Cancellation On Fuzzy Projective Modules And Schanuel's Lemma Using Its Conditioned Class

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Abstract:–As an extension, the current study looks at fuzzy projective module cancellation and fuzzy module equivalence in specific situations. While addressing cancellation, we provide the necessary and sufficient criteria for fuzzy projective modules to fulfill cancellation over the polynomial ring and ring R. Furthermore, using fuzzy p-poor modules, we have established an intriguing result in Schanuel's lemma, claiming that for any two fuzzy exact sequences of fuzzy R-modules $0 \to \mu_1 \xrightarrow{\bar{f_1}} \eta_1 \xrightarrow{\bar{g_1}} \mu \to 0$ and $0 \to \mu_2 \xrightarrow{\bar{f_2}} \eta_2 \xrightarrow{\bar{g_2}} \mu \to 0$. If η_1 and η_2 are fuzzy p-poor modules then $\mu_1 \oplus \eta_2 \cong \mu_2 \oplus \eta_1$. The same is reinforced by an acceptable illustration of fuzzy p-poor module.

Key-words:- fuzzy modules, fuzzy projective module, fuzzy projective poor-module, fuzzy subprojective poor-module, schanuel's lemma.

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1 Introduction

Throughout the study, rings are commutative with identity, and modules are unitary. Authors like Gilmer[6] studied the ring whose ideals meet cancellation characteristics independently. According to his research, every ring ideal is confined cancellation if and only if the ring is a nearly Dedekind domain or a primary ring. D.D. and D.F. Anderson[2] verified a similar finding and investigated it further. Mijbass[16] generalized this notion to modules. Many researchers worked on its various types, as mentioned in [5], [8], [25]. Also, weak cancellation modules by Naoum and Mijbas[18] proved some properties of them as well as their relations with other types of modules, such as projective and flat modules, and provided some conditions under which projective and flat modules act as weak cancellation modules. Zhang and Tong[24] also worked on the characterization of the cancellation property for projective modules and demonstrated that Dedekind domains contain it. Bothaynah, Khalaf and Mahmood investigated purely and weakly purely cancellation modules in [3] and developed equivalent criteria for each kind. Mahmood, Bothaynah and Rasheed[15] investigated comparable cancellation modules and discovered some connections between them and cancellation modules. They also looked at the impact of module localisation and tracing on this sort of module. On the Laurent polynomial ring, authors like Mishra[17] studied cancellation modules. Later, as illustrated in [11], [12] and [4] cancellation modules such as purely, restricted, weakly restricted, fully and naturally were fuzzified.

Since then, the current work has focused on either the classical version of cancellation on projective modules or various sorts of cancellation fuzzy modules. Thus, the current work addresses the gap, and we extend the existing situation by examining cancellation on fuzzy projective modules and demonstrating the equivalence of fuzzy modules using Schanuel's lemma. To demonstrate Schanuel's lemma exemption for fuzzy projective modules, we constructed a new structure called the fuzzy p-poor module. The current research is organised as follows. In Section 2, the basic definitions are given for a better understanding of the reader. Section 3 is motivated by [17] and deals with the cancellation of fuzzy projective modules over polynomial rings. In it, while extending the interesting results to their fuzzy framework we have discussed the fuzzy version of Schanuel's Lemma which shows the equivalence of two fuzzy modules μ_1 and μ_2 provided that there exist two fuzzy projective modules η_1 and η_2 such that $\mu_1 \oplus \eta_2 \cong \mu_2 \oplus \eta_1 \Rightarrow \mu_1 \cong \mu_2$. This lemma shows how far modules are being projective and also is useful in defining the Hellar operator in the stable category and giving an elementary description of the dimension shifting. In addition to the above necessary and sufficient conditions for which a fuzzy projective module has a cancellation property are discussed in the section. Finally, section 4 draws the attention of the reader to the introduction of a fuzzy structure called the fuzzy p-poor module, which exempts the requirement of fuzzy projective modules in the Schanuel's lemma and also discusses the few relevant and interesting properties of the same.

2 Preliminaries

The following sections outline the definitions and outcomes used in this research.
This paper's ***Terminology*** is as follows:

1. $_RM$ and M_R denote the left and right R – module respectively for each module M.

2. $\exists$ means there exists

3. μ_M denotes the fuzzy module μ over module M.

4. $\Rightarrow$ means implies

5. $\mu(m)$ represents the arbitrary element of fuzzy set μ_M.

6. μ_t denotes the level subset of a fuzzy module μ.

Definition 2.1.[13] *If the following conditions are met, a fuzzy subset μ_M is called a fuzzy submodule of module M:*

(i) $\mu(m+n) \geq min\{\mu(m), \mu(n)\}$

(ii) $\mu(xm) \geq \mu(m)$, for all $m, n \in M$ and $x \in R$

(iii) $\mu(-x) = \mu(x)$ for all $x \in M$

(iv) $\mu(0) = 1$

Definition 2.2 [14] *A fuzzy R-module μ_P is called projective if and only if for every surjective fuzzy R-homomorphism $\bar{f} : \mu_A \rightarrow \mu_B$ and for every fuzzy R-homomorphism $\bar{g} : \mu_P \rightarrow \mu_B$ there exists a fuzzy R-homomorphism $\bar{h} : \mu_P \rightarrow \mu_A$ such that the figure below commutes that is : $\bar{f}\bar{h} = \bar{g}$*

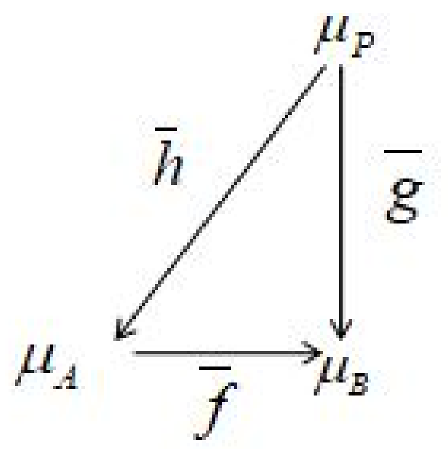

Fig.1 Fuzzy Projective Module

Note : We can also decipher the above definition as μ_P is μ_A- projective.

