,\gamma\geq 0).$$

A function $f\in\mathcal{A}$ is called λ-pseudo-starlike function in U if (see [7])

$$Re\left\{\frac{z\,(f'(z))^{\lambda}}{f(z)}\right\}>0,\quad(z\in U,\lambda\geq 1).$$

The elementary distributions such as the Poisson, the Pascal, the Logarithmic, the Binomial, the beta negative binomial have been partially studied in Geometric Function Theory from a theoretical point of view (see for example [6, 11, 22, 24, 40]).

Very recently, Wanas and Khuttar [42] introduced the following power series whose coefficients are probabilities of the Borel distribution:

$$\mathcal{M}(\tau,z)=z+\sum_{n=2}^{\infty}\frac{(\tau(n-1))^{n-2}\,e^{-\tau(n-1)}}{(n-1)!}z^n$$
$$(z\in U;\ 0<\tau\leq 1).$$

We note by the familiar Ratio Test that the radius of convergence of the above series is infinity.

The linear operator $\mathcal{B}_\tau : \mathcal{A} \longrightarrow \mathcal{A}$ is defined as follows (see [42])

$$\mathcal{B}_\tau f(z) = \mathcal{M}(\tau, z) * f(z) =$$

$$z + \sum_{n=2}^{\infty} \frac{(\tau(n-1))^{n-2} e^{-\tau(n-1)}}{(n-1)!} a_n z^n z \in U,$$

where $(*)$ indicate the Hadamard product of two series.

According to the Koebe One-Quarter Theorem [10] every function $f \in S$ has an inverse f^{-1} defined by $f^{-1}(f(z)) = z$, $(z \in U)$ and $f(f^{-1}(w)) = w$, $(|w| < r_0(f), r_0(f) \geq \frac{1}{4})$, where

$$g(w) = f^{-1}(w) = w - a_2 w^2 + \left(2a_2^2 - a_3\right) w^3 -$$

$$\left(5a_2^3 - 5a_2 a_3 + a_4\right) w^4 + \cdots . \tag{2}$$

A function $f \in \mathcal{A}$ is said to be bi-univalent in U if both f and f^{-1} are univalent in U. Let Σ stands for the class of bi-univalent functions in U given by (1).

Srivastava et al. [29] have actually revived the study of analytic and bi-univalent functions in recent years, it was followed by such works as those by Bulut [8], Adegani et al. [2], Güney et al. [12], Srivastava and Wanas [30] and others (see, for example [9, 16, 19, 23, 25, 27, 31, 34, 35, 36, 37, 38, 44]).

We notice that the class Σ is not empty. For example, the functions z, $\frac{z}{1-z}$, $-\log(1-z)$ and $\frac{1}{2}\log\frac{1+z}{1-z}$ are members of Σ. However, the Koebe function is not a member of Σ. Until now, the coefficient estimate problem for each of the following Taylor-Maclaurin coefficients $|a_n|$, $(n = 3, 4, \cdots)$ for functions $f \in \Sigma$ is still an open problem.

Let the functions f and g be analytic in U. We say that the function f is subordinate to g, if there exists a Schwarz function ω analytic in U with $\omega(0) = 0$ and $|\omega(z)| < 1$ $(z \in U)$ such that $f(z) = g(\omega(z))$. This subordination is denoted by $f \prec g$ or $f(z) \prec g(z)$ $(z \in U)$. It is well known that (see [21]), if the function g is univalent in U, then $f \prec g$ if and only if $f(0) = g(0)$ and $f(U) \subset g(U)$.

The Horadam polynomials $h_n(r)$ are defined by the following repetition relation (see [14]):

$$h_n(r) = pr h_{n-1}(r) + q h_{n-2}(r) \tag{3}$$

$$(r \in \mathbb{R}, n \in \mathbb{N} = \{1, 2, 3, \cdots\}),$$

with $h_1(r) = a$ and $h_2(r) = br$, for some real constant a, b, p and q. The characteristic equation of repetition relation (3) is $t^2 - prt - q = 0$. This equation has two real roots $x = \frac{pr+\sqrt{p^2r^2+4q}}{2}$ and $y = \frac{pr-\sqrt{p^2r^2+4q}}{2}$.

