Structures of Fibers of Groups Actions on Graphs

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Abstract.- In this paper we prove that if G is a group acting on a tree X such that G is fixing no vertex of X, the stabilizers of the edges of X are finite, and the stabilizers G_v of the vertices of X act on trees X_v where $X_v \neq X$, $X_u \neq X_v$ for all vertices u,v of X, where u ≠ v, and the stabilizer G_e of each edge contains no edge x of the tree $X_{o(x)}$ such that g(x) = $\bar{x}$ for every edge $g \in G_x$, then there exists a tree denoted $\tilde{X}$ and is called the fiber of X such that G acts on $\tilde{X}$.

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1 Introduction

Let G be a group and H≤G such that H acts on the set X. Define the relation ≡ on G×X as follows. If f,g∈G and u,v∈X let (f,u) ≡ (g,v) if there exists an element h∈H such that f = gh, and u = h^{-1}(v). It is clear that ≡ is an equivalent relation on on G×X. The equivalent class containing the element (g,v)∈G×H is denoted by $g \otimes_H v$. Thus,
$g \otimes_H v = \{(gh, h^{-1}(v)|h \in H\} \subseteq (gH) \times H(v)$, where
gH = {gh|h∈H} is the left coset of g and H(v) is the orbit of v under the action of H on X.
Let $g \in G, x \in X, A \subseteq G$, and $Y \subseteq X$. We have the following notation.
(1) $g \otimes_H$ Y = { $g \otimes_H$ a $|a \in Y\}$,
(2) $g \otimes_H$ X = { $g \otimes_H$ b $|b \in X\}$,
(3) $A \otimes_H$ x { $c \otimes_H$ x $|c \in A\}$,
(4) $A \otimes_H$ Y = { $a \otimes_H$ y $|y \in Y\}$,
(5)$G \otimes_H$ Y $= \{ g \otimes_H$ y $|g \in G, y \in Y\}$,
(6)$G \otimes_H$ X $= \{ g \otimes_H$ x $|g \in G, x \in X\}$,
(7)$G \otimes_H (H/X) = \{ G \otimes_H O | O \in H/X\}$, where $H/X = \{H(x)|x \in X\}$, the set of the orbits of the action of H on X, and $H(x) = \{h(x)|h \in H\}$, the orbit that contains the element x∈X. It is clear that if $g \in G$, $h \in H$, and $x \in X$, then $g \otimes_H$ h(x) = $gh \otimes_H$x and
$g \otimes_H$ x = $gh^{-1} \otimes_H h$(x).

The aim of this paper is to use above notation to show that groups acting on trees with inversions, fixing no vertex of the tree and of given trees on which the stabilizers of the vertices act and of finite edges stabilizers induce a new tree called the fiber tree of the group.

2 Concepts of Graphs

A graph X is the disjoint union of vertices V(X) and edges E(X). An edge e is called a loop if the initial vertex o(e) equals its terminal vertex t(e). If all edges in a graph are loop we call the graph a loop graph. Moving on, if a graph has at least on loop then it is called quasi-graph. A graph that all its initial vertices and terminals and inverses in a graph X is called a subgraph of X, say Y. Define $\overline{Y}$ to be the set $\overline{Y} = \{\bar{e}|e \in E(Y)\}$ where $\bar{e}$ is the invers of e.
Let $e_1, e_2, \ldots, e_n$ be edges in the graph X.
P = (e_1, e_2, ..., e_n) is called a path in X if
t(e_i) = o(e_{i+1}) for i = 1,2, ..., n-1. If u = o(e_1) and

$v = t(e_n)$ then P is called a path in X joining (or liking) the vertices u and v, or a path in X from u to v. If $o(e_1) = t(e_n)$ then P is called a closed path in X. If $e_{i+1} \neq \overline{e_i}$ for i = 1,2, ..., n, then P is called a reduced path in X. $o(P) = o(e_1)$ is the initial of P, $t(P) = t(e_n)$ is the terminal of P, $\overline{P} = (\overline{e_n}, \overline{e_{n-1}}, ..., \overline{e_2}, \overline{e_1})$ is the inverse of P. It is clear that $\overline{P}$ is a path in X joining the vertices t(P) and o(P). The edges in the path $P = (e_1, e_2, ..., e_n)$ are called the edges of P. If Q is a path in X such that t(P) = o(Q), then PQ is a path in X such that o(PQ) = o(P) and t(PQ) = t(Q). |P| = n is called the length of P. It is clear that if $e\in E(X)$ such that t(e) = o(P), then $(e, P) = (e, e_1, e_2, ..., e_n)$ is a path in X. A path P is called a simple circuit if it is close and contains no repeated edges. The set of all paths in the graph X is denoted by Path(X). We recommend readers to [2, 9] for the structures of groups acting on graphs without inversions and [1, 4, 5, 6] for with inversions, when an edge of the graph equals its inverse is allowed. For further studies see [10, 11].

A group G acts on a graph X if there exists a unique element denoted by $g(x)\in X$ for every $g\in G$ and every $x\in X$. G acts on X with inversions if there exist an element $g\in G$ and an edge $e\in E(X)$ such that $g(e) = \bar{e}$.

Remark. We write **(G;X)** to mean that G is a group acting on the tree X.

Definition 2.1. A subtree T is called a tree of representatives for the action of G on X if T has a unique vertex from each vertex orbit. A subtree Y is called a transversal it has a unique edge y such that $\bar{y}$ move in different orbit than y. The pair (T;Y) is called a **cover** or (a fundamental domain). See [3].

The following are some properties of the tree of the representatives T and the transversal Y.

(1) For any $v\in V(X)$, we have a unique vertex denoted v^* where $v^*\in V(T)$ and $G(v) = G(v^*)$. That is, $v = g(v^*)$, $g\in G$, where $G(v) = \{g(v): g\in G\}$ is the orbit containing v.

(2) For every $v\in V(X)$ we have $g\in G$ where $g(v^*) = v$.

(3) $v^* = v$ for all $v\in V(T)$.

(4) $(v^*)^* = v^*$ for all $v\in V(X)$.

(5) $(g(v))^* = v^*$ for all $g\in G$ and all $v\in V(X)$.

(6) If $g\in G$ where g(u) = v then u = v.

(7) If $g\in G$ and $u,v\in V(X)$ where g(u) = v then $(g(u))^* = u^* = v^*$.

(8) If $e\in E(Y)$ where $o(e)\in V(T)$, then $(o(e))^* = o(e)$, and if $t(e)\in V(T)$, then $(t(e))^* = t(e)$.

(9) If $e\in E(T)$, then $(o(e))^* = o(e)$, $(t(e))^* = t(e)$ and $o(e) \neq t(e)$.

(10) For every $a\in E(X)$ we have $g\in G$ and $b\in E(Y)$ where $a = g(b)$.

(11) If $g\in G$ and $a,b\in E(Y)$ on which $g(a) = b$, then $a = b$ or $a = \bar{b}$.

For the rest of this section G will be a group acting on a tree X of cover (T;Y).
The proofs of the following propositions are straight forward.

Proposition 2.2.
The edges E(Y) of Y can be split in to the following sets of edges, called the sets of splitting edges of Y.
(1) $E_0(Y) = \{m\in E(Y): o(m), t(m)\in E(T)\} = E(T)$.
(2) $E_1(Y) = \{y\in E(Y): o(y)\in E(T), t(y)\notin E(T), G(y) \neq G(\bar{y})\}$.
(3) $E_2(Y) = \{x\in E(Y): o(x)\in E(T), t(x)\notin E(T), G(x) = G(\bar{x})\}$.

Proposition 2.3. For $e\in E(Y)$, $o(e)\in V(T)$, there exists an element denoted $[e]\in G$ where $[e]((t(e))^*) = t(e)$. We choose [e] = 1 in case $e\in E(T)$ and $[e](e) = \bar{e}$ if $G(\bar{e}) = G(e)$.