Definition 2.3[**Classical Version**][5] *A module M is said to have a cancellation property if for all modules H and K, $A \oplus H \cong A \oplus K$ implies $H \cong K$.*

Lemma 2.4[**Classical Version of Schanuel's Lemma in Projective Modules**] [10] *Let R be a ring. Then for any given exact sequences of R-modules $0 \rightarrow M_1 \xrightarrow{f_1} P_1 \xrightarrow{g_1} M \rightarrow 0$ and $0 \rightarrow M_2 \xrightarrow{f_2} P_2 \xrightarrow{g_2} M \rightarrow 0$ with P_1 and P_2 projective we have $M_1 \oplus P_2 \cong M_2 \oplus P_1$.*

Definition 2.5[19] *A module M is defined to be projectively poor(or p-poor) if its domain of projectivity contains only semisimple modules. Where $\mathscr{N}(M) = [N \in M_R | \ M$ is N-projective] is defined as a projectivity domain of module M.*

Definition 2.6[23] *The sequence .... $\rightarrow \mu_{n-1} \xrightarrow{\bar{f_{n-1}}} \mu_n \xrightarrow{\bar{f_n}} \mu_{n+1} \rightarrow$ .... of R- fuzzy module homomorphism is termed as fuzzy exact if and only if $Im\bar{f_{n-1}} = Ker\bar{f_n}$ for every n. Here $Im\bar{f_{n-1}}$ and $Ker\bar{f_n}$ means $\mu_n \mid Imf_{n-1}$ and $\mu_n \mid Kerf_n$ that is*

μ_n *is restricted to image and kernel respectively.*
Definition 2.7[23] *The exact sequence of the form* $0 \to \mu_A \xrightarrow{f} \eta_B \xrightarrow{g} \nu_B \to 0$ *is called as fuzzy short exact sequence.*
Definition 2.8[25] *A finitely generated projective R-module P is said to be cancellative if* $P \oplus R^n \cong Q \oplus R^n$ *implies* $P \cong Q$
Definition 2.9[25] *Let R be a ring and P be a projective R-module. An element* $p \in P$ *is called unimodular if there is a surjective R-linear map* $\phi : P \to R$ *such that* $\phi(p) = 1$.
Proposition 2.10[8] *Let* μ *be a fuzzy module of an R- module M,then* μ *is a fuzzy cancellation module if and only if* μ_t *is a cancellation module.*

3 Cancellation on Polynomial Rings

Let R be a ring and μ_M be the fuzzy R-module. Let μ_M[X] be the fuzzy polynomial module over polynomial ring R[X] where μ_M[X] = [$\Sigma a_i X^i$, where, a_i are fuzzy numbers]. If $\bar{g} : \mu_M \to \eta_N$ is a fuzzy homomorphism of fuzzy R-modules then it induces a homomorphism $\Psi : \mu_M[X] \to \eta_N[X]$ defined as $\Psi(\Sigma a_i X^i) = \Sigma \bar{g}(a_i) X^i$. Given any fuzzy R-module μ and $\bar{f} \in$ End(μ). We can make μ as R[X] module whose scalar multiplication is defined as $(m \Sigma a_i X^i) = a_i \Sigma \bar{f}^n(m)$ and denote the R[X] module as ${}_{\bar{f}}\mu$. Then there is canonical R[X] surjection $\phi_{\bar{f}} : \mu[X] \to {}_{\bar{f}}\mu$ defined as $\phi_{\bar{f}}(\Sigma a_i X^i) = \Sigma \bar{f}^n(m)$.

We have extended Schanuel's lemma (described in 2.4) to its fuzzy environment, having fuzzy projective-modules, in the following lemma. It is linked to the equivalence of two fuzzy modules μ_{M_1} and μ_{M_2} if two fuzzy projective modules μ_{P_1} and μ_{P_2} are present such that $\mu_{M_1} \oplus \mu_{P_2} \cong \mu_{M_2} \oplus \mu_{P_1}$.

Lemma 3.1[**Fuzzier form of Schanuel's Lemma**] *Given the two sequences of fuzzy R-modules*
$0 \to \mu_1 \xrightarrow{\bar{f_1}} \eta_1 \xrightarrow{\bar{g_1}} \mu \to 0$ *and* $0 \to \mu_2 \xrightarrow{\bar{f_2}} \eta_2 \xrightarrow{\bar{g_2}} \mu \to 0$. *If they are fuzzy exact with* η_1 *and* η_2 *are fuzzy projective modules then* $\mu_1 \oplus \eta_2 \cong \mu_2 \oplus \eta_1$.

Proof. Fuzzy direct sum $\eta_1 \oplus \eta_2$ can be formed using fuzzy R-modules η_1 and η_2. Next $\nu =$
$\eta_1 \oplus \eta_2 = [(\eta_1(x_1), \eta_2(x_2)) \in \eta_1 \oplus \eta_2 : \bar{g_1}(\eta_1(x_1)) = \bar{g_2}(\eta_2(x_2))]$. Clearly, $\nu \subseteq \eta_1 \oplus \eta_2$ and ν is nonempty set. Then for each $(\eta_1(x_1), \eta_2(x_2))$ and $(\eta_1(y_1), \eta_2(y_2))$ in ν and r in R we have

$$\begin{aligned}
\bar{g_1}[(\eta_1(x_1)) + (\eta_1(y_1))] &= \bar{g_1}(\eta_1(x_1)) + \bar{g_1}(\eta_1(y_1)) \\
&= \bar{g_2}(\eta_2(x_2)) + \bar{g_2}(\eta_2(y_2)) \\
&= \bar{g_2}[(\eta_2(x_2)) + (\eta_2(y_2))] \\
&\Rightarrow [(\eta_1(x_1)) + (\eta_1(y_1))] \in \nu.
\end{aligned}$$

and $[(\eta_1(x_1))r + (\eta_2(x_2))r] \in \nu$ or in other words we can say that ν is a submodule $\eta_1 \oplus \eta_2$. Next we have $\bar{g_1}$ is surjective homomorphism so $\bar{g_1}(\eta_1) = \mu$ therefore for each $\bar{g_1}(\eta_1) \in \mu \ \exists \ \eta_2(x) \in \eta_2$ such that $\bar{g_1}(\eta_1(x)) = \bar{g_2}(\eta_2(x))$. Defined homomorphism $\bar{\pi}_1 : \nu \to \eta_1$ defined as $\bar{\pi}_1(\eta_1, \eta_2) = \eta_1$. Then we have

$$\begin{aligned}
ker\bar{\pi}_1 &= [(\eta_1, \eta_2) : \bar{\pi}_1(\eta_1, \eta_2) = 0] \\
&= [(\eta_1, \eta_2) : \eta_1 = 0] \\
&= [(0, \eta_2) : g_1(\bar{\eta_2}) = 0 \\
&= ker\bar{g}_2 \\
&= Im\bar{f}_2.
\end{aligned}$$

