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Determining the volatility in option pricing from degenerate parabolic equation

YILIHAMUJIANG YIMAMU
Department of Mathematics, Lanzhou Jiaotong University, Lanzhou, Gansu, 730070,
PEOPLE'S REPUBLIC OF CHINA

Abstract: This contribution deals with the inverse volatility problem for a degenerate parabolic equation from numerical perspective. Being different from other inverse volatility problem in classical parabolic equations, the model in this paper is degenerate parabolic equation. Due to solve the deficiencies caused by artificial truncation and control the volatility risk with precision, the linearization method and variable substitutions are applied to transformed the inverse principal term coefficient problem for classical parabolic equation into the inverse source problem for degenerate parabolic equation in bounded region. An iteration algorithm of Landweber type is designed to obtain the numerical solution of the inverse problem. Some numerical experiments are performed to validate that the proposed algorithm is robust and the unknown coefficient is recovered quite well.

Keywords: Inverse volatility problem, Linearization method, Landweber iteration, Numerical experiments
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1. Introduction

In the last 20 years, the parameter inversion problem in option pricing field has been extensively studied by many scholars, and the results of these studies have all relied on the famous Black-Sholes model. An important parameter in the Black-Sholes model is the volatility of the underlying asset associated with the option, which has a significant impact on market value of the options, and as such many scholars and practitioners in the financial industry have focused intensively on the volatility of an underlying asset in option pricing.

The derivation of the Black-Scholes partial differential equations builds on the basic components of derivatives theory, such as delta hedging and no arbitrage. One of the erroneous assumptions of the Black-Sholes model is that volatility of the underlying asset is a constant. Empirical research on implied volatility shows that implied volatility depends on strike prices. The value of a call option is obviously a function of various parameters in the contract, such as strike price K and expiration time $T-t$, where T is the expiration time and t is the current time. For our inverse problem, we will just use $u(s,t;K,T)$ for the option value.

Problem P1: Considering the option on the stock without paying dividend, it is well-known that $u(s,t;K,T)$ for a call option satisfies the following Black-Sholes equation

$$\begin{cases} \frac{\partial u}{\partial t}+L_{BS}=0, & (s,t)\in R^+\times(0,T), \\ u(s,T)=(s-K)^+=\max(0,s-K), & s\in R^+ \end{cases} \quad (1)$$

Here, s is the price of underlying stock, K is the strike price, T is the time of expiry, and μ and r are, respectively, the risk-neutral drift and the risk-free interest rate which are assumed to be constants. The Black-Sholes operator L_{BS} is given by

$$L_{BS}=\frac{1}{2}\sigma^2(s)s^2\frac{\partial^2 u}{\partial s^2}+s\mu\frac{\partial u}{\partial s}-ru,$$

The parameter $\sigma(s)$ is the volatility coefficient to be identified. We assume that

$$\frac{1}{2}\sigma^2(s)=\frac{1}{2}\sigma_0^2+g(s),$$

where $g(s)$ is small perturbation of constant σ_0. Given the following additional condition:

$$u(s^*,0,K,T)=u^*(K,T), \quad K\in R^+, \quad (2)$$

where s^* is market price of the stock at time $t^*=0$, and $u^*(K,T)$ indicates market price of the option with strike K at a given expiry time T. The inverse problem is to determine the functions u and σ satisfying (1.1) and (1.2), respectively.