Remark 2.1. By selecting the particular values of a, b, p and q, the Horadam polynomial $h_n(r)$ reduces to several polynomials. Some of them are illustrated below:

1. Taking $a = b = p = q = 1$, we obtain the Fibonacci polynomials $F_n(r)$;
2. Taking $a = 2$ and $b = p = q = 1$, we attain the Lucas polynomials $L_n(r)$;
3. Taking $a = q = 1$ and $b = p = 2$, we have the Pell polynomials $P_n(r)$;
4. Taking $a = b = p = 2$ and $q = 1$, we get the Pell-Lucas polynomials $Q_n(r)$;
5. Taking $a = b = 1$, $p = 2$ and $q = -1$, we obtain the Chebyshev polynomials $T_n(r)$ of the first kind;
6. Taking $a = 1$, $b = p = 2$ and $q = -1$, we have the Chebyshev polynomials $U_n(r)$ of the second kind.

These polynomials, the families of orthogonal polynomials and other special polynomials, as well as their generalizations, are potentially important in a variety of disciplines in many of sciences, specially in the mathematics, statistics and physics. For more information associated with these polynomials see [13, 14, 17, 18].

The generating function of the Horadam polynomials $h_n(r)$ (see [15]) is given by

$$\Pi(r, z) = \sum_{n=1}^{\infty} h_n(r) z^{n-1} = \frac{a + (b - ap)rz}{1 - prz - qz^2}. \tag{4}$$

Srivastava et al. [28] have studied the Horadam polynomials in a similar context involving analytic and bi-univalent functions, it was followed by such works as those by Al-Amoush [3], Wanas and Alb Lupaş [43], Abirami et al. [1] and others (see, for example, [4, 5, 20, 32, 33, 39, 45]).

In this paper we define a subclass $\mathcal{G}_\Sigma(\delta, \gamma, \lambda, \tau, r)$ of normalized analytic and bi-univalent function using Borel distribution and Horadam polynomial $h_n(r)$. We obtain Taylor-Maclaurin coefficient inequalities for functions belonging to the defined subclass $\mathcal{G}_\Sigma(\delta, \gamma, \lambda, \tau, r)$ and study the famous Fekete-Szegö problem.

2 Main Results

We begin this section by defining the family $\mathcal{G}_\Sigma(\delta, \gamma, \lambda, \tau, r)$ as follows:

Definition 2.1. For $0 \leq \delta \leq 1$, $\gamma \geq 0$, $\lambda \geq 1$, $0 < \tau \leq 1$ and $r \in \mathbb{R}$, a function $f \in \Sigma$ is said to be in the family $\mathcal{G}_\Sigma(\delta, \gamma, \lambda, \tau, r)$ if it satisfies the subordinations:

$$(1-\delta)\frac{z^{1-\gamma}\left(\mathcal{B}_\tau f(z)\right)'}{\left(\mathcal{B}_\tau f(z)\right)^{1-\gamma}}+\delta\frac{z\left(\left(\mathcal{B}_\tau f(z)\right)'\right)^\lambda}{\mathcal{B}_\tau f(z)} \prec \Pi(r,z)+1-a$$

and

$$(1-\delta)\frac{w^{1-\gamma}\left(\mathcal{B}_\tau g(w)\right)'}{\left(\mathcal{B}_\tau g(w)\right)^{1-\gamma}}+\delta\frac{w\left(\left(\mathcal{B}_\tau g(w)\right)'\right)^\lambda}{\mathcal{B}_\tau g(w)} \prec \Pi(r,w)+1-a$$

where a is real constant and the function $g=f^{-1}$ is given by (2).

Note: $\theta=(1-\delta)(\gamma+1)+\delta(2\lambda-1)$ is used throyghout the paper unless othewise mentioned.

Theorem 2.1. For $0\le\delta\le1$, $\gamma\ge0$, $\lambda\ge1$, $0<\tau\le1$ and $r\in\mathbb{R}$, let $f\in\mathcal{A}$ be in the family $\mathcal{G}_\Sigma(\delta,\gamma,\lambda,\tau,r)$. Then

$$|a_2|\le\frac{e^\tau|br|\sqrt{2|br|}}{\sqrt{\left|\left[\varphi(\delta,\gamma,\lambda,\tau)b-2p\theta\right]br^2-2qa\theta^2\right|}}$$

and

$$|a_3|\le\frac{e^{2\tau}|br|}{\tau\left[(1-\delta)(\gamma+2)+\delta(3\lambda-1)\right]}+\frac{e^{2\tau}b^2r^2}{\theta^2},$$

where

$$\varphi(\delta,\gamma,\lambda,\tau)=(1-\delta)(\gamma+2)(4\tau+\gamma-1)$$

$$+2\delta\left(2\tau\left(3\lambda-1\right)+2\lambda(\lambda-2)+1\right) \quad (5)$$