Proposition 2.4. Let $m\in E_0(Y)$, $y\in E_1(Y)$, and $x\in E_2(Y)$. Then [m] = 1, $[\bar{y}] = [y]^{-1}$, $[\bar{x}] = [x]$, and $[x]^2\in G_x$.

Proposition 2.5. ([4]) The element $g\in G$, $g \neq 1$ can be written as a product
$g = g_0[e_1]g_1[e_2]g_2 ..., g_{n-1}[e_n]g_n$, where $e_1, e_2, ..., e_n$ are edges of Y and $g_0, g_1, g_2 ..., g_{n-1}, g_n$, are elements of G such that $(t(e_i))^* = (o(e_{i+1}))^*$ for i = 1, 2, ..., n-1, $g_0\in G_{(t(e_i))^*}$ and $g_i\in G_{(o(e_{i+1}))^*}$ for i = 1, 2, ..., n.

Definition 2.6. For $e\in E(Y)$ define the sign +e of e to be the edge +e = e if $o(e)\in V(T)$ and +e = [e](e) if $t(e) \in V(T)$.
It is clear that if $o(p)\in V(T)$ and $t(p) \in V(T)$, then $p\in E(T)$, [p] = 1 and +p = p.

Proposition 2.7. (1) For $e\in E(Y)$ we have the following.
(i) $o(+e) = (o(e))^*$, $t(+e) = [e]((t(e))^*)$, and $\overline{+e} = [e](+\bar{e})$.

(ii) $G_{+e} \leq G_{(o(e))^*}$ and $[e]^{-1}G[e] = G_{[e]^{-1}(e)} = G_{+e}$.
(iii) If $e \in V(T)$ or $G(e) = G(\bar{e})$ then $G_{+e} = G_e$.
(2) If $p,q \in E(Y)$ on which $+p = +q$ then $p = q$ or $p = \bar{q}$.
(3) If $g \in G$ and, $p,q \in E(Y)$ on which $g(+p) = +q$ then $+p = +q$.
(4) If $m \in E_0(Y)$, $y \in E_1(Y)$, $x \in E_2(Y)$ and $g \in G$, then $+m = m$, $+y = y$, $+\bar{y} = [\bar{y}](\bar{y}) = \overline{[y]^{-1}(y)}$, $+x = x$, and $+\bar{x} = x$.

Definition 2.8. Let $g \in G$ and $e \in E(X)$. The sum of g and e is denoted by $g \oplus e$ and is defined to be the pair $g \oplus e = (gG_{+e}, +e)$.
Let X* be the set $X^* = \{g \oplus e | g \in G, e \in E(Y)\}$.

We have the following facts. The proofs are clear.
(1) $g \oplus m = (gG_m, m)$, $g[m] \oplus \bar{m} = (gG_m, \bar{m}) = g \oplus \bar{m}$
(2) $g \oplus y = (gG_y, y)$, $g \oplus \bar{y} = (gG_{[y]^{-1}(y)}, [y]^{-1}(\bar{y}))$, $g[y] \oplus \bar{y} = (g[y]G_{[y]^{-1}(y)}, \overline{[y]^{-1}(y)})$.
(3) $g \oplus x = g[x] \oplus \bar{x} = (g[x]G_x, x) = g[x] \oplus x$.
(4) $X^* = \{g \oplus m, g \oplus y, g \oplus \bar{y}, g \oplus x | m \in E_0(Y), y \in E_1(Y), x \in E_2(Y)\}$.
(5) If $f,g \in G$ and $p,q \in E(Y)$ such that $f \oplus p = g \oplus q$, then $f = gh$, where $h \in G_{+p}$ and $+p = +q$.

Proposition 2.9. $X^* \approx E(X)$.
Proof. It is clear that the mapping $\theta: X^* \to E(X)$ giving by $\theta(g \oplus e) = g(+e)$ is one-one and onto.

3 Inversion Elements

Definition 3.1. If G is a group acting on a graph X, $g \in G$ and $e \in E(X)$ where $g(e) = \bar{e}$, we say that g is an inversion element of G and e is called an inversion edge of X under g. It is clear that if X is a quasi-graph on which G acts then we have $e \in E(X)$ on which $\bar{e} = e$. Then $1_G(e) = \bar{e}$. In this case 1_G is an inversion element of G and e is an inversion edge of E(X) under 1_G, the identity element of G
Proposition 3.2. Let X be a graph where the group G acts. Then the following imply each other.
(1) The action of G on X is with inversions.
(2) E(X) has an inversion edge and G has an inversion element.
(3) The orbit space G/X is a quasi-graph.
Proposition 3.3. Let X be a graph on which the group G acts such that G has inversion element $g \in G$ and $e \in E(X)$ be an inversion edge of X under g. Let $u \in \{o(e), t(e)\}$. Then
(1) $\bar{e}$ is an inversion under g, $g^2 \in G_e$ and $g^2 \in G_u$.
(2) $g \notin G_u$ if X is a tree.
Proof. Clear.
Lemma 3.4. Let (G;X) and $H \leq G$. Then
(i) If H has an element that is an inversion, then H is not contained in the stabilizer of any vertex of X.
(ii) If H is finite and contains no inversion element then H is contained in a stabilizer of a vertex of X, H fixes a vertex of X, and has a trivial orbit for the action of H on X . Moreover, if $u,v \in V(X)$ are two vertices of X such that $H \leq G_u$ and $H \leq G_v$, then $H \leq \cap_e G_e$, where e is an edge of the reducing path in X joining u and v.
Proof. (i) Let $g \in H$ be an inversion element. Then there exists an inversion edge $e \in E(X)$ of X under g. So $g(e) = \bar{e}$. Let $u \in \{o(e), t(e)\}$. Since X is a tree, Proposition 3.3-(2) implies that $g \notin G_u$. If u = v we are done. Now assume that $u \neq v$. We need to show that $g \notin G_v$. X being a tree implies that there exists a unique reduced path $P = (e_1, e_2, ..., e_n) \in Path(X)$ joining u and v. So the edges $e_1, e_2, ..., e_n$ are distinct and $n \geq 1$. The properties of groups acting trees imply that Q: $g(e_1), g(e_2,), ..., g(e_n) \in Path(X)$, where Q is a unique reduced liking g(u) and g(v) of length $n \geq 1$. Assume that $g \in G_v$. Then g(v) = v. Let u = o(e). We consider the following cases.
Case 1. $e = e_1$. So P is the path $P = (e, e_2, ..., e_n)$. The property $g(e) = \bar{e}$ implies that Q is the reduced path $Q = (\bar{e}, g(e_2,), ..., g(e_n)) \in Path(X)$ linking g(u) and g(v) = v. Since $t(\bar{e}) = o(e) = u = o(e_2)$, then $o(g(e_2,)) = g(u)$ and $R = (g(e_2,), ..., g(e_n)) \in Path(X)$ such that R is reduced and linking g(u) and g(v) = v and of length n-1. Hence Q and R are two reduced paths in Path(X) joining g(u) and g(v) = v of different lengths n and n-1. This is impossible because X is a tree. This implies that $g \notin G_v$.
Case 2. $e \neq e_1$. Then $(\bar{e}, e_1, e_2, ..., e_n) \in Path(X)$ such that it is reduced and linking $o(\bar{e}) = t(e)$ and v. Then $(g(e), e_1, e_2, ..., e_n) \in Path(X)$ is reduced and linking t(e) and v. As X is a tree, $S = (g^2(e), g(e_1), g(e_2,), ..., g(e_n)) \in Path(X)$ is a unique and reduced linking g(t(e)) and g(v) = v. Since u = o(e) and $g(\bar{e}) = g^2(e)$, therefore by Proposition 3.3-(2), $g^2 \in G_u$. So $g^2(u) = u$. So S is a reduced path in X joining u and v. Thus, R and S are two distinct reduced paths in X joining u and v. Since X is a tree, this contradicts a property of a tree that two distinct vertices of a tree are joined by exactly one reduced path. This implies that $g \notin G_v$.

Let $u = t(e)$. Then $u = o(\bar{e})$ and by adjusting the cases above yields that $g \notin G_v$. Hence H is not contained in any stabilizer G_v. for any vertex $v \in V(X)$ of X.