The inverse volatility problem for the Black-Scholes equation has been discussed intensively in the literature. The inverse problem was first considered by Dupire in [4]. He applied the symmetric property of the transition probability density function to replace the option pricing inverse problem with an equation containing parameters K,T, which has duality, and proposed Dupire's formula for calculating implied volatility. Although this formula is seriously ill-posed, Dupire's solution lays an important foundation for later scholars to study this problem. In [5], the authors reduce the identification of volatility to an inverse parabolic problem with terminal observation and establish uniqueness and stability results by using Carleman estimates. This approach produces a nonlinear Fredholm integral equation in which the approximated solution is obtained from solving the integral equation iteratively. In [6], a time-dependent and a space-dependent volatility have been studied, respectively. A class of non-Gaussian stochastic processes has been generated in the study of spatially correlated volatility. The problem is transformed into a known inverse coefficient problem with final observations and uniqueness and stability theorems are established by using the dual equations. In [7], L.S. Jiang used an optimal control framework to determine the implied volatility, and carried out a rigorous mathematical analysis of the inverse problem, proving that the approximate optimal solution converges to an appropriate solution to the original problem. Similar results are derived in [15]. In [8]-[9], the inverse problem of identifying the principal coefficient is investigated when the solution is known, and a well-posed approximation algorithm to identify the coefficient is proposed. The existence, uniqueness and stability of such solutions are proved. The Tikhonov regularization method has always been an important tool for solving ill-posed problems. In [9], a new two-dimensional numerical differentiation method is proposed through Tikhonov regularization. Convergence analysis and numerical examples are given. The authors of [10] studied the stable identification problem of the local volatility surface $\sigma(S,t)$ in the Black-Scholes/Dupire equation from the market price of European options. The stability and convergence of the approximation obtained by Tikhonov regularization. In case of a known term-structure of volatility, based on the assumption that the volatility is constant in time $\sigma(S,t)=\sigma(S)$, the convergence rate under simple smoothness and decay conditions on the true volatility is proved. In recent years, linearization techniques have been applied to the inverse problem of option pricing. In [11,12,14], linearization techniques are applied to transform the problem into an inverse source problem, from which unknown volatility can be recovered. A stable numerical solution to the inverse problem is obtained by using the integral equation method and the Landweber iteration method. Both the theoretical analysis and the numerical examples demonstrate the effectiveness of the proposed method.

It is worth mentioning that the aforementioned scholars and their research have made outstanding contributions to the inverse volatility problem in option pricing. However, there are some deficiencies in these studies that need to be improved. One of the significant deficiencies is to consider that the original problem is in the unbounded region, so many scholars conduct numerical simulations by artificial truncation. There is a potential trouble in this approach, that is, if we truncate the interval too large, it will increase the amount of calculation, and if the truncation interval is too small, it will increase the error. In practical applications, this approach will fail to precisely control the volatility risk. In order to solve this deficiency, our main objective in this paper is to introduce a new variable substitutionand the linearization method that transforms original problem into the degenerate problem. Since the problem is ill-posedness, we adapt tikhonov regularization and design the landweber-type iteration to solve the invese valatility problem.

2. Preliminary Knowledge

We shall assume that the option price premium $u(\cdot,\cdot;K,T)$ satisfies the equation dual to the Black-Sholes equation (1) with respect to the strike price K and expiry time T:

$$\frac{\partial u}{\partial T}-\frac{1}{2}K^2\sigma^2(K)\frac{\partial^2 u}{\partial K^2}+\mu K\frac{\partial u}{\partial K}+(r-\mu)u=0. \qquad (3)$$

The equation (3) was found by Dupire in [4].

In the lognormal variables

$$y=\ln\left(\frac{K}{s^*}\right), \qquad \tau=T,$$

$$a(y)=\sigma(s^*e^y), \qquad U(y,\tau)=u(s^*e^y,\tau), \qquad (4)$$

The inverse problem P1 transforms into

$$\begin{cases} U_\tau-LU=0, & \tau>0,\\ U(y,0)=s^*(1-e^y)^+, y\in R, \end{cases} \qquad (5)$$

where

$$LU=\frac{1}{2}a^2(y)U_{yy}-\left(\frac{1}{2}a^2(y,\tau)+\mu\right)U_y-(r-\mu)U,$$

with the additional market data

$$U(y,T)=U(y), \quad y\in R, \qquad (6)$$

where

$$U(y)=u^*(s^*e^y,T). \qquad (7)$$

The goal is to recover the unknown space-dependent volatility coefficient $a(y)$ from market data $U(y)$.