Im $\bar{f}_2 = \mu_2$ because $\bar{f}_2$ is an injective homomorphism. As a result, we have Ker $\bar{\pi}_1 = \mu_2$ is obtained. Thus, a fuzzy short exact sequence can be created $0 \to \mu_2 \to \nu \xrightarrow{\bar{\pi}_1} \eta_1 \to 0$————-(1)
Since η_1 is fuzzy projective equation (1) splits thus $\exists \ \bar{h} : \eta_1 \to \nu$ such that $\bar{\pi}_1$ o $\bar{h} = Id_{\eta_1}$. Hence by [22] $\nu = \eta_1 \oplus \mu_2$. A fuzzy short exact sequence $0 \to \mu_1 \to \nu \xrightarrow{\bar{\pi}_2} \eta_2 \to 0$———(2) can be generated in a similar way to give $\nu = \eta_2 \oplus \mu_1$. Therefore
$\mu_1 \oplus \eta_2 \cong \mu_2 \oplus \eta_1$. □

Proposition 3.2 *Let* μ *and* μ' *be the fuzzy projective R[X] modules and* $\bar{\phi} : \mu' \to \mu$, $\bar{\psi} : \mu \to \mu'$ *be the fuzzy injective homomorphism. If R[X],* $\mu/\bar{\phi}\bar{\psi}\mu$, $\mu'/\bar{\psi}\bar{\phi}\mu'$ *are fuzzy projective over R then* $\mu/\bar{\phi}\mu'$ *and* $\mu'/\bar{\psi}\mu$ *are also fuzzy projective over R.*

Proof. : Since μ and μ' are fuzzy R[X] projective and R[X] is R projective, there exists fuzzy R projective module η and fuzzy R[X] projective modules μ_1 and μ_1' and. We now, get for some positive integers n and m

$$\begin{aligned}\mu \oplus \mu_1 &\cong R[X]^n \\ &= \mu' \oplus \mu'_1 \cong R[X]^m \\ &= R[X] \oplus \eta \cong \sum R \\ &.. \\ &.. \\ &.. \\ &= R[X]^n \oplus \eta^n \cong \sum R \\ &= \mu \oplus \mu_1 \oplus \eta^n \cong \sum R \\ &= \mu \oplus \eta_1^n \cong \sum R where \mu_1 \oplus \eta^n \\ &= \eta_1^n \\ &.\end{aligned}$$

Therefore μ is fuzzy R projective. Similarly μ' is fuzzy R projective. Since $0 \to \mu \xrightarrow{\bar{\beta}} \mu' \to \mu'/\bar{\psi}\mu \to 0$, pd$(\mu'/\bar{\psi}\mu) \leq 1$. Similarly, pd$(\mu/\bar{\phi}\mu')$ ≤ 1. Now isomorphism between μ and $\bar{\psi}\mu$ induces $\mu/\bar{\phi}\mu' \cong \bar{\psi}\mu/\bar{\psi}\bar{\phi}\mu'$ we have exact sequences

$$\begin{tikzcd} 0 \arrow[r] & \mu \arrow[r] & \mu' \arrow[r] & \mu'/\bar{\psi}\mu \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & \bar{\psi}\mu/\bar{\psi}\bar{\phi}\mu' \arrow[r] & \mu'/\bar{\psi}\bar{\phi}\mu' \arrow[r] & \mu'/\bar{\psi}\mu \arrow[r] & 0 \end{tikzcd}$$

Fig.2 Fuzzy Exact sequences

By Lemma 3.1 $\mu'/\bar{\psi}\bar{\phi}\mu' \oplus \mu \cong \bar{\psi}\mu/\bar{\psi}\bar{\phi}\mu' \oplus \mu'$ so that $\mu'/\bar{\psi}\bar{\phi}\mu' \oplus \mu \cong \mu/\bar{\phi}\mu' \oplus \mu'$. Direct sum of fuzzy projectives are also fuzzy projective and $\mu'/\bar{\psi}\bar{\phi}\mu'$, μ are fuzzy R-projective, $\mu'/\bar{\psi}\bar{\phi}\mu' \oplus \mu$ are fuzzy R-projective. Therefore $\mu/\bar{\phi}\mu' \oplus \mu'$ are fuzzy R-projective. Thus, $\mu/\bar{\phi}\mu' \oplus \mu' \oplus \mu_0 \cong R^n$ comes from the definition of fuzzy R-projective module where μ_0 is a fuzzy R-module. Let $\bar{\mu} = \mu' \oplus \mu_0$ be a fuzzy R-module. Then $\mu/\bar{\phi}\mu' \oplus \bar{\mu} \cong R^n$. Hence $\mu/\bar{\phi}\mu'$ is a fuzzy R-projective. Similarly, $\mu'/\bar{\psi}\mu$ is also fuzzy R-projective. □

Corollary 3.3 *Let μ and μ' be fuzzy projective R[X] modules with $\mu \supset \mu' \supset f\mu$, where f is a monic polynomial of polynomial ring. If R[X] and R[x]/f R[X] are R-projective then μ/μ' is fuzzy projective.*

Proof. Let us assume the inclusion map $\bar{\phi}:\ \mu' \to \mu$ and $\bar{\psi}$ the multiplication by f from $\mu \to \mu'$. Then $\mu/\bar{\phi}\bar{\psi}\mu = \mu/f\mu$ and $\mu'/\bar{\psi}\bar{\phi}\mu' = \mu'/f\mu'$. Since μ and μ' are fuzzy projective R[X] projective and R[X], R[X]/fR[X] are R-projective, thus $\mu/$fμ and $\mu'/$fμ' are R[X]/fR[X] are projective. Hence $\mu/$fμ and $\mu'/$fμ' are fuzzy R-projective. Thus, $\mu/\bar{\phi}\bar{\psi}\mu$ and $\mu'/\bar{\psi}\bar{\phi}\mu'$ are also fuzzy R-projective. By preposition 3.2 fuzzy module μ/μ' is fuzzy projective. □

Proposition 3.4 *Let μ be a fuzzy R-module and $\bar{f} \in End(\mu)$. Then $0 \to \mu[X] \xrightarrow{X.1_{\mu[X]}-f[X]} \mu[X] \xrightarrow{\bar{\phi_f}} {}_{\bar{f}}\mu \to 0$ is an fuzzy exact sequence of R[X] modules.*