(ii) Since H is finite and contains no inversion element of G, by [2, Theorem 8.1, p. 27], there exists a vertex $v \in V(X)$ such that $H \leq G_v$. Then the stabilizer H_v of the vertex $v \in V(X)$ is $H \leq G_v$ The case $H \leq G_v$ implies that $H = H_v$. So H fixes the vertex v. Since H is finite, the stabilizer H_v and the orbit $H(v) = \{h(v): h \in H\}$ of v under the action of H on X are finite. By the Orbit-Stabilizer Theorem [8, Lemma 4.11, p. 72], the orders of $|H|$, $|H_v|$, and $|H(v)|$ satisfy the equation $|H| = |H_v||H(v)|$. The case $H = H_v$ implies that $|H| = |H_v|$. So $|H(v)| = 1$. So H has a trivial orbit for the action of H on X. If $u,v \in V(X)$ are two vertices of X such that $H \leq G_u$ and $H \leq G_v$ then $G_u = G_v$ or $H \leq G_u \cap G_v$ and by Theorem 4.3 of [7], H is contained in the intersection of the stabilizers of the edges of the reduce path in X joining u and v.

Corollary 3.5. Let (G;X), $y \in E(X)$, $o(y) = v$, X_v be a tree where $(G_v;X_v)$ and is finite and contains no inversion element of G_v. Then we have $w(y) \in V(X_v)$ of X_v on which $G_y \leq (G_v)_{w(y)}$, $(G_v)_{w(y)}$ is the stabilizer of the vertex w(y) under the action of G_v on X_v.

Proof. By Lemma 3.4-(ii).

4 Basics of the Fibers

For the rest of this section, we have (G;X) of a cover (T;Y) of the following assumptions.

(a) For each $v \in V(T)$ let X_v be a graph such that $X_u \cap X_v = \emptyset$ for all $u \in V(T)$, $u \neq v$, and the stabilizer of v G_v acts on X_v.

(b) For $g \in G$, $v \in V(T)$ let $g \otimes_{G_v} X_v = \{g \otimes_{G_v} x | x \in X_v\}$ and $G \otimes_{G_v} X_v = \cup_{f \in G} f \otimes_{G_v} X_v = \{g \otimes_{G_v} x | g \in G, x \in X_v\}$.

(c) Let $\widehat{X} = \cup_{v \in V(T)}[G \otimes_{G_v} X_v]$, and $\widetilde{X} = X^* \cup \widehat{X}$, where $X^* = \{g \oplus e | g \in G, e \in E(Y)\}$ and $g \oplus e = (G_{+e}, +e)$ of Definition 2.8.

Definition 4.1. For $e \in E(Y)$, let $w(e) \in V(X_{(o(e))^*})$ be chosen so that $G_{+e} \leq (G_{(o(e))^*})_{w(e)}$, where $(G_{(o(e))^*})_{w(e)}$ is the stabilizer of w(e). So $w(\bar{e}) \in V(X_{(t(e))^*})$ and $G_{+\bar{e}} \leq (G_{(t(e))^*})_{w(\bar{e})}$.

Proposition 4.2. Let $u,v \in V(T)$ and $f,g \in G$. Then

(1) If $u_1 \in V(X_u)$ and $v_1 \in V(X_v)$ where $f \otimes_{G_u} u_1 = g \otimes_{G_v} v_1$ then $u = v$, $G_u = G_v$, and we have $h \in G_u$ where $f = gh$ and $v_1 = h(u_1) \in X_u$.

(2) If $u \neq v$ then $[f \otimes_{G_u} X_u] \cap [g \otimes_{G_v} X_v] = \emptyset$ and $[G \otimes_{G_u} X_u] \cap [G \otimes_{G_v} X_v] = \emptyset$.

Lemma 4.3. (1) $\widetilde{X}$ is a graph where $V(\widetilde{X}) = \cup_{v \in V(T)}[G \otimes_{G_v} V(X_v)]$ and $E(\widetilde{X}) = X^* \cup [\cup_{v \in V(T)}(G \otimes_{G_v} E(X_v))]$, where the ends of $\alpha \in E(\widetilde{X})$ are

If $\alpha \in X^*$, then $\alpha = g \oplus e = (gG_{+e}, +e)$, where $g \in G$, and $e \in E(Y)$. Let $o(\alpha) = o(g \oplus e) = g \otimes_{G_{(o(e))^*}} w(e)$, $t(\alpha) = t(g \oplus e) = g[e] \otimes_{G_{(t(e))^*}} w(\bar{e})$, and $\bar{\alpha} = \overline{g \oplus e} = g[e] \oplus \bar{e}$. If $\alpha \in (\cup_{v \in V(T)}[G \otimes_{G_v} E(X_v)]$, then the ends of α are defined as follows. $o(\alpha) = o(g \otimes_{G_v} e) = g \otimes_{G_v} o(e)$, $t(\alpha) = t(g \otimes_{G_v} e) = g \otimes_{G_v} t(e)$, and, $\bar{\alpha} = \overline{g \otimes_{G_v} e} = g \otimes_{G_v} \bar{e}$, where $e \in E(X_v)$ and, o(e), t(e), and $\bar{e}$ are the initial, the terminal and the inverse of the edge $e \in E(X_v)$.

(2) $G \otimes_{G_v} X_v$, $v \in V(X)$, and $\widehat{X}$ form subgraphs of $\widetilde{X}$.

Proof. First we show that $\widetilde{X}$ forms a graph. Since X_v is a graph, this implies that $V(X_v) \cap E(X_v) = \emptyset$. If $[G \otimes_{G_v} V(X_v)] \cap [G \otimes_{G_v} E(X_v)] \neq \emptyset$, then there exists an element $a \in [G \otimes_{G_v} V(X_v)] \cap [G \otimes_{G_v} E(X_v)]$. So $a = f \otimes_{G_v} x = g \otimes_{G_v} e$, where $f,g \in G$, $x \in V(X_v)$, and $e \in E(X_v)$. From $\otimes_{G_v}$ we have $h \in G_v$ where $f = gh$ and $e = h(x)$. The case $h(x) \in V(X_v)$, because G_v acts on X_v, implies that $e \in V(X_v)$ which contradicts above that $V(X_v) \cap E(X_v) = \emptyset$. So $[G \otimes_{G_v} V(X_v)] \cap [G \otimes_{G_v} E(X_v) = \emptyset$.

Since $X^* \cap (\cup_{v \in V(T)}[G \otimes_{G_v} V(X_v)]) = \emptyset$, we have $(\cup_{v \in V(T)}[G \otimes_{G_v} V(X_v)]) \cap [X^* \cup (\cup_{v \in V(T)}[G \otimes_{G_v} E(X_v)])] = \emptyset$. By taking the set of vertices $V(\widetilde{X})$ to be $V(\widetilde{X}) = \cup_{v \in V(T)}[G \otimes_{G_v} V(X_v)]$ and the set of edges $E(\widetilde{X})$ to be $E(\widetilde{X}) = X^* \cup (\cup_{v \in V(T)}[G \otimes_{G_v} E(X_v)])$ we see that $V(\widetilde{X}) \cap E(\widetilde{X}) = \emptyset$.

Now we show that for $\alpha \in \widetilde{X}$ we have $o(\bar{\alpha}) = t(\alpha)$, $t(\bar{\alpha}) = o(\alpha)$, and $\bar{\bar{\alpha}} = \alpha$.

Let $\alpha \in E(\widetilde{X})$.

Case 1. $\alpha \in X^*$. Then $\alpha = g \oplus e = (gG_{+e}, +e)$, where $g \in G$, and $e \in E(Y)$.