Proof Let $f\in\mathcal{G}_\Sigma(\delta,\gamma,\lambda,\tau,r)$. Then there are two analytic functions $u,v:U\longrightarrow U$ given by

$$u(z)=u_1z+u_2z^2+u_3z^3+\cdots \quad (z\in U) \quad (6)$$

and

$$v(w)=v_1w+v_2w^2+v_3w^3+\cdots \quad (w\in U), \quad (7)$$

with $u(0)=v(0)=0$, $|u(z)|<1$, $|v(w)|<1$, $z,w\in U$ such that

$$(1-\delta)\frac{z^{1-\gamma}\left(\mathcal{B}_\tau f(z)\right)'}{\left(\mathcal{B}_\tau f(z)\right)^{1-\gamma}}+\delta\frac{z\left(\left(\mathcal{B}_\tau f(z)\right)'\right)^\lambda}{\mathcal{B}_\tau f(z)}=\Pi(r,u(z))+1-a$$

and

$$(1-\delta)\frac{w^{1-\gamma}\left(\mathcal{B}_\tau g(w)\right)'}{\left(\mathcal{B}_\tau g(w)\right)^{1-\gamma}}+\delta\frac{w\left(\left(\mathcal{B}_\tau g(w)\right)'\right)^\lambda}{\mathcal{B}_\tau g(w)}=\Pi(r,v(w))+1-a.$$

Or, equivalently

$$(1-\delta)\frac{z^{1-\gamma}\left(\mathcal{B}_\tau f(z)\right)'}{\left(\mathcal{B}_\tau f(z)\right)^{1-\gamma}}+\delta\frac{z\left(\left(\mathcal{B}_\tau f(z)\right)'\right)^\lambda}{\mathcal{B}_\tau f(z)}=$$

$$1+h_1(r)+h_2(r)u(z)+h_3(r)u^2(z)+\cdots \quad (8)$$

and

$$(1-\delta)\frac{w^{1-\gamma}\left(\mathcal{B}_\tau g(w)\right)'}{\left(\mathcal{B}_\tau g(w)\right)^{1-\gamma}}+\delta\frac{w\left(\left(\mathcal{B}_\tau g(w)\right)'\right)^\lambda}{\mathcal{B}_\tau g(w)}=$$

$$1+h_1(r)+h_2(r)v(w)+h_3(r)v^2(w)+\cdots. \quad (9)$$

Combining (6), (7), (8) and (9) yields

$$(1-\delta)\frac{z^{1-\gamma}\left(\mathcal{B}_\tau f(z)\right)'}{\left(\mathcal{B}_\tau f(z)\right)^{1-\gamma}}+\delta\frac{z\left(\left(\mathcal{B}_\tau f(z)\right)'\right)^\lambda}{\mathcal{B}_\tau f(z)}=$$

$$1+h_2(r)u_1z+\left[h_2(r)u_2+h_3(r)u_1^2\right]z^2+\cdots \quad (10)$$

and

$$(1-\delta)\frac{w^{1-\gamma}\left(\mathcal{B}_\tau g(w)\right)'}{\left(\mathcal{B}_\tau g(w)\right)^{1-\gamma}}+\delta\frac{w\left(\left(\mathcal{B}_\tau g(w)\right)'\right)^\lambda}{\mathcal{B}_\tau g(w)}=$$

$$1+h_2(r)v_1w+\left[h_2(r)v_2+h_3(r)v_1^2\right]w^2+\cdots. \quad (11)$$

It is quite well-known that if $|u(z)|<1$ and $|v(w)|<1$, $z,w\in U$, then

$$|u_i|\le1 \quad and \quad |v_i|\le1 \; forall\; i\in\mathbb{N}. \quad (12)$$

Comparing the corresponding coefficients in (10) and (11), after simplifying, we have

$$\left[(1-\delta)(\gamma+1)+\delta(2\lambda-1)\right]e^{-\tau}a_2=h_2(r)u_1, \quad (13)$$

$$2\tau\left[(1-\delta)(\gamma+2)+\delta(3\lambda-1)\right]e^{-2\tau}a_3+$$
$$\left[\frac{1}{2}(1-\delta)(\gamma+2)(\gamma-1)+\delta\left(2\lambda(\lambda-2)+1\right)\right]e^{-2\tau}a_2^2$$
$$=h_2(r)u_2+h_3(r)u_1^2, \quad (14)$$