If

$$\frac{1}{2}a^2(y)=\frac{1}{2}\sigma_0^2+g(y),$$

we have

$$U=V_0+V+v,$$

where V_0 is the solution of (5) when $a=\sigma_0$ and v is quadratically small with respect functions g . The principal linear term V satisfies

$$\begin{cases} V_\tau-AV=\alpha_0(y,\tau)g(y),\\ V(y,0)=0, \quad y\in R, \end{cases} \qquad (8)$$

where

$$AV=\frac{1}{2}\sigma_0^2V_{yy}-\left(\frac{1}{2}\sigma_0^2+\mu\right)V_y-(r-\mu)V,$$

with the additional final data

$$V(y,T)=V(y)=U(y)-V_0(y,T), \qquad y\in R, \qquad (9)$$

in which

$$\alpha_0(y,\tau)=s^*\frac{1}{\sigma_0\sqrt{2\pi\tau}}e^{-\frac{y^2}{2\tau\sigma_0^2}+c'y+d\tau},$$

$$c'=\frac{1}{2}+\frac{\mu}{\sigma_0^2}, \qquad d=-\frac{1}{2\sigma_0^2}\left(\frac{\sigma_0^2}{2}+\mu\right)^2+\mu-r,$$

is known.

The recovery of g in (5) and (6) is a linear inverse source problem. However, this is a matter of unbounded areas, we proposed some new variable substitutions here, so that the above problem is transformed into linear inverse source degenerate parabolic problem on bounded areas.

Taking

$$x=\arctan y, \qquad W=V. \qquad (10)$$

Problem P: Consider the following parabolic equation:

$$\begin{cases} W_\tau - BW = \alpha(x,\tau)g(x), (x,\tau) \in Q = \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \times (0,T], \\ W(x,0) = 0, \qquad -\frac{\pi}{2} < x < \frac{\pi}{2}, \end{cases} \tag{11}$$

where

$$\begin{aligned} BW &= \frac{1}{2}\sigma_0^2 cos^4 x W_{xx} \\ &- \left[\sigma_0^2 sinx cos^3 x + \left(\frac{1}{2}\sigma_0^2 + \mu\right) cos^2 x\right] W_x \\ &- (r-\mu)W, \end{aligned}$$

with the additional final data

$$V(y,T) = W(\tan x, T) = w(x), \quad x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), \tag{12}$$

in which

$$\alpha(x,\tau) = s^* \frac{1}{\sigma_0\sqrt{2\pi\tau}} e^{-\frac{(\tan x)^2}{2\tau\sigma_0^2} + c' * \tan x + d*\tau},$$

$$c' = \frac{1}{2} + \frac{\mu}{\sigma_0^2}, \quad d = -\frac{1}{2\sigma_0^2}\left(\frac{\sigma_0^2}{2} + \mu\right)^2 + \mu - r,$$

For the sake of analysis, we take

$$a(x) = \sigma_0^2 \cos^4 x,$$

$$b(x) = \sigma_0^2 \sin x \cos^3 x + \left(\frac{1}{2}\sigma_0^2 + \mu\right)\cos^2 x,$$

$$c = (r-\mu),$$

$$W(x,0) = \varphi(x),$$

$$f(x,\tau) = \alpha(x,\tau)g(x),$$

where $a(x), b(x), c, \varphi(x)$ and $\alpha(x,\tau)$ are given smooth function on $(-\frac{\pi}{2}, \frac{\pi}{2})$ which satisfies

$$a(-\frac{\pi}{2}) = a(\frac{\pi}{2}) = 0, \quad a(x) > 0, \quad x \in (-\frac{\pi}{2}, \frac{\pi}{2}), \tag{13}$$

$$b(-\frac{\pi}{2}) = b(\frac{\pi}{2}) = 0,$$

and $g(x)$ is an unknown source term in (11). We shall determine the functions W and g satisfying (11) and (12).

3. Landweber Iterations

There are many tools such as optimal control frameworks that can analyze the uniqueness and stability of solutions to such inverse problems. In this paper, we solve the inverse source problem from a numerical perspective. Since the inverse problem P is ill-posedness, the Tikhonov regularization method based on the L^2 gradient norm should be adopted. We consider the following linear system:

$$Tx = y, \quad x \in X, \ y \in Y,$$

where X and Y are Hilbert spaces, and $T : X \to Y$ is a bounded linear operator. Then Tikhonov functional can be written as follows:

$$J(x) := \|Tx - y\|^2 + \alpha\|x\|^2, \quad x \in X.$$

Obviously, the minimal element of the functional described above is equivalent to the solution of the following equation

$$\alpha x + T^*Tx = T^*y,$$

which is

$$x = (\alpha I + T^*T)^{-1}T^*y.$$

However, this method is not suitable for solving the problem of this article. There are two main difficulties. First, we do not know the specific form of $(\alpha I + T^*T)^{-1}$. In fact, we can write the specific form of $(\alpha I + T^*T)^{-1}$ only for individual operators, for example, T is a matrix. Second, a second derivative term of the unknown function appears in the form of Euler's equation, which makes the numerical simulation process very complicated.