Then $o(\bar{\alpha}) = o(\overline{g \oplus e}) = o(g[e] \oplus \bar{e}) = g[e] \otimes_{G_{(o(\bar{e}))^*}} w(\bar{e}) = g[e] \otimes_{G_{(t(e))^*}} w(\bar{e}) = t(\alpha)$,

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$t(\overline{\alpha}) = t(\overline{g \oplus e}) = t(g[e] \oplus \overline{e}) = g[e][\overline{e}] \otimes_{G_{(t(\overline{e}))^*}} w(\overline{e}) = g \otimes_{G_{(o(e))^*}} w(e) = o(\alpha)$, because $[e][\overline{e}] \in G_{+e} \leq G_{(o(e))^*}$ and $\overline{\overline{e}} = e$, $\overline{\overline{\alpha}} = \overline{\overline{g \oplus e}} = \overline{g[e] \oplus \overline{e}} = g[e][\overline{e}] \oplus \overline{\overline{e}} = g \oplus e = \alpha$.

Case 2. $\alpha \in (\cup_{v\in V(T)}[G \otimes_{G_v} E(X_v)]$, then $\alpha = g \otimes_{G_v} e$, where $g \in G$ and $e \in E(X_v)$ and, $o(\overline{\alpha}) = o(\overline{g \otimes_{G_v} e}) = o(g \otimes_{G_v} \overline{e}) = g \otimes_{G_v} o(\overline{e}) = g \otimes_{G_v} t(e) = t(g \otimes_{G_v} e) = t(\alpha)$. Similarly, $t(\overline{\alpha}) = t(\overline{g \otimes_{G_v} e}) = t(g \otimes_{G_v} \overline{e}) = g \otimes_{G_v} t(\overline{e}) = g \otimes_{G_v} o(e) = o(g \otimes_{G_v} e) = o(\alpha)$. Furthermore, $\overline{\overline{\alpha}} = \overline{\overline{g \otimes_{G_v} e}} = \overline{g \otimes_{G_v} \overline{e}} = g \otimes_{G_v} \overline{\overline{e}} = g \otimes_{G_v} e = \alpha$. Then $\widetilde{X}$ forms a graph.

(2) From above we have $V(G \otimes_{G_v} X_v) \cap E(G \otimes_{G_v} X_v) = \emptyset$. If $a \in E(G \otimes_{G_v} X_v)$ is an edge of $G \otimes_{G_v} X_v$, then $a = g \otimes_{G_v} e$, where $g \in G$, $e \in E(X_v)$. It is clear that $o(a) = g \otimes_{G_v} o(e)$, $t(a) = g \otimes_{G_v} t(e)$, and $\overline{a} = \overline{g \otimes_{G_v} e} = g \otimes_{G_v} \overline{e}$ are the ends of a, where $o(e)$, $t(e)$, and $\overline{e}$ are the ends of $e \in E(X_v)$. So $G \otimes_{G_v} X_v$ forms a subgraph of $\widetilde{X}$. Since $G \otimes_{G_v} X_v \subseteq \widetilde{X}$, therefore $G \otimes_{G_v} X_v$ forms a subgraph of $\widetilde{X}$. Since $V(\widetilde{X}) = V(\widehat{X})$ and $\widehat{X} \subseteq \widetilde{X}$, this shows that $\widehat{X}$ is a subgraph of $\widetilde{X}$.

Lemma 4.4. ($G; \widetilde{X}$) where if $f, g \in G$, $v \in V(T)$, $x \in V(X_v)$, $p \in E(X_v)$, and $e \in E(Y)$ then $f(g \otimes_{G_v} x) = fg \otimes_{G_v} x$, $f(g \otimes_{G_v} p) = fg \otimes_{G_v} p$, and $f(g \oplus e) = fg \oplus e$. $(G; \widetilde{X})$ is with inversions if the action of $(G;X)$ is with inversions.

Corollary 4.5. For each $g \in G$, $x \in V(X_v)$, $p \in E(X_v)$, and $e \in E(Y)$, the stabilizers of the elements $g \otimes_{G_v} x \in V(\widetilde{X})$, $g \otimes_{G_v} p \in E(\widetilde{X})$, and $g \oplus e \in E(\widetilde{X})$ are the followings. $G_{g \otimes_{G_v} x} = g(G_v)_x g^{-1}$, $G_{g \otimes_{G_v} p} = g(G_v)_p g^{-1}$, $G_{g \oplus e} = gG_{+e}g^{-1}$.

Proposition 4.6. If the stabilizer of every element of X is finite and the stabilizer of every element of X_v under the action of G_v on X_v is finite, then the stabilizer of every element of $\widetilde{X}$ is finite.

Definition 4.7. For $v \in V(T)$ and $e \in E(Y)$, let $L_v = (G/G_v) \times \{v\}$ and $L_e = (G/G_{+e}) \times \{+e\}$.

Lemma 4.8. If $g \in G$, $v \in V(T)$, $x \in V(X_v)$, $p \in E(X_v)$, and $e \in E(Y)$ then the orbits of $g \otimes_{G_v} x \in V(\widetilde{X})$, $g \otimes_{G_v} p \in E(\widehat{X})$, and $g \oplus e \in E(\widetilde{X})$ are the following. $G(g \otimes_{G_v} x) = G \otimes_{G_v} G_v(x)$, and $G(g \otimes_{G_v} p) = G \otimes_{G_v} G_v(p)$, $G(g \otimes_{G_v} p) = G \otimes_{G_v} G_v(p)$, and, $G(g \oplus e) = L_e$.

Corollary 4.9. $G/\widetilde{X} = \cup_{v\in V(T)}[G \otimes_{G_v} (G_v/X_v)] \cup [\cup_{e\in E(Y)} L_e]$, where $G_v/X_v = \{G_v(a) | a \in X_v\}$, the orbit space $G \otimes_{G_v} (G_v/X_v) = \{G \otimes_{G_v} G_v(a) | a \in X_v\}$.

Corollary 4.10. If G/X, G_v/X_v, $v \in V(T)$, $[G, G_a]$ are finite $a \in X$, then $G/\widetilde{X}$ is finite.

Proof. It is clear that L_v and L_e are finite, $v \in V(T)$, $e \in E(Y)$. So $G/\widetilde{X}$ is finite.

Lemma 4.11. For $v \in V(T)$ and $X_v = \{v\}$ be the trivial graph of one vertex v and no edges. Let $\widehat{X}$ and $\widetilde{X}$ be the graphs defined above. Then

(1) $V(\widehat{X}) = \{L_v | v \in V(T)$ and $E(\widehat{X}) = \emptyset$.

(2) For $e \in E(Y)$, $w(e) = (o(e))^*$, $w(\bar{e}) = (t(e))^*$.

(3) For $g \in G$, $e \in E(Y)$, and $g \oplus e$, $o(g \oplus e) = g \otimes_{G_{(o(e))^*}} (o(e))^*$ and $t(g \oplus e) = g[e] \otimes_{G_{(t(e))^*}} (t(e))^*$.

(4) $V(\widetilde{X}) = V(\widehat{X}) = \{G/G_v | v \in V(T)\}$ and $E(\widetilde{X}) = X^* = \{g \oplus e | g \in G, e \in E(Y)\}$, where $g \oplus e = (G_{+e}, +e)$.

(5) For $g \in G$, $v \in V(T)$ and $e \in E(Y)$, the stabilizers, the orbits of the vertex $g \otimes_{G_v} v \in V(\widetilde{X})$ and the edge $g \oplus e \in E(\widetilde{X})$ are $G_{g \otimes_{G_v} v} = gG_v g^{-1}$, a conjugate of G_v in G, $G_{g \oplus e} = gG_{+e}g^{-1}$.

(6) The orbit space $G/\widehat{X}$ is the set $G/\widetilde{X} = \{L_v, L_e | v \in V(T), e \in E(Y)\}$.

(7) $(G; \widetilde{X})$ is with inversions if $(G;X)$ is with inversions.

Proof. $X_v = \{v\}$ is a trivial graph of one vertex v and no edge for each vertex $v \in V(T)$. That is, $V(X_v) = \{v\}$ and $E(X_v) = \emptyset$. G_v acts on X_v trivially.