$$-\left[(1-\delta)(\gamma+1)+\delta(2\lambda-1)\right]e^{-\tau}a_2=h_2(r)v_1 \quad (15)$$

and

$$2\tau\left[(1-\delta)(\gamma+2)+\delta(3\lambda-1)\right]e^{-2\tau}\left(2a_2^2-a_3\right)+$$

$$\left[\frac{1}{2}(1-\delta)(\gamma+2)(\gamma-1)+\delta\left(2\lambda(\lambda-2)+1\right)\right]e^{-2\tau}a_2^2$$

$$=h_2(r)v_2+h_3(r)v_1^2. \qquad (16)$$

It follows from (13) and (15) that

$$u_1=-v_1 \qquad (17)$$

and

$$2\theta^2e^{-2\tau}a_2^2=h_2^2(r)(u_1^2+v_1^2). \qquad (18)$$

If we add (14) to (16), we find that

$$\varphi(\delta,\gamma,\lambda,\tau)e^{-2\tau}a_2^2=h_2(r)(u_2+v_2)+h_3(r)(u_1^2+v_1^2), \qquad (19)$$

where $\varphi(\delta,\gamma,\lambda,\tau)$ is given by (5).

Substituting the value of $u_1^2+v_1^2$ from (18) in the right hand side of (19), we deduce that

$$a_2^2=\frac{e^{2\tau}h_2^3(r)(u_2+v_2)}{h_2^2(r)\varphi(\delta,\gamma,\lambda,\tau)-2h_3(r)\theta^2}. \qquad (20)$$

Further computations using (3), (12) and (20), we obtain

$$|a_2|\le\frac{e^{\tau}\,|br|\sqrt{2\,|br|}}{\sqrt{\left|\left[\varphi(\delta,\gamma,\lambda,\tau)b-2p\theta^2\right]br^2-2qa\theta^2\right|}}.$$

Next, if we subtract (16) from (14), we can easily see that

$$2\tau\left[(1-\delta)(\gamma+2)+\delta(3\lambda-1)\right]e^{-2\tau}(a_3-a_2^2)=$$

$$h_2(r)(u_2-v_2)+h_3(r)(u_1^2-v_1^2). \qquad (21)$$

In view of (17) and (18), we get from (21)

$$a_3=\frac{e^{2\tau}h_2(r)(u_2-v_2)}{2\tau\left[(1-\delta)(\gamma+2)+\delta(3\lambda-1)\right]}+\frac{e^{2\tau}h_2^2(r)(u_1^2+v_1^2)}{2\theta^2}.$$

Thus applying (3), we obtain

$$|a_3|\le\frac{e^{2\tau}\,|br|}{\tau\left[(1-\delta)(\gamma+2)+\delta(3\lambda-1)\right]}+\frac{e^{2\tau}b^2r^2}{\theta^2}.$$

This completes the proof of Theorem 2.1.

In the next theorem, we discuss the Fekete-Szegö problem for the family $\mathcal{G}_\Sigma(\delta,\gamma,\lambda,\tau,r)$.

Theorem 2.2. For $0\le\delta\le1$, $\gamma\ge0$, $\lambda\ge1$, $0<\tau\le1$ and $r,\mu\in\mathbb{R}$, let $f\in\mathcal{A}$ be in the family $\mathcal{G}_\Sigma(\delta,\gamma,\lambda,\tau,r)$. Then

$$\left|a_3-\mu a_2^2\right|\le$$

$$\begin{cases}\frac{e^{2\tau}|br|}{\tau[(1-\delta)(\gamma+2)+\delta(3\lambda-1)]}; \\ \qquad for\ |\mu-1|\le\frac{\left|\left[\varphi(\delta,\gamma,\lambda,\tau)b-2p\theta^2\right]br^2-2qa\theta^2\right|}{2\tau b^2r^2[(1-\delta)(\gamma+2)+\delta(3\lambda-1)]}, \\ \frac{2e^{2\tau}|br|^3|\mu-1|}{|[\varphi(\delta,\gamma,\lambda,\tau)b-2p\theta^2]br^2-2qa\theta^2|}; \\ \qquad for\ |\mu-1|\ge\frac{\left|\left[\varphi(\delta,\gamma,\lambda,\tau)b-2p\theta^2\right]br^2-2qa\theta^2\right|}{2\tau b^2r^2[(1-\delta)(\gamma+2)+\delta(3\lambda-1)]}.\end{cases}$$