Proof. Clearly $\bar{\phi_f}$ is surjective so we have

$$\begin{aligned}&\bar{\phi_f}(X.1_{\mu[X]} - f[X])(\sum A_iX^i) \\ &= \bar{\phi_f}(\sum(A_iX^{i+1} - f(A_i)X^i \\ &= \bar{\phi_f}[\sum(A_iX^{i+1} - \sum f(A_i)X^i] \\ &= \sum(f^{i+1}(A_i) + f^i(f(A_i))) \\ &= \sum(f^{i+1}(A_i) + f^{i+1}(A_i)) \\ &= 0.\end{aligned}$$

Thus, Im$(X.1_{\mu[X]} - f[X]) \subseteq$ Ker$\bar{\phi_f}$. Now, to show Ker$\bar{\phi_f} \subseteq$ Im$(X.1_{\mu[X]} - f[X])$. Let $\sum A_iX^i$ $\in$ Ker$\bar{\phi_f} \Rightarrow \bar{\phi_f}(\sum A_iX^i) = 0$. Then,
$Z = Z - \sum f^i(A_i)$
$= \sum(A_iX^i - f^i(A_i)$
$= \sum(X^i.1_{\mu[X]} - f^i)A_i$
$= (X^i.1_{\mu[X]} - f[X])[..1/(X-f)(X^i - f^i)/X^if^i - 1/(X-f)(X^{i-1} - f^{i-1})/X^{i-1}f^{i-1}.. - ..1/(X - f)(X-f)/(X-f) + 0 + (X-f)/(X-f) + ..]A_i$
$= (X^i.1_{\mu[X]} - f[X])\sum h_i(A_i)$
Z $\in$ Im$(X.1_{\mu[X]} - f[X])$.
Hence, Ker$\bar{\phi_f}$ $\subseteq$Im$(X.1_{\mu[X]} - f[X])$. Thus, Ker$\bar{\phi_f} =$ Im$(X.1_{\mu[X]} - f[X])$. □

Theorem 3.5 *Let μ and μ' be finitely generated fuzzy projective R[X] modules. Suppose $\mu \supset \mu' \supset f\mu$ for some monic polynomial $f \in R[X]$. Then μ and μ' are stably isomorphic. In Particular, if $\mu_f \cong \mu'_f$, then μ and μ' are stably isomorphic.*

Proof. : Take $\eta = \mu/\mu'$. R[X]$/f$ is a free R-module since f is a monic polynomial. As a result, $\mu/f\mu$ is R-projective, with corollary 3.3 indicating that η is fuzzy R-projective. We have a fuzzy exact sequence of R[X] modules

$$\begin{tikzcd} 0 \arrow[r] & \mu' \arrow[r, "\bar{i}"] & \mu \arrow[r, "\bar{\pi}"] & \eta \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & \eta[X] \arrow[r] & \eta[X] \arrow[r] & \eta \arrow[r] & 0 \end{tikzcd}$$

Fig.3 Fuzzy Exact Sequence

Since $\mu' \subset \mu$ and $\eta = \mu/\mu'$. The first sequence is fuzzy exact since $\bar{i}$ and $\bar{\pi}$ are inclusive and surjective maps respectively. By preposition 3.4 the second sequence is also fuzzy exact. Since η is fuzzy projective, η[X] is fuzzy R[X] projective. Thus by Schanuel's Lemma $\mu \oplus \eta[X] \cong \mu' \oplus \eta[X]$. Hence μ and μ' are stably isomorphic. ☐

3.1 Cancellation on Ring R

All rings considered in this section are associative with identity and modules are unital right modules.

Example 3.1.1. *Let $\mu : Z \to [0,1]$ is defined as*

$$\mu(z) = \begin{cases} 1, if\ z \in 2Z \\ 0, elsewhere \end{cases}$$

Then μ is a fuzzy module for all $z \in Z$. Furthermore, because $\mu_t = 2Z$ is a cancellation module, μ can be regarded as a cancellation fuzzy module[proposition 2.10].

We've now defined the necessary and sufficient requirements for fuzzy projective modules to have the cancellation property.

Proposition 3.1.2 Let R be a ring, μ be a fuzzy projective R-module and $\bar{\phi} = \mathrm{End}(\mu)$. If $\eta \cong \oplus R/\nu$ for some index set N and ν fuzzy submodule of $\oplus$ R then the following are indistinguishable:

1. For any fuzzy R module ψ
   $\mu \oplus \eta \cong \mu \oplus \psi \Rightarrow \eta \cong \psi$.

2. Whenever $\bar{\theta}\bar{\lambda} + \bar{\alpha}\bar{\sigma} = 1_\mu \in \bar{\phi}$ with $\bar{\alpha}(\bar{\nu}) = 0$. Where $\bar{\theta}$, $\bar{\lambda} \in \bar{\phi}$, $\bar{\alpha} = (\mu_1, \mu_2, ...\mu_i, ...) \in \prod \mu_i \cong \mathrm{Hom}(\oplus R, \mu)$ and $\bar{\sigma} \in \mathrm{Hom}(\mu, \oplus R)$, there are $\bar{\tau}_1 \in \prod \mu$ and $\bar{h} \in \mathrm{Hom}(\oplus R, \oplus R)$ with $\bar{\tau}_1(\nu) = 0$ and $\bar{h}(\nu) \subseteq \nu$ satisfying the following conditions:
   (i) $\bar{\theta}\ \bar{\tau}_1 + \bar{\alpha}\bar{h} = 0$.
   (ii) If $\bar{\theta}(\mu_1) + \bar{\alpha}$(r) $= 0$ where ($\mu_1 \in \mu$ and r $\in \oplus R$ then there is z $\in \oplus R$ such that $\mu_1 \in \bar{\tau}_1(z)$ and r- $\bar{h}(z) \in \nu$.
   (iii) $\bar{\tau}_1(r) = 0$ and $\bar{h}(r) \in \nu \Rightarrow$ r $\in \nu$ for any r $\in \oplus R$.