(1) $V(\widehat{X}) = \cup_{v\in V(T)}[G \otimes_{G_v} V(X_v)] = \cup_{v\in V(T)}[G \otimes_{G_v} \{v\}] = \{g \otimes_{G_v} v | g \in G, v \in V(T)\}$. The case $g \otimes_{G_v} v = \{(gh, h^{-1}(v)) | h \in G_v\} = \{(gh, v) | h \in G_v\} = [G/G_v] \times \{v\} = L_v$ implies that $V(\widehat{X}) = \{L_v | v \in V(T)\}$. Since $E(X_v) = E(\{v\}) = \emptyset$, therefore $E(\widehat{X}) = \cup_{v\in V(T)}[G \otimes_{G_v} E(X_v)] = \{g \otimes_{G_v} e | g \in G, e \in E(X_v)\} = \{g \otimes_{G_v} \emptyset | g \in G\} = \emptyset$. So $\hat{X}$ is a null graph.

(2) $w(e) \in X_{(o(e))^*} = \{(o(e))^*\}$ and $w(\bar{e}) \in X_{(t(e))^*} = \{(t(e))^*\}$. Therefore $w(e) = (o(e))^*$ and $w(\bar{e}) = (t(e))^*$.

(3) For $g\in G$, $e\in E(Y)$, the initial and the terminal of $g\oplus e$ are $o(g\oplus e) = g\otimes_{G_{(o(e))^*}} w(e) = g\otimes_{G_{(o(e))^*}}(o(e))^*$, $t(g\oplus e) = g[e]\otimes_{G_{(t(e))^*}} w(\bar{e}) = g[e]\otimes_{G_{(t(e))^*}} (t(e))^*$.
(4) From (1), $V(\widetilde{X}) = V(\widehat{X}) = \{G/Gv|v\in V(T)\}$ and, $E(\widetilde{X}) = E(\hat{X})\cup X^* = \emptyset\cup X^* = X^* = \{g\oplus e|g\in G, e\in E(Y)\}$, where $g\oplus e = (G_{+e},+e)$.
(5) By Corollary 4.5, $G_{g\otimes_{G_v} v} = g(G_v)_v g^{-1} = gG_v g^{-1}$ because $(G_v)_v = G_v$ and $G_{g\oplus e} = gG_{+e}g^{-1}$, a conjugate of G_{+e}. By Lemma 4.8, $G(g\otimes_{G_v} v) = G\otimes_{G_v} G_v(v) = G\otimes_{G_v}\{v\} = \{g\otimes_{G_v} v|g\in G\} = (G/G_v)\times\{v\} = L_v$, and $G(g\oplus e) = \{f(g\oplus e)|f\in G\} = \{fg\oplus e|f\in G\} = \{(fgG_{+e},+e)|f\in G\} = (G/G_{+e})\times\{+e\} = L_e$.
(6) From above $G/\widetilde{X} = \{G(g\otimes_{G_v} v), G(g\oplus e) |v\in V(T), |e\in E(Y)\} = \{L_v, L_e|v\in V(T), e\in E(Y)\}$.
(7) By Lemma 4.4.

Corollary 4.12. For $v\in V(T)$ let $X_v = \{v\}$ such that the index of the stabilizer G_v in G is of finite. Then $\widetilde{X} \approx X$.

5 Paths in the fiber graph $\widetilde{X}$

Again, in this section, G will be a group acting on a connected graph X of fundamental domain (T;Y) such that for each $v\in V(T)$, X_v is a graph such that $X_u\cap X_v = \emptyset$, $u\in V(T)$, $u \neq v$, and the stabilizer G_v acts on X_v. Furthermore, for $e\in E(Y)$, $w(e)$ is a vertex $w(e)\in V(X_{(o(e))^*})$ such that $G_{+e}\leq(G_{(o(e))^*})_{w(e)}$ and $w(\bar{e})\in V(X_{(t(e))^*})$ where $G_{+\bar{e}}\leq(G_{(t(e))^*})_{w(\bar{e})}$. Now we state and prove relations in the graphs X and $\widetilde{X}$.

Definition 5.1. Assume that $g\in G$ and $v\in V(T)$, $a,b\in V(X_v)$, $e_1, e_2, ..., e_n\in E(X_v)$. Let $P = (e_1, e_2, ..., e_n)$. Define $g \otimes_{G_v} P = (g\otimes_{G_v} e_1, g\otimes_{G_v} e_2, ..., g\otimes_{G_v} e_n)$.

Lemma 5.2. (1) $P\in Path(X_v)$ if and only if $g \otimes_{G_v} P\in Path(g \otimes_{G_v} X_v)$. If $o(P) = a$ and $t(P) = b$, then $o(g \otimes_{G_v} P) = g \otimes_{G_v} a$ and $t(g \otimes_{G_v} P) = g \otimes_{G_v} b$.
(2) P is closed if and only if $g \otimes_{G_v} P$ is closed.
(3) P is reduced if and only if $g \otimes_{G_v} P$ reduced.
(4) P is a simple circuit if and only if $g \otimes_{G_v} P$ is a simple circuit.

Proof. (1) By the definition of $g \otimes_{G_v} X_v$, $g \otimes_{G_v} a$, $g \otimes_{G_v} b\in V(g \otimes_{G_v} X_v)$, and, $g\otimes_{G_v} e_1, g\otimes_{G_v} e_2, ..., g\otimes_{G_v} e_n\in E(g \otimes_{G_v} X_v) = g \otimes_{G_v} E(X_v)$. Let $P\in Path(X_v)$. Then for each i we have $o(e_{i+1}) = t(e_i)$. This implies that $t(g\otimes_{G_v} e_i) = g\otimes_{G_v} t(e_i) = g\otimes_{G_v} o(e_{i+1}) = o(g\otimes_{G_v} e_{i+1})$. So $g \otimes_{G_v} P\in Path(g \otimes_{G_v} X_v)$. Conversely, if $g \otimes_{G_v} P\in Path(g \otimes_{G_v} X_v)$, then $g\otimes_{G_v} t(e_i) = g\otimes_{G_v} o(e_{i+1})$. By the definition of $\otimes_{G_v}$ we have $f\in G_v$ on which $g = gf$ and $f^{-1}(t(e_i)) = o(e_{i+1})$. So $g = 1$ and $t(e_i) = o(e_{i+1})$. This implies that $P\in Path(X_v)$. If $o(P) = a$ then $o(e_1)$.
(2), (3), and **(4)** are clear.

Proposition 5.3. Let $f, g\in G$, $v\in V(T)$, $e\in E(Y)$, $P\in Path(f \otimes_{G_v} X_v)$, and the edge $a = g\oplus e$ of X^*. Then
(1) There exist two vertices denoted α_P and β_P of $V(X_v)$ such that the initial of P is $o(P) = f \otimes_{G_v}\alpha_P$ and the terminal of P is $t(P) = f \otimes_{G_v}\beta_P$.
(2) If $o(a) = t(P)$, then $v = (o(e))^*$ and we have $h_e\in G_{(o(e)^*}$ on which $g = fh_e$, $h_e(w(e)) = \beta_P$.
(3) If $t(a) = o(P)$, then $v = (t(e))^*$ and there exists an element $k_e\in G_{(t(e))^*}$ such that $g[e] = fk_e$ and $k_e(w(\bar{e})) = \alpha_P$.
Proof. (1) Since $f \otimes_{G_v} X_v$ is a graph and $P\in Path(f \otimes_{G_v} X_v)$, therefore $o(P)$ and $t(P)$ are in $V(f \otimes_{G_v} X_v) = f \otimes_{G_v} V(X_v)$. Then $o(P) = f \otimes_{G_v}\alpha_P$ and $t(P) = f \otimes_{G_v}\beta_P$, where $\alpha_P,\beta_P\in V(X_v)$. **(2)** If $o(a) = t(P)$ then $o(a) = g\otimes_{G_{(o(e))*}} w(e) = t(P) = f\otimes_{G_v}\beta_P$. Then $v = (o(e))^*$ and $G_{(o(e)^*} = G_v$ and $g\otimes_{G_{(o(e)^*}} w(e) = f\otimes_{G_{(o(e)^*}}\beta_P$. This implies we have $h_e\in G_{(o(e)^*}$, $g = fh_e$ and $h_e(w(e)) = \beta_P$.
(3) Similar to (2), $v = (t(e))^*$, $t(a) = g[e]\otimes_{G_{G_{(t(e)^*}}} w(\bar{e}) = o(P) = f \otimes_{G_{(t(e)^*}}\alpha_P$ and we have $k_e\in G_{(t(e))^*}$ on which $g[e] = fk_e$ and $k_e(w(\bar{e})) = \alpha_P$.