Therefore, we use the iterative method to solve the inverse problem P. In this article, we particularly use the Landweber iteration method to get the numerical results.

Define the linear operator as shown below:

$$K: \ L^2(-\frac{\pi}{2}, \frac{\pi}{2}) \to H_a^1(-\frac{\pi}{2}, \frac{\pi}{2}), \tag{14}$$

$$Kg = W(\cdot, T) = w(x), \tag{15}$$

where W is the solution of equation (11) under the following initial value condition:

$$\varphi(x) \equiv 0, \quad x \in (-\frac{\pi}{2}, \frac{\pi}{2}). \tag{16}$$

Noticed that (15) can also be written as the following form:

$$g = (I - \alpha K^*K)g + \alpha K^*w. \tag{17}$$

for $\alpha > 0$, K^* is the adjoint operator of K, so the following iterative format can be used to solve equation (17):

$$\begin{cases} g = 0, \\ g_m = (I - \alpha K^*K)g_{m-1} + \alpha K^*w, \quad m = 1, 2, 3, \cdots. \end{cases} \tag{18}$$

It is easy to verify that equation (18) is the fastest descent method to solve the following equation, where α is the step size.

$$\phi(g) = \frac{1}{2}\|Kg - w\|. \tag{19}$$

Besides, there is the following lemma for the conjugate adjoint K^*.

Lemma 1. *For any given $h(x) \in H_a^1(-\frac{\pi}{2}, \frac{\pi}{2})$, let $\omega(x,0) = K^*h$, which is*

$$K^*: H_a^1(-\frac{\pi}{2}, \frac{\pi}{2}) \to L^2(-\frac{\pi}{2}, \frac{\pi}{2}),$$

$$K^*h = \omega(x,0),$$

then ω satisfies the following parabolic equation:

$$\begin{cases} -\omega_\tau - (a\omega)_{xx} - (b\omega)_x + c\omega \\ = \alpha(x,\tau)h(x), \quad (x,\tau) \in Q, \\ \omega(x,T) = 0. \end{cases}$$

The proof of lemma 1 is similar to the references(see [15]).

We have already known the K is a linear mapping, so problem (11)-(12) can be transformed into the first type of operator equation:

$$Kg = w - H\varphi. \tag{20}$$

Noticing the (20) can be rewritten as the following form:

$$g = (I - \alpha K^*K)g + \alpha K^*(w - H\varphi). \tag{21}$$

so the following iterative format can be used to solve equation (21):

$$\begin{cases} g = 0, \\ g_m = (I - \alpha K^*K)g_{m-1} + \alpha K^*(w - H\varphi), \\ m = 1, 2, 3, \cdots. \end{cases} \tag{22}$$

equation (22) is the fastest descent method to solve the following equation, where α is the step size.

$$\phi(g) = \frac{1}{2}\|Kg - (w - H\varphi)\|. \tag{23}$$

from the definition of H and (22), we have:

$$\begin{aligned} g_m =& g_{m-1} - \alpha K^*(Kg_{m-1} - (w - H\varphi)) \\ =& g_{m-1} - \alpha K^*(W_{m-1} - (\cdot, T) - w), \end{aligned} \tag{24}$$

where W_{m-1} is the solution of (11)-(12), when $g = g_{m-1}$. From equation (24) and the defined K^*, the following adjoint equation is introduced:

$$\begin{cases} -\omega_\tau - (a\omega)_{xx} - (b\omega)_x + c\omega \\ = W(x, T) - w(x), \quad (x, \tau) \in Q, \\ \omega(x, T) = 0. \end{cases} \tag{25}$$

Assuming that the true solution $w(x)$ is available, that is, there is a $g(x) \in L^2(-\frac{\pi}{2}, \frac{\pi}{2})$ such that

$$W(x, T; g) = w(x),$$

and the noise of the observation data has an upper bound δ, that is,

$$\|w^\delta - w\|_{L^2(-\frac{\pi}{2}, \frac{\pi}{2})} \leq \delta.$$

In summary, the calculation steps of the iteration format can be stated as follows:

Step one: Choose an initial iterative function $g = g(x)$. The initial function can be selected arbitrarily, for the convenience of calculation. We generally choose $g(x) = 0, \ x \in (-\frac{\pi}{2}, \frac{\pi}{2})$;

Step two: $W_0(x, \tau)$ is obtained by solving the initial boundary value problem (11), where $g = g(x)$;

Step three: Solve the equation (25) to get $\omega_0(x, \tau)$, where $W(x, T) = W_0(x, T)$;

Step four: Let $g_1(x) = g(x) - \alpha\omega_0(x, T)$, where $\alpha \geq 0$, and let $W_1(x, \tau)$ be the solution of (11) when $g = g_1(x)$;

Step five: Choose an arbitrarily small normal number ε as the error limit, calculate $\|W_1(x, T) - w(x)\|$ and compare the size with ε, if:

$$\|W_1(x, T) - w(x)\| < \varepsilon,$$

then terminate the iteration, and take $g = g_1(x)$ at this time. Otherwise, continue to execute step three, and let $g_1(x)$ be the new initial value of the iteration and continue to execute the inductive criterion until the iteration meets the termination condition.

Normally, if the input data is accurate, the more iterations of the Landweber iterative method, the higher the accuracy of the output data. However, in the case of noise, there will be errors in the initial iteration process. This calculation error will initially decrease as the number of iterations increases, but when a certain threshold is reached, it will increase rapidly as the number of iterations increases. Therefore, the iteration must be terminated at the appropriate time. In other words, in order to balance accuracy and stability, a compromise solution must be found, that is, a suitable parameter must be selected so that the iteration format is both accurate and stable method.

If $x \in (K^*K)^r(X), \ r \in N$, then the following error estimation formula can be obtained:

$$\left\|x^{N(\delta),\delta} - x\right\| \leq CM^{\frac{1}{2r+1}}\delta^{\frac{2r}{2r+1}},$$

where M is the boundary of $(K^*K)^r x$. Therefore, in this case it is different from Tikhonov's regularization method.

4. Numerical Experiments

We would like to give some numerical examples to test the validity of the proposed methods in section 2. The simulated data are generated by using the standard finite difference method to solve the direct problems (11) and (12) under some appropriate boundary conditions. We use $T = 1$, for numerical convenience, x interval $(-\frac{\pi}{2}, \frac{\pi}{2})$ is divided into 100 equal interval and on x axis we show number of an interval. It is same for the case of τ.

Example 1. Take

$$g(x) = \begin{cases} \cos x, & x \in [-\frac{\pi}{2}, \frac{\pi}{2}], \\ 0, & others. \end{cases}$$

and

$$\alpha(x, \tau) = 1 + \frac{1}{2}\tau\cos^4 x - \tau(\sin^2 x\cos^2 x + \sin x\cos x),$$
$$r = \mu = 0.5,$$

the numerical results are shown in Fig. 1, where the iteration number $k = 1000$. It can be seen from this figure that the main shape of unknown functions is recovered well.

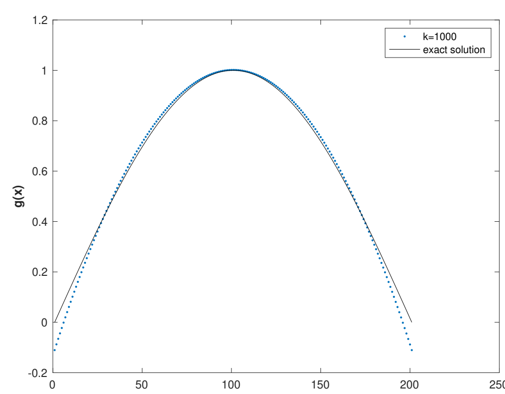

Fig. 1. Numerical solution of source function $g(x)$ for Example 1, where $k = 1000$

We also consider the case of noisy input data to test the stability of our algorithm. The noisy data are generated in the following form:

$$w(x) = W^{\delta}(x,T) = W(x,T)[1+\delta\times random(x)],\ x \in [-\frac{\pi}{2}, \frac{\pi}{2}],$$

with $\delta = 0.001, 0.01, 0.08$. The reconstruction result is displayed in Fig. 2 in which satisfactory approximation is obtained under the case of noisy data as well. As $\delta = 0.001, 0.01, 0.08$, the corresponding iteration numbers are $k = 1500, 1200$ and $k = 1000$, respectively. Since the observation data contains error, to obtain stable numerical results, we shall cease the iteration at some suitable time.