Proof. : Write F $= \oplus$ R. Since $\eta = F/\nu$, there is surjective homomorphism $\bar{q} : F \to \eta$ with Ker$\bar{q} = \nu$. So we have the following fuzzy exact sequence. Since $0 \to \nu \to$ F $\xrightarrow{\bar{q}} \eta \to 0$. (i) $\Rightarrow$ (ii) Let us assume $\bar{\theta}\bar{\lambda} + \bar{\alpha}\bar{\sigma} = 1_\mu \in \bar{\phi}$ with $\bar{\alpha}(\bar{\nu}) = 0$ then there is $\bar{p} in$ Hom (η, μ) such that $\bar{\alpha} = \bar{p}\bar{q}$, so $\bar{\theta}\bar{\lambda} + \bar{p}\bar{q}\bar{\sigma}$ =1. Set $\bar{\pi} = (\bar{\theta}, \bar{p}) \in$ Hom $(\mu \oplus \eta, \mu)$ and $\bar{\pi} = (\bar{\lambda}, \bar{q}\bar{\sigma}) \in$ Hom $(\mu, \mu \oplus \eta)$ then $\bar{\pi}\bar{\phi} = 1_\mu$ which mean the following fuzzy exact sequence splits.$0 \to ker\pi \to \mu \oplus \eta \xrightarrow{\bar{\pi}} \mu \to 0$. Hence $\mu \oplus \eta \cong \mu \oplus ker\bar{\pi}$. By (1) we have $\eta \cong ker\bar{\pi}$ which implies there is a homomorphism $\bar{\tau} \in$ Hom $(\eta, \mu \oplus \eta)$ such that following sequence is fuzzy exact. $0 \to \eta \xrightarrow{\bar{\tau}} \mu \oplus \eta \xrightarrow{\bar{\pi}} \mu \to 0$—–(1). Write $\bar{\tau} = (\bar{\tau}_1', \bar{\tau}_2')$ for some $\bar{\tau}_1' \in$ Hom (η, μ) and $\bar{\tau}_2' \in$ Hom (η, η). Take $\bar{\tau}_1 = \bar{\tau}_1'\bar{q} \in$ Hom (F, μ) and $\bar{\tau}_2 = \bar{\tau}_2'\bar{q} \in$ Hom (F, μ). Then $\bar{\tau}_1\nu = 0$. Since F is projective, there is $\bar{h} \in$ Hom(F, F) such that $\bar{\tau}_2 = \bar{\tau}_2'\bar{q} = \bar{q}\bar{h}$. Thus, $\bar{q}\bar{h}(\nu) = \bar{\tau}_2'\bar{q}(\nu) = 0$ so $\bar{h}(\nu) \subseteq$ Ker $\bar{q} = \nu$. Since the sequence (1) is fuzzy exact, we have $\bar{\pi}\bar{\tau} = 0$ that is $\bar{\theta}\bar{\tau}_1' + \bar{p}\bar{\tau}_2' = 0$. So $\bar{\theta}\bar{\tau}_1' + \bar{\alpha}\bar{h} = \bar{\theta}\bar{\tau}_1' + \bar{p}\bar{q}\bar{h} = \bar{\theta}\bar{\tau}_1'\bar{q} + \bar{p}\bar{\tau}_2'\bar{q} = 0$. Thus (1) holds

Now, ker$\bar{\pi} \subseteq \bar{\tau}(\eta)$. Suppose $\mu_1(x) \in \mu$ and r $\in$ F with $\bar{\theta}(\mu_1) + \bar{p}(\bar{q}(r)) = 0$, that is $(\mu_1(x) + \bar{q}(r) \in$ Ker$\bar{\pi} = \bar{\tau}(\eta)$. So there is $\bar{q}(z) \in \eta$ with z $\in$ F. Such that $\mu_1(x) = \bar{\tau}_1'(\bar{q}(z)) = \bar{\tau}_1(z)$ and

$$\begin{aligned} \bar{q}(r) &= \bar{\tau}_2'(\bar{q}(z)) \\ &= \bar{q}\bar{h}(z) \\ &= r - \bar{h}(z) \in ker\bar{q} \\ &= \nu \end{aligned}$$

Thus(ii) holds. Moreover, $\bar{\tau}$ is a monomorphism. For any r $\in$ F with $\bar{\tau}_1$(r) $= 0$ and $\bar{h}(r) \in \nu$, we have $\bar{\tau}_1\bar{q}$(r) $= (\bar{\tau}_1(r), \bar{q}\bar{h}(r)) = 0$. So, $\bar{q}$(r) $= 0$. Thus, (iii) holds.
(2) $\Rightarrow$(1) Suppose $\mu \oplus \eta \cong \mu \oplus \psi$. Then there is a fuzzy split exact sequence $0 \to \psi \to \mu \oplus \eta \xrightarrow{\bar{\pi}} \mu \to 0$ with $\bar{\pi}\bar{\xi} = 1_\mu$ for some $\xi \in \mathrm{Hom}(\mu, \mu \oplus \eta)$. Set $\bar{\pi} = (\bar{\theta}, \bar{p})$ and $\bar{\xi} = (\bar{\lambda}, \bar{\phi}_1)$ for some $\bar{\theta}, \bar{\lambda} \in \bar{\phi}$. $\bar{p} \in$ Hom (η, μ) and $\bar{\phi}_1 \in$ Hom (μ, η). Since we have μ is fuzzy projective $\exists \bar{\sigma} \in$ Hom $(\mu$, F) such that $\bar{\phi}_1 = \bar{q}\bar{\sigma}$. Take $\bar{\alpha} = \bar{p}\bar{q} \in$ Hom $(\mu$, F) $\cong \prod \mu_i, i \in N$

then $\bar{\alpha}(\nu) = \bar{p}\bar{q}(\nu) = 0$ and

$$\begin{aligned} &\bar{\theta}\bar{\lambda} + \bar{\alpha}\bar{\sigma} = \bar{\theta}\bar{\lambda} + \bar{p}(\bar{q}\bar{\sigma}) \\ &= \bar{\theta}\bar{\lambda} + \bar{p}\bar{\phi}_1 \\ &= \bar{\pi}\bar{\xi} \\ &= 1_\mu \end{aligned}$$

By hypothesis there is $\bar{\tau}_1 \in$ Hom(F, μ) and $\bar{h} \in$ Hom(F, F) with $\bar{\tau}_1(\nu) = 0$ and $\bar{h}(\nu)$ is a fuzzy submodule of ν satisfying the condition(i)-(iii). Since $\bar{\tau}_1(\nu) = 0$ we have $\bar{\tau}'_1 \in$ Hom (η, μ) such that $\bar{\tau}_1 = \bar{\tau}'_1\bar{q}$. Now,

$$\begin{aligned} &\bar{h}(\nu) \subset \nu \\ &\Rightarrow \bar{q}\bar{h}(\nu) = 0 \end{aligned}$$

So there exists $\bar{\tau}'_2 \in \mathrm{Hom}(\eta, \eta)$ such that $\bar{q}\bar{h} = \bar{\tau}'_2\bar{q}$. If $\bar{\tau} = (\bar{\tau}'_1, \bar{\tau}'_2) \in \mathrm{Hom}(\eta, \mu \oplus \eta)$ then conditions(i) –(iii) gives the sequence $0 \to \eta \xrightarrow{\bar{\tau}} \mu \oplus \eta \xrightarrow{\bar{\pi}} \mu \to 0$ is fuzzy exact. Therefore $\eta \cong \mathrm{Ker}\bar{\pi} \cong \psi$. □

We can derive the following theorem from the above preposition :

Theorem 3.1.3 Let R be a ring, μ be a fuzzy projective R-module, and $\bar{\phi} = \mathrm{End}(\mu)$ for some index set N then the following are indistinguishable:

1. For any fuzzy R module ψ and any fuzzy N-generated R module η, $\mu \oplus \eta \cong \mu \oplus \psi \Rightarrow \eta \cong \psi$.