Lemma 5.4. Let $P\in Path(\widetilde{X})$. Then
(i) If $v\in V(T)$ and $g\in G$ such that $P\in Path(g \otimes_{G_v} X_v)$, then we have the edges $e_1, e_2, ..., e_n\in E(X_v)$ such that $P = (g\otimes_{G_v} e_1, g\otimes_{G_v} e_2, ..., g\otimes_{G_v} e_n)$, $o(P) = g \otimes_{G_v} o(e_1)$ and $t(P) = g \otimes_{G_v} t(e_n)$.
(ii) If $P\notin Path(g \otimes_{G_v} X_v)$ for all $v\in V(T)$, $g\in G$, then there exist elements $f_1, f_2, ..., f_n, f_{n+1}, g_1, g_2, ..., g_n$ of

G, vertices $v_1, v_2, ..., v_n, v_{n+1}$ of V(T), edges $e_1, e_2, ..., e_n$ of E(Y), and paths $P_1, P_2, ..., P_n$ of $f_1 \otimes_{G_{v_1}} X_{v_1}$, $f_2 \otimes_{G_{v_2}} X_{v_2}$, ..., $f_n \otimes_{G_{v_n}} X_{v_n}$ such that
$P = (P_1, g_1\oplus e_1, P_2, g_2\oplus e_2, ..., P_n, g_n\oplus e_n, P_n)$.
Furthermore, the following properties of P hold.
(1) $v_i = (o(e_i))^*$, $v_{i+1} = (t(e_i))^*$ and $(t(e_i))^* = (o(e_{i+1}))^*$.
(2) $P_i \in Path(f_i \otimes_{G_{(o(e_i))^*}} V(X_{(o(e_i))^*}))$ and there exist vertices $\alpha_{P_i}, \beta_{P_i} \in V(X_{(o(e_i))^*})$ such that
$o(P_i) = f_i \otimes_{G_{(o(e_i))^*}} \alpha_{P_i}$ and $t(P_i) = f_i \otimes_{G_{(o(e_i))^*}} \beta_{P_i}$.
(3) $o(P) = o(P_1) = f_1 \otimes_{G_{(o(e_1))^*}} \alpha_{P_1}$ and $t(P) = t(P_n) = f_n \otimes_{G_{(t(e_n))^*}} \beta_{P_n}$. So P joins the graphs $f_1 \otimes_{G_{(o(e_1))^*}} V(X_{(o(e_1))^*})$ and $f_n \otimes_{G_{(t(e_n))^*}} V(X_{(o(e_n))^*})$. That is, $P \in Path(\widetilde{X})$ linking the vertices $f_1 \otimes_{G_{(o(e_1))^*}} \alpha_{P_1}$ and $f_n \otimes_{G_{(t(e_n))^*}} \beta_{P_n}$ of $V(\widetilde{X})$.
(4) $o(g_i\oplus e_i) = t(P_i) = f_i \otimes_{G_{(o(e_i))^*}} \beta_{P_i}$, and $t(g_i\oplus e_i) = o(P_{i+1}) = f_{i+1} \otimes_{G_{(o(e_{i+1}))^*}} \alpha_{P_{i+1}}$.
(5) We have $h_i, k_i \in G_{(o(e_i))^*}$ and
$g_i = f_i h_i$, $g_i[e_i] = f_i k_i = g_{i+1} h_i$,
$w(e_i) = h_i(\alpha_{P_i})$, and $w(\overline{e_i}) = k_i(\beta_{P_i})$.
(6) If P is closed, then $(o(e_1))^* = (t(e_n))^*$, $f_1 = f_{n+1}h$, and $g_1 = g_n[e_n]l^{-1}hk$where $h,k,l \in G_{(t(e_n))^*}$.
(7) If P is reduced, then P_i is reduced and $+e_{i+1} \neq +\overline{e_i}$
Proof. (i) From Proposition 5.3-(1) and Lemma 5.2 where $o(e_1) = \alpha_P$ and $t(e_n) = \beta_P$ the result follows.
(ii) Since $E(\widetilde{X}) = E([\cup_{v\in V(T)}(G \otimes_{G_v} X_v])\cup X^* = [\cup_{v\in V(T)}(G \otimes_{G_v} E(X_v)]\cup X^*$, the edges of P consist of edges of the forms $g \otimes_{G_v} p \in (G \otimes_{G_v} E(X_v)$ and edges of the form $g\oplus e \in X^*$. By (i) above, P consists of edges from both of $\cup_{v\in V(T)}(G \otimes_{G_v} E(X_v)]$ and X^*. So the edges of P consist of the edges of paths in $g \otimes_{G_v} X_v$, $v\in V(T)$, $g\in G$ and edges of X^*. This gives the required structure of P introduced above. Now the proofs of (1)-(7) of the lemma as follows.
(1) Follows from Proposition 5.3 -(2).
(2) Follows from Proposition 5.3-2.
(3) From (2) above.
(4) From (3) above.
(5) From Proposition 5.3-(3).
(6) Since P is closed, therefore o(P) = t(P). So $o(P_1) = t(P_n)$. Since $o(P_1) = f_1 \otimes_{G_{v_1}} \alpha_{P_1}$, $t(P_n) = f_{n+1} \otimes_{G_{v_{n+1}}} \beta_{P_n}$, $v_1 = (o(e_1))^*$ and $v_{n+1} = (t(e_n))^*$, therefore $f_1 \otimes_{G_{v_1}} \alpha_{P_1} = f_{n+1} \otimes_{G_{v_{n+1}}} \beta_{P_n}$ and $v_1 = v_{n+1}$. Then $(o(e_1))^* = (t(e_n))^*$, $f_1 = f_{n+1}h$ and $h(\beta_{P_n}) = \alpha_{P_1}$, where $h\in G_{(o(e1))^*}$. Since P is a path of $\widetilde{X}$, therefore $t(P_1) = o(g_1\oplus e_1)$ and $t(g_n\oplus e_n) = o(P_n)$, therefore Then $f_1 \otimes_{G_{(o(e_1))^*}} \beta_{P_1} = g_1 \otimes_{G_{(o(e_1))^*}} w(e_1)$ and $g_n[e_n] \otimes_{G_{(t(e_n))^*}} w(\overline{e_n}) = f_{n+1} \otimes_{G_{(t(en))^*}} \alpha_{P_n}$. So $g_1 = f_1 k$ and $g_n[e_n] = f_{n+1} l$, where $k,l \in G_{(t(e_n))^*}$.
From above, $g_1 = g_n[e_n]l^{-1}hk$.
(7) Since P is a reduced path, no edge of P is the inverse of its previous edge. So, if a,b are adjacent edges of a path P_i, then $b \neq \bar{a}$. So the path P_i is reduced. If for some i, i = 1, 2, ..., n-1, we have $+e_{i+1} = +\overline{e_i}$, then $(t(e_i))^* = (o(e_{i+1}))^*$ and have $g_{i+1} = g_i[e_i]h$, $h\in G_{(t(e_i))^*}$. This implies that$g_{i+1}\oplus e_{i+1} = (g_{i+1}G_{+e_{i+1}}, +e_{i+1}) = (g_i[e_i]hG_{+\overline{e_i}}, +\overline{e_i}) = (g_i[e_i]G_{+\overline{e_i}}, +\overline{e_i})$, because $h\in G_{(t(e_i))^*}$ and $G_{+\overline{e_i}} \leq G_{(t(e_i))^*}$. This implies that $g_{i+1}\oplus e_{i+1} = g_i[e_i]\oplus\overline{e_i} = \overline{g_i \oplus e_i}$. Then P contains the edge $g_i\oplus e_i$ and its inverse $\overline{g_i \oplus e_i} = g_i[e_i]\oplus\overline{e_i}$. Contradiction, because P is reduced path.