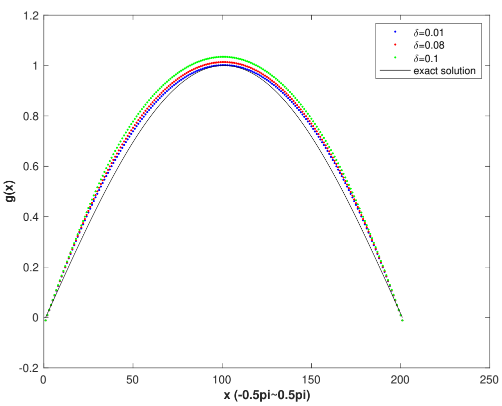

Fig. 2. Numerical solution of source function $g(x)$ for Example 1 with noisy input data.

Example 2. In the second numerical experiment, we take

$$T = 1,\ \ \alpha(x,\tau) = x^2 + 6\tau\cos^4 x + 4x\tau(\sin x\cos^3 x + \cos^2 x),$$

$$r = \mu = 0.5,$$

$$g(x) = \begin{cases} x^2, & x \in [-\frac{\pi}{2}, \frac{\pi}{2}], \\ 0, & others. \end{cases}$$

the numerical results are shown in Fig. 3, where the iteration number $k = 800$.

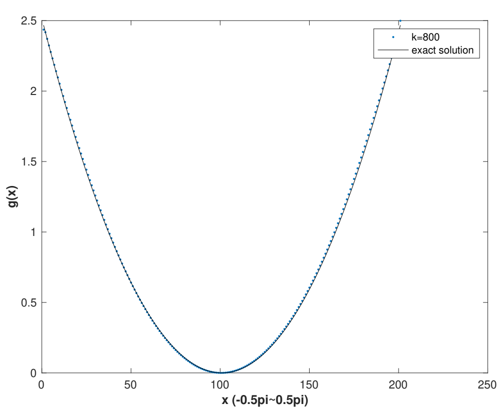

Fig. 3. Numerical solution of source function $g(x)$ for Example 2,where $k = 800$.

Generally speaking, it is not easy to reconstruct the information of unknown function near the boundary of parabolic equations. From this figure, we can see the main error which appears near the boundary is very small. Analogously, we also consider the noisy case, where the noisy levels are same to those in Example 1, i.e., $\delta = 0.001, 0.01, 0.08$. The corresponding numerical result is displayed in Fig. 4. One can see that for the noisy case, our algorithm is still stable and the unknown function is reconstructed very well. As $\delta = 0.001, 0.01, 0.08$, the corresponding iteration numbers are $k = 980, 820$ and 500, respectively.

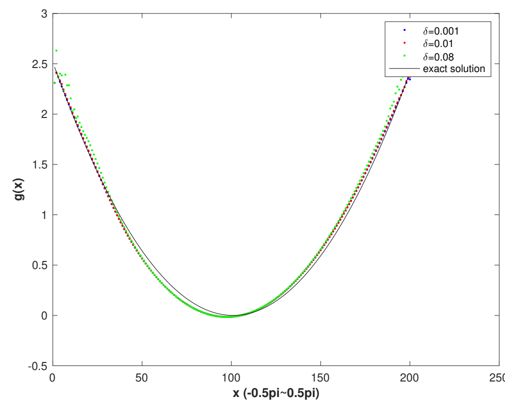

Fig. 4. Numerical solution of source function $g(x)$ for Example 2 with noisy input data.

5. Conclusion

We investigate the inverse volatility problem from numerical perspective. We apply the linearization method and variable substitutions to transform the inverse principal term coefficient problem for classical parabolic equation into the inverse source problem for a degenerate parabolic equation. We design an iteration of Landweber-type to obtain the numerical solution of the inverse problem and present several experiments to show that the proposed algorithm is robust.

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