2. Whenever $\bar{\theta}\bar{\lambda} + \bar{\alpha}\bar{\sigma} = 1_\mu \in \bar{\phi}$ with $\bar{\alpha}(\bar{\nu}) = 0$. Where $\bar{\theta}$, $\bar{\lambda} \in \bar{\phi}$, $\bar{\alpha} = (\mu_1, \mu_2, ...\mu_i, ...) \in \prod \mu_i \cong \mathrm{Hom}(\oplus R, \mu)$ and $\bar{\sigma} \in \mathrm{Hom}(\mu, \oplus R)$, if ν is fuzzy submodule of Ker $\bar{\alpha}$, there is $\bar{\tau}_1 \in \prod \mu$ and $\bar{h} \in \mathrm{Hom}(\oplus R, \oplus R)$ with $\bar{\tau}_1(\nu) = 0$ and $\bar{h}(\nu) \subseteq \nu$ satisfying the following conditions:
(i) $\bar{\theta}\ \bar{\tau}_1 + \bar{\alpha}\bar{h} = 0 \in \mathrm{Hom}(\oplus R, \mu)$.
(ii) If $\bar{\theta}(\mu_1) + \bar{\alpha}(\mathrm{r}) = 0$ where $\mu_1 \in \mu$ and r $\in \oplus R$ then there is z $\in \oplus R$ such that $\mu_1 \in \bar{\tau}_1(z)$ and r - $\bar{h}(z) \in \nu$.
(iii) $\bar{\tau}_1(r) = 0$ and $\bar{h}(r) \in \nu \Rightarrow$ r $\in \nu$ for any r $\in \oplus R$.

Note : A fuzzy projective R module μ satisfy the cancellation property if and only if μ satisfies condition (2) of Theorem 3.1.3 for any N.

4 Fuzzy P-Poor Modules

This section presents a few intriguing results in addition to the pertinent result where fuzzy p- poor modules are shown as an alternative to fuzzy projective modules in equivalence of modules.

Here Mod-FR denotes the category of all fuzzy right R-modules over the ring R and SS Mod-FR stands for fuzzy semisimple right R-modules.

Definition 4.1 *A fuzzy R-module μ_P is called p-poor if and only if for every fuzzy semisimple module μ_A satisfies for each surjective fuzzy R-homomorphism $\bar{f} : \mu_A \to \mu_B$ and for every fuzzy R-homomorphism $\bar{g} : \mu_P \to \mu_B$ there exist a fuzzy R-homomorphism $\bar{h} : \mu_P \to \mu_A$ such that the figure below commutes that is : $\bar{f}\bar{h} = \bar{g}$.*

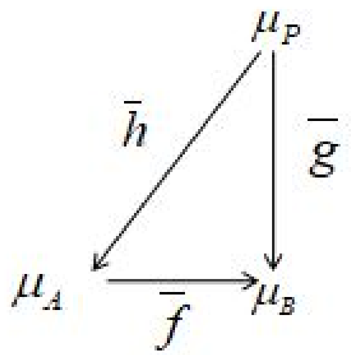

Fig.4 Fuzzy p-poor module

Example 4.2 *Let μ, η and ϕ be fuzzy modules that are defined over $Q\sqrt{2}$, $Q\sqrt{3}$ and $Q\sqrt{3}/Q$ respectively as*

$$\mu(x + y\sqrt{2}) = \begin{cases} 1, \textit{if } x,\ y\ = 0 \\ 4/5, \textit{if } x \neq 0,\ y = 0 \\ 1/2, \textit{if } y \neq 0 \end{cases}$$

,

$$\eta(x + y\sqrt{3}) = \begin{cases} 1, \textit{if } x,\ y\ = 0 \\ 1/2, \textit{if } x \neq 0,\ y = 0 \\ 1/3, \textit{if } y \neq 0 \end{cases}$$

and $\phi[(x + y\sqrt{3})+Q] = \eta(x + y\sqrt{3})\ \forall\ x,\ y\ \epsilon\ Q.$

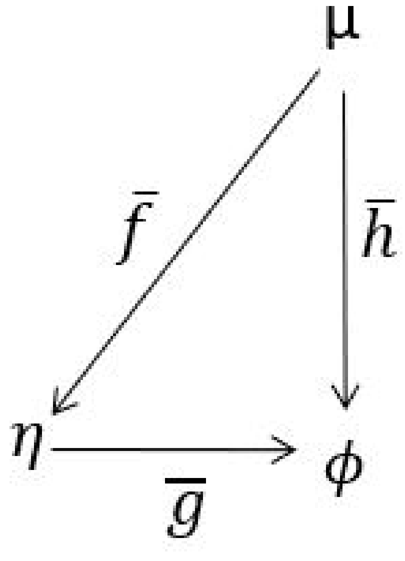

Fig.5 μ is fuzzy projective

Then μ is fuzzy projective, where the mappings $\bar{f}$, $\bar{h}$ and $\bar{g}$ are $\bar{f}[\mu(x + y\sqrt{2})] = \eta(x + y\sqrt{3})$, $\bar{g}[\eta(x+y\sqrt{3})] = [\phi(x+y\sqrt{3})+Q]$ and $\bar{h}[\mu(x+y\sqrt{2})] = [\phi(x + y\sqrt{3})+Q]$ respectively.