Corollary 5.5. Let $P = (P_1, g_1\oplus e_1, P_2, g_2\oplus e_2, ..., P_n, g_n\oplus e_n, P_n)$ be the path in Lemma 5.4. Then $(g_1\oplus e_1, g_2\oplus e_2, ..., g_n\oplus e_n)$ is a path in the trivial fiber graph $\widetilde{X_1}$ where $X_v = \{v\}$ for all $v\in V(T)$.

Proposition 5.6. Let $P = (P_1, g_1\oplus e_1, P_2, g_2\oplus e_2, ..., P_n, g_n\oplus e_n, P_n)$ be the path of Lemma 5.4. Let
$P^* = (g_1(+e_1), g_2(+e_2), ..., g_n(+e_n))$. Then
(1) $P^* \in Path(X)$.
(2) If P is closed so P* is closed.
(3) If P is reduced so P* is reduced.
(4) If P is a simple circuit, so P* is a simple circuit.
Proof. Clear.

Lemma 5.7. Let $g = g_0[e_1]g_1[e_2]g_2 ..., g_{n-1}[e_n]g_n$ be a product of the element g. For i = 1, 2, ..., n, let
$f_i = g_0[e_1]g_1[e_2]g_2 ..., g_{i-2}[e_{i-1}]g_{i-1}$ with convention that $f_1 = g_0$, let $q_i = f_i(+e_i)$, and $p_i = f_i\oplus e_i$. Then
(1) $q = (q_1, q_2, ..., q_n) \in Path(X)$ linking $(o(e_1))^*$ to $g((t(e_n))^*)$.
(2) $o(p_i) \in f_i \otimes_{G_{(o(e_i))^*}} X_{(o(e_i))^*}$ and
$t(p_i) \in f_{i+1} \otimes_{G_{(o(e_{i+1}))^*}} X_{(o(e_{i+1}))^*}$.
(3) For i = 1, 2, ..., n, assume that

$P_i \in Path(f_i \otimes_{G_{(o(e_i))^*}} X_{(o(e_i))^*}$ such that $t(P_1) = o(p_1)$, $o(P_i) = t(p_i)$, $t(P_i) = o(p_{i+1})$ for i = 2, ..., n-1. Then $P = (P_1, p_1, ..., p_{n-1}, P_n)$ is a path in $\widetilde{X}$ joining the vertices $o(P_1)$ and $t(P_n)$.
(4) $P^* = q$.
Proof. (1) We need to show that $o(q) = (o(e_1))^*$, $t(q) = g((t(e_n))^*)$, $t(q_i) = o(q_{i+1})$, i = 1,2, ..., n-1. Now $o(q) = o(q_1) = o(f_1(+e_1)) = f_i(o(+e_i)) = f_1((o(e_1))^*) = (o(e_1))^*$ because $f_1 = g_0 \in G_{(o(e_1))^*}$, $t(q) = t(q_n) = t(f_n(+e_n)) = t(g_0[e_1]g_1[e_2]g_2 ..., g_{n-1}[e_n]$ $t(+e_n)) = g_0[e_1]g_1[e_2]g_2 ..., g_{n-1}[e_n]g_n((t(e_n))^*) = g((t(e_n))^*)$, and $t(q_i) = t(f_i(+e_i) = f_i(t(+e_i)) = f_i[e_i]((t(e_i))^*) = f_i[e_i]((o(e_{i+1}))^*) = o(q_{i+1})$.
(2) $o(p_i) = o(f_i \oplus e_i) =$
$f_i \otimes_{G_{(o(e_i))^*}} w(e_i) \in f_i \otimes_{G_{(o(e_i))^*}} X_{(o(e_i))^*}$ and,
$t(p_i) = t(f_i \oplus e_i) = f_i[e_i] \otimes_{G_{(t(e_i))^*}} w(\bar{e_i}) =$
$f_i[e_i]g_{i+1} \otimes_{G_{(t(e_i))^*}} w(\bar{e_i}) =$
$f_{i+1} \otimes_{G_{(o(e_{i+1}))^*}} w(\bar{e_i})$ because $(t(e_i))^* = (o(e_{i+1}))^*$ and $g_{i+1} \in G_{(o(e_{i+1}))^*}$. This shows that
$t(p_i) \in f_{i+1} \otimes_{G_{(o(e_{i+1}))^*}} X_{(o(e_{i+1}))^*}$.
(3) By Lemma 5.4-(ii), the cases $o(p_i) \in f_i \otimes_{G_{(o(e_i))^*}} X_{(o(e_i))^*}$ and $t(p_i) \in f_{i+1} \otimes_{G_{(o(e_{i+1}))^*}} X_{(o(e_{i+1}))^*}$ of (2) above implies that P is a path in $\widetilde{X}$ linking $o(P_1)$ and $t(P_n)$.
(4) This follows from Proposition 5.4.

Definition 5. 8. The path P of Lemma 5.7 is called the path in $\widetilde{X}$ obtained from the product of the element $g = g_0[e_1]g_1[e_2]g_2 ..., g_{n-1}[e_n]g_n$ and q is the path obtained by collapsing the vertices of P.