Proof It follows from (20) and (21) that

$$\begin{aligned}a_3-\mu a_2^2&=\frac{e^{2\tau}h_2(r)(u_2-v_2)}{2\tau\left[(1-\delta)(\gamma+2)+\delta(3\lambda-1)\right]}+(1-\mu)\,a_2^2\\&=\frac{e^{2\tau}h_2(r)(u_2-v_2)}{2\tau\left[(1-\delta)(\gamma+2)+\delta(3\lambda-1)\right]}\\&\quad+\frac{e^{2\tau}h_2^3(r)(u_2+v_2)\,(1-\mu)}{h_2^2(r)\varphi(\delta,\gamma,\lambda,\tau)-2h_3(r)\theta^2}\\&=h_2(r)\left[\left(\psi(\mu,r)+\frac{e^{2\tau}}{2\tau\left[(1-\delta)(\gamma+2)+\delta(3\lambda-1)\right]}\right)u_2\right.\\&\quad\left.+\left(\psi(\mu,r)-\frac{e^{2\tau}}{2\tau\left[(1-\delta)(\gamma+2)+\delta(3\lambda-1)\right]}\right)v_2\right],\end{aligned}$$

where

$$\psi(\mu,r)=\frac{e^{2\tau}h_2^2(r)\,(1-\mu)}{h_2^2(r)\varphi(\delta,\gamma,\lambda,\tau)-2h_3(r)\theta^2}.$$

According to (3), we find that

$$\left|a_3-\mu a_2^2\right|\le$$

$$\begin{cases}\frac{e^{2\tau}|br|}{\tau[(1-\delta)(\gamma+2)+\delta(3\lambda-1)]};\ 0\le|\psi(\mu,r)|\le\frac{e^{2\tau}}{2\tau[(1-\delta)(\gamma+2)+\delta(3\lambda-1)]}, \\ 2\,|br|\,|\psi(\mu,r)|;\quad |\psi(\mu,r)|\ge\frac{e^{2\tau}}{2\tau[(1-\delta)(\gamma+2)+\delta(3\lambda-1)]}.\end{cases}$$

After some computations, we obtain

$$\left|a_3-\mu a_2^2\right|\le$$

$$\begin{cases}\frac{e^{2\tau}|br|}{\tau[(1-\delta)(\gamma+2)+\delta(3\lambda-1)]}; \\ \qquad for\ |\mu-1|\le\frac{\left|\left[\varphi(\delta,\gamma,\lambda,\tau)b-2p\theta^2\right]br^2-2qa\theta^2\right|}{2\tau b^2r^2[(1-\delta)(\gamma+2)+\delta(3\lambda-1)]}, \\ \frac{2e^{2\tau}|br|^3|\mu-1|}{|[\varphi(\delta,\gamma,\lambda,\tau)b-2p\theta^2]br^2-2qa\theta^2|}; \\ \qquad for\ |\mu-1|\ge\frac{\left|\left[\varphi(\delta,\gamma,\lambda,\tau)b-2p\theta^2\right]br^2-2qa\theta^2\right|}{2\tau b^2r^2[(1-\delta)(\gamma+2)+\delta(3\lambda-1)]}.\end{cases}$$

Putting $\mu = 1$ in Theorem 2, we obtain the following result:

Corollary 2.1. For $0 \leq \delta \leq 1$, $\gamma \geq 0$, $\lambda \geq 1$, $0 < \tau \leq 1$ and $r \in \mathbb{R}$, let $f \in \mathcal{A}$ be in the family $\mathcal{G}_\Sigma(\delta, \gamma, \lambda, \tau, r)$. Then

$$\left|a_3 - a_2^2\right| \leq \frac{e^{2\tau}\,|br|}{\tau\left[(1-\delta)(\gamma+2)+\delta(3\lambda-1)\right]}.$$

3 Conclusion

The fact that we can find many unique and effective uses of a large variety of interesting functions and specific polynomial in Geometric Function Theory provided the primary inspiration for our analysis in this article. The primary objective was to create a new family $\mathcal{G}_\Sigma(\delta, \gamma, \lambda, \tau, r)$ of normalized analytic and bi-univalent function defined by Borel distribution and also using the Horadam polynomial $h_n(r)$, which are given by the recurrence relation (3) and generating function $\Pi(r, z)$ in (4). We generate Taylor-Maclaurin coefficient inequalities for functions belonging to this newly introduced bi-univalent function family $\mathcal{G}_\Sigma(\delta, \gamma, \lambda, \tau, r)$ and viewed the famous Fekete-Szegö problem.

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All authors have read and agreed to the published version of the manuscript.

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