Example 4.3 *Using the same module μ as in example 4.2 above, where $M = Q\sqrt{2} = Q \oplus \sqrt{2}Q$ is semi simple. Furthermore, define μ_1 over Q as,*

$$\mu_1(x) = \begin{cases} 1, \text{if } x = 0 \\ 4/5, \text{if } x \neq 0 \end{cases}$$

and μ_2 over $\sqrt{2}Q$ as

$$\mu_2(x) = \begin{cases} 1, \text{if } x = 0 \\ 4/5, \text{if } x \neq 0 \end{cases}$$

*Then μ_1 and μ_2 are respectively fuzzy modules over Q and $\sqrt{2}Q$. Furthermore, $\mu = \mu_1 \oplus \mu_2$ establishes that μ is a semi-simple R-module over M. This is now known as the **fuzzy p-poor module**.*

Definition 4.4 *For a fuzzy module μ_P, $\mathscr{P}(\mu_P) = [\mu_A|\ \mu_P$ is μ_A-projective] is defined as a projectivity domain of μ_P.*

Note : Recalling the following definitions given in [9]

(a)**Definition 4.5** *μ_M is said to be simple fuzzy left module if it has no proper submodules.*

(b)**Definition 4.6** *μ_M is said to be semi-simple fuzzy left module if whenever for ν_N, a strictly proper fuzzy submodule of μ_M there exist a strictly proper fuzzy submodule η_P of μ_M such that $\mu_M = \nu_N \oplus \eta_P$.*

Note : A ring is said to be semi-simple if, every left-module over it is semi-simple.

Definition 4.7 *A ring R is called fuzzy semisimple artinian if any of the following equivalent conditions hold: (i) ${}_RM$ is semisimple*
(ii) M_R is semisimple
(iii) SS Mod-FR = Mod-FR

Remark : *The fuzzy p-poor module is a special case of the fuzzy projective module as the projectivity domain of it consists of only fuzzy semisimple modules over ring R.*

Lemma 4.8 *Let μ_M be a finitely generated fuzzy R-module which has a projective direct summand of rank $f > d$ = dim of Y (where Y is the space whose each element is the fuzzy maximal ideal of R) and let ν_Q be a finitely generated fuzzy p-poor module. Then if $\eta_{M'}$ is another fuzzy R-module we have $\nu_Q \oplus \mu_M \cong \nu_Q \oplus \eta_{M'} \Rightarrow \mu_M \cong \eta_{M'}$.*

Proof. : Since $\mu_Q \oplus \theta_{Q'} \cong R^n$ for some n and $\theta_{Q'}$. We can reduce, by induction on n to the case R $= \mu_Q$. Using the given isomorphism to identify R $\oplus\ \mu_M$ with R $\oplus\ \eta_{M'}$, we can write βR $\oplus\ \mu_M$ = αR$\oplus\ \eta_{M'}$ with β and α being unimodular. Since α is unimodular then there exists a $\tau \in$ group of R automorphism of (βR$\oplus\ \mu_M$) with $\tau\alpha = \beta$. Therefore, $\mu_M \cong (\beta\text{ R} \oplus \mu_M)/\beta\text{ R} = \tau(\alpha\text{ R} \oplus \eta_{M'})/\tau(\alpha\text{ R}) \cong (\alpha\text{ R} \oplus \eta_{M'})/\alpha\text{ R} \cong \eta_{M'}$. □

Lemma 4.9[Schanuel's Lemma using fuzzy p-poor modules] *Given the two sequences of fuzzy R-modules $0 \to \mu_1 \xrightarrow{\bar{f_1}} \eta_1 \xrightarrow{\bar{g_1}} \mu \to 0$ and $0 \to \mu_2 \xrightarrow{\bar{f_2}} \eta_2 \xrightarrow{\bar{g_2}} \mu \to 0$. If they are fuzzy exact with η_1 and η_2 are fuzzy p-poor modules then $\mu_1 \oplus \eta_2 \cong \mu_2 \oplus \eta_1$.*

Proof. : A fuzzy direct sum $\eta_1 \oplus \eta_2$ can be formed using fuzzy p-poor modules η_1 and η_2.
Next $\nu = \eta_1 \oplus \eta_2 = [(\eta_1(x_1), \eta_2(x_2)) \in \eta_1 \oplus \eta_2 : \bar{g_1}(\eta_1(x_1)) = \bar{g_2}(\eta_2(x_2))]$———-(1)
Clearly, $\nu \subseteq \eta_1 \oplus \eta_2$ and ν is a non-empty set. Then for each $(\eta_1(x_1), \eta_2(x_2))$ and $(\eta_1(y_1), \eta_2(y_2))$ in ν and r in R we have

$$\begin{aligned} \bar{g_1}[(\eta_1(x_1)) + (\eta_1(y_1))] &= \bar{g_1}(\eta_1(x_1)) + \bar{g_1}(\eta_1(y_1)) \\ &= \bar{g_2}(\eta_2(x_2)) + \bar{g_2}(\eta_2(y_2))[by(1)] \\ &= \bar{g_2}[(\eta_2(x_2)) + (\eta_2(y_2))] \end{aligned}$$

implying $[(\eta_1(x_1))+(\eta_1(y_1))] \in \nu$ [by the definition of ν] and $[(\eta_1(x_1))r + (\eta_2(x_2))r] \in \nu$ or in other words we can say that ν is a submodule $\eta_1 \oplus \eta_2$. Since every fuzzy submodule of a fuzzy semisimple module is fuzzy semisimple, then we can say ν is fuzzy semisimple. Next, we have $\bar{g_1}$ is surjective homomorphism so $\bar{g_1}(\eta_1) = \mu$ therefore for each $\bar{g_1}(\eta_1) \in \mu \ \exists \ \eta_2(x) \in \eta_2$ such that $\bar{g_1}(\eta_1(x)) = \bar{g_2}(\eta_2(x))$. Define homomorphism $\bar{\pi}_1 : \ \nu \to \eta_1$ as $\bar{\pi}_1(\eta_1, \eta_2) = \eta_1$———-(2). Then we have

$$\begin{aligned} ker\bar{\pi}_1 &= [(\eta_1, \eta_2) : \bar{\pi}_1(\eta_1, \eta_2) = 0] \\ &= [(\eta_1, \eta_2) : \eta_1 = 0][by(2)] \\ &= [(0, \eta_2) : g_2(\bar{\eta_2}) = 0][since\eta_1 = 0] \end{aligned}$$

$= ker\bar{g}_2$ [by definition of kernel]
$= Im\bar{f_2}$ [Since the equation is exact]. Now, as $\bar{f_2}$ is injective homomorphism we can write Im $\bar{f_2} = \mu_2$. As a result, $Ker\bar{\pi}_1 = \mu_2$. So, a fuzzy short exact sequence can be formed

$$0 \to \mu_2 \to \nu \xrightarrow{\bar{\pi}_1} \eta_1 \to 0 \text{————-}(3)$$

Since η_1 is a fuzzy p-poor module equation (3) splits thus $\exists \ \bar{h} : \ \ \eta_1 \to \nu$ such that $\bar{\pi}_1$ o $\bar{h} = Id_{\eta_1}$. Hence by [17], we have $\nu = \eta_1 \oplus \mu_2$. In an analogous way, another fuzzy short exact sequence can be formed

$$0 \to \mu_1 \to \nu \xrightarrow{\bar{\pi}_2} \eta_2 \to 0 \text{ ———}(4)$$

to give $\nu = \eta_2 \oplus \mu_1$. Therefore $\mu_1 \oplus \eta_2 \cong \mu_2 \oplus \eta_1$. □