Lemma 5.9. (I) If X and X_v are connected, $v \in V(T)$, so $\widetilde{X}$ is connected.
(II) If X and X_v are trees, $v \in V(T)$, so $\widetilde{X}$ is a tree.
Proof. (I) The following steps imply that $\tilde{X}$ is connected.
(1) $f \otimes_{G_v} X_v = \{f \otimes_{G_v} x | x \in X_v\}$ is connected , $f \in G$ $v \in V(T)$.
(2) $1 \otimes_{G_u} X_u$ and $1 \otimes_{G_v} X_v$ are linked by a path in $\widetilde{X}$, $u,v \in V(T)$.
(3) $1 \otimes_{G_v} X_v$ and $g \otimes_{G_v} X_v$ are linked by a path in $\widetilde{X}$, $g \in G$, $v \in V(T)$.
(4) $f \otimes_{G_u} X_u$ and $g \otimes_{G_v} X_v$ are linked by a path in $\widetilde{X}$, $f,g \in G$, $u,v \in V(T)$.
(1) Let $p,q \in V(f \otimes_{G_v} X_v) = f \otimes_{G_v} V(X_v)$, $q \neq q$ be two distinct vertices of $f \otimes_{G_v} X_v$. By the definition of $f \otimes_{G_v} X_v$, we have vertices $a,b \in V(X_v)$ where $p = f \otimes_{G_v} a$ and $q = f \otimes_{G_v} b$. If a equals b, then p = q. This contradicts the assumption that $q \neq q$. Since X_v is a connected graph, we have $P \in Path(X_v)$ on which $o(P) = a$ and $t(P) = b$. By Lemma 5.2, $f \otimes_{G_v} P \in Path(f \otimes_{G_v} X_v)$, $o(f \otimes_{G_v} P) = f \otimes_{G_v} a = p$ and, $t(f \otimes_{G_v} P) = f \otimes_{G_v} b = q$. So $f \otimes_{G_v} X_v$ is connected.
(2) For $u,v \in V(T)$, there exist edges $e_1, e_2, ..., e_n \in E(T)$ such that $p = (e_1, e_2, ..., e_n) \in Path(T)$, $o(p) = u$, $t(p) = v$, $t(e_i) = o(e_{i+1})$, and $w(\bar{e_i}) = w(e_{i+1})$, i = 1, 2, ..., n-1. Then for each $e \in \{e_1, e_2, ..., e_n\}$, $[e] = 1$, $+e = e$, $+\bar{e} = \bar{e}$, $(o(e))^* = o(e)$, and $(t(e))^* = t(e)$, $w(e) \in V(X_{o(e)})$, $w(\bar{e}) \in V(X_{t(e)})$, $G_{+e} = G_e$, $G_{+\bar{e}} = G_{\bar{e}} = G_e$. Consider the edges $1 \oplus e_1, 1 \oplus e_2, ..., 1 \oplus e_n$ of X^*. Then
$o(1 \oplus e_1) = 1 \otimes_{G_{o(e_1)}} w(e) = 1 \otimes_{G_u} w(e) \in 1 \otimes_{G_u} X_u$,
$t(1 \oplus e_n) = [e_n] \otimes_{G_{t(e_n)}} w(\overline{e_n}) = 1 \otimes_{G_v} w(\overline{e_n}) \in 1 \otimes_{G_v} X_v$,
and, $t(1 \oplus e_i) = [e_i] \otimes_{G_{t(e_i)}} w(\bar{e_i}) = 1 \otimes_{G_{o(e_{i+1})}} w(e_{i+1})$.
So $Q = (1 \oplus e_1, 1 \oplus e_2, ..., 1 \oplus e_n) \in Path(\widetilde{X})$ and liking the subgraphs $1 \otimes_{G_u} X_u$ and $1 \otimes_{G_v} X_v$ of $\widetilde{X}$.
(3) If g = 1, we have case (1). Assume that $g \neq 1$. By Proposition 2.5, the element g has the product $g = g_0[e_1]g_1[e_2]g_2 ..., g_{n-1}[e_n]g_n$ where $(o(e_1))^* = (t(e_n))^* = v$. Then $P_g = (P_1, p_1, ..., p_{n-1}, P_n)$ of Lemma 5.8, $\in Pth(\widetilde{X})$ and liking the subgraphs $1 \otimes_{G_v} X_v$ and $g \otimes_{G_v} X_v$. Similarly, for $f \in G$, $u \in V(T)$ we have $P_f \in Path(\widetilde{X})$ liking the subgraphs $1 \otimes_{G_u} X_v$ and $f \otimes_{G_u} Xu$.
(4) Let P_f^{-1} be the converse of the path P_f of (3) above. Then the composition $P_f^{-1} Q P_g$ of the paths P_f^{-1}, Q and P_g, $\in Path(\widetilde{X})$ liking the subgraphs $f \otimes_{G_u} X_u$ and $g \otimes_{G_v} X_v$ of $\widetilde{X}$. Consequently, $\widetilde{X}$ is a connected graph.
(II) First we show that for $g \in G$, $v \in V(T)$, the subgraph $g \otimes_{G_v} X_v$ forms a subtree. If $g \otimes_{G_v} X_v$ contains a loop, then there exists an edge $\gamma \in E(g \otimes_{G_v} X_v) = g \otimes_{G_v} E(X_v)$ such that $o(\gamma) = t(\gamma)$. Then $\gamma = g \otimes_{G_v} e$ where $e \in E(X_v)$. For the case $o(\gamma) = t(\gamma)$ we have
$o(\gamma) = o(g \otimes_{G_v} e) = g \otimes_{G_v} o(e) = t(\gamma) = t(g \otimes_{G_v} e) = g \otimes_{G_v} t(e)$. The definition of $\otimes_{G_v}$ implies that

o(e) = t(e). So e is a loop in the tree X_v. This contradicts the assumption that X_v is a tree. If $g \otimes_{G_v} X_v$ contains a simple circuit $P = (P_1, P_2, ..., P_n) \in Path(g \otimes_{G_v} X_v)$, then $o(P_1) = t(P_n)$, $t(P_i) = o(P_{i+1})$ and $P_{i+1} \neq \overline{P_i}$ for i = 1, 2, ..., n-1. $g \otimes_{G_v} X_v$ being a subgraph of $\widetilde{X}$ implies that there exist edges $e_1, e_2, ..., e_n \in E(X_v)$ such that $P_i = g \otimes_{G_v} e_i$, i = 1, 2, ..., n. Then $o(e_1) = t(e_n)$, $t(e_i) = o(e_{i+1})$, and $e_{i+1} \neq \overline{e_i}$, i = 1, 2, ..., n. This implies that $(e_1, e_2, ..., e_n) \in Path(X_v)$ is a simple circuit. This is a contradiction because X_v is a tree. *So* $g \otimes_{G_v} X_v$ is a subtree of $\widetilde{X}$. If $P \in Path(\widetilde{X})$ is a simple circuit, then from above, $P \notin Path(g \otimes_{G_v} X_v)$. Then P is the path of the form of Lemma 5.4-(ii). Then Lemma 5.7 shows that the path P* obtained by collapsing the vertices of P is a simple circuit in X. Since X is a tree, we get contradiction because a tree contains no simple circuits. Hence $\widetilde{X}$ is a tree.

6 The Main Result

Theorem 6.1. Assume (G;X) of a given cover (T;Y) where X_v is a tree, $X_u \cap X_v = \emptyset$ for all $u \in V(T)$, $u \neq v$. Furthermore, for $d \in E(Y)$, assume that G_d of d is finite and containing no inversions of the tree $X_{(o(d))^*}$. Then

(1) There exists $v(d) \in V(X_{(o(d))^*})$ where $G_{+d} \leq (G_{(o(d))^*})_{v(d)}$, and v (d) = w(d), w(d) is the vertex of Definition 4.1.

(2) The fiber $\widetilde{X}$ is a tree.

(3) If (G;X) is with inversions or for $v \in V(T)$, if $(G_v; X_v)$ is with inversions, then $(G; \widetilde{X})$ is with inversions.

(4) The structures of the stabilizers of the elements of $\widetilde{X}$ are $G_{f \otimes_{G_v} x} = f(G_v)_x f^{-1}$, $G_{f \otimes_{G_v} p} = f(G_v)_p f^{-1}$, and $G_{f \oplus d} = fG_{+d} f^{-1}$ for all $f \in G$, $z \in V(X_v)$, $p \in E(X_v)$, and $d \in E(Y)$.

(5) structures for the orbits of the elements of $\widetilde{X}$ are

$G(f \otimes_{G_v} z) = G \otimes_{G_v} G_v(z)$,

$G(f \otimes_{G_v} p) = G \otimes_{G_v} G_v(p)$, and, $G(f \oplus d) = (G/G_{+d}) \times \{+d\}$, $f \in G$, $z \in V(X_v)$, $p \in E(X_v)$, and $d \in E(Y)$.

(6) The orbit space $G/\widetilde{X}$ has the form $G/\widetilde{X} = \cup_{v \in V(T)}[G \otimes_{G_v} (G_v/X_v)] \cup [\cup_{d \in E(Y)} (G/G_{+d}) \times \{+d\}]$.

The edges of $\widetilde{X}$ have the properties that

$o(f \otimes_{G_v} p) = f \otimes_{G_v} o(p)$, $t(f \otimes_{G_v} p) = f \otimes_{G_v} t(p)$, and, $\overline{f \otimes_{G_v} p} = f \otimes_{G_v} \bar{p}$, and $o(f \oplus d) = f \otimes_{G_{(o(d))^*}} v(d)$, $t(f \oplus d) = f[d] \otimes_{G_{(t(d))^*}} v(\bar{d})$ and $\overline{f \oplus d} = f[d] \oplus \bar{d}$ for all $f \in G$ $p \in E(X_v)$, and $d \in E(Y)$.

Proof.

(1) Since the stabilizer of each edge $e \in E(Y)$ is finite, therefore G_{+e} is finite. Since $G_{+e} \leq G_{(o(e))^*}$, and G_{+e} contains no inverter edges of the tree $X_{(o(e))^*}$, therefore by Corollary 2.5, there exists a vertex denoted v(e) where $G_{+e} \leq ((G_{(o(e)^*})_{v(e)}$. Since w(e) is arbitrary, we take w(e) = v(e). (2) Th assumptions that X and X_v, $v \in V(Y)$ are trees, Lemma 5.9-(II) implies that the fiber $\widetilde{X}$ is a tree. (3) Lemma 4.4. (4) Corollary 4.5. (5) Lemma 4.8. (6) Corollary 4.9.

Corollary 6.2. If (G;X) is without inversions and G_d, $d \in E(Y)$ is finite, then $\widetilde{