Lemma 4.10 *For any ring R, $\bigcap \mathscr{P}(\mu_P)$ = SS Mod-FR where μ_P is fuzzy right R-module.*

Proof. : The containment $\supseteq$ is obvious. Let $\mu_B \in \bigcap \mathscr{P}(\mu_P)$ and μ_C is a fuzzy submodule of μ_B. Then μ_B/μ_C is μ_B-projective. This implies μ_C is a fuzzy direct summand of μ_B. Hence μ_B is fuzzy semisimple. □

Lemma 4.11 *Let μ_M be a fuzzy p-poor module. Then for every ν_N, $\mu_M \oplus \nu_N$ is fuzzy p-poor.*

Proof. : Let ν_N be in Mod-FR and $\mu_M \oplus \nu_N$ is η_T-projective. Then μ_M is η_T-projective. Since μ_M is fuzzy p-poor, η_T must be fuzzy semisimple. Thus, $\mu_M \oplus \nu_N$ is fuzzy p-poor. □

Lemma 4.12 *If $\mu_M \oplus \nu_N$ is fuzzy p-poor and μ_M is fuzzy projective then ν_N is fuzzy p-poor.*

Proof. : Let ν_N is η_T-projective. Then $\mu_M \oplus \nu_N$ is η_T-projective. Hence η_T is a fuzzy semisimple. □

Lemma 4.13 *For every ring R the following are equivalent:*
(i) R is fuzzy semisimple artinian.
(ii) Every fuzzy module μ_M is p-poor.
(iii) There exists a fuzzy projective p-poor R-module.

Proof. : Let μ_M belong to Mod-FR. Then SS Mod-FR $\subseteq \mathscr{P}(\mu_M) \subseteq$ Mod-FR = SS Mod-FR. Hence (i) $\Rightarrow$ (ii). Also (ii) $\Rightarrow$ (iii) is clear. Assuming (iii) we can have μ_M as a fuzzy projective p-poor module. Thus, SS Mod-FR = $\mathscr{P}(\mu_M)$ = Mod-FR which implies (i). □

Proposition 4.14 *For every ring R the following are equivalent:*
(i) R is fuzzy semisimple artinian.
(ii) All fuzzy p-poor right(left)R-modules are fuzzy semisimple.
(iii) Non zero direct summmands of fuzzy p-poor right(left) R-modules are fuzzy p-poor.

Proof. : If R is fuzzy semisimple Artinian then (ii) and (iii) holds good. If (ii) or (iii) holds true then every fuzzy module is fuzzy p-poor, since a fuzzy p-poor module exists and a direct sum of any fuzzy module with a fuzzy p-poor module is again a p-poor [lemma 4.11]. Thus, R is a fuzzy semisimple Artinian [lemma 4.13]. □

Proposition 4.15 *If $Hom_R(\mu_M, \mu_A) = 0$ then μ_A belongs to projectivity domain of μ_M.*

Proof. : If $Hom_R(\mu_M, \mu_A) = 0$ then given any surjective homomorphism $\bar{g} : \mu_C \to \mu_A$ and if we suppose $\bar{h} : \mu_M \to \mu_C$ be a zero mapping then $\bar{g}\bar{h} = 0$. This implies μ_A belongs to the projectivity domain of μ_M. □

5 Future Scope

The behavior of fuzzy projective modules in various rings, such as the Laurent polynomial, QF and Artinian Rings can always be investigated to add another dimension to section 3. With the help of [7], one may always try to apply the concepts produced during this study to the research stated in [1] of fuzzy semirings, and can also try to exclude fuzzy projective modules using fuzzy projective semi modules supplied in [21]. Section 4 of current study act as a necessary catalyst to ponder one to choose an alternate perspective of fuzzy projectivity, namely fuzzy subprojectively poor modules whose domain contains precisely fuzzy projective modules only. In the same vein, fuzzy sub injectively poor modules can be studied to give fuzzy dimension to the research mentioned in [5] and [20].

Also, the research mentioned in [3, 4, 8, 11] and [12] can be extended using the current cancellation study on fuzzy projective modules. In addition, Schanuel's Lemma, which is being explored in a fuzzy setting is useful in giving a fresh and fuzzy direction to many vital classical notions, like the uniqueness of syzygy modules of a module up to free summands and uniqueness of cosyzygy modules of a module up to injective summands.

6 Conclusion

The concept of cancellation of the fuzzy module over the polynomial ring is initiated during this study, along with which the necessary and sufficient condition for the fuzzy projective module to conciliate cancellation is discussed. We have also introduced the concept of fuzzy p-poor modules, which has proven to be a viable alternative to employ fuzzy projective modules during the equivalence of fuzzy modules. To make it reader-friendly, the study done in this paper is diagrammatically summarized below:

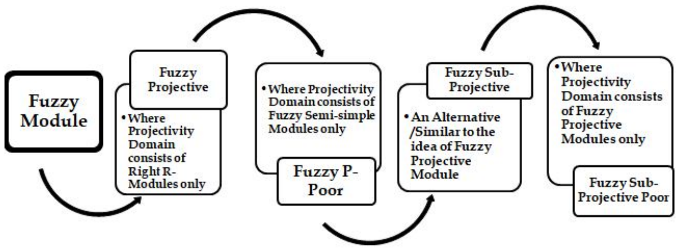

Fig.6 Summarizing the types of Fuzzy Modules studied in the paper

NOTE FOR FIGURE 6

(i) In definition 2.2 the fuzzy R-module μ_P can also, be called μ_A-projective or projective relative to μ_A.

(ii) Projectivity domain of a module μ_P is the set of all μ_A's such that μ_P is μ_A-projective.

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Contribution of individual authors to the creation of a scientific article (ghostwriting policy)

Author Contributions:

Amarjit kaur sahni formulated the manuscript after the necessary literature review. Constructed the examples, lemmas, theorems are given and developed the concept of Schanuel's Lemma using fuzzy p-poor modules.

Jayanti Tripathi Pandey suggested bridging the gap between fuzzy module cancellation and fuzzy projective module cancellation. In addition, the notion of cancellation over polynomial rings and ring R was advised to be studied. She also reviewed the manuscript's overall structure by giving helpful suggestions.

Ratnesh Kumar Mishra proposed the idea of cancellation on fuzzy projective modules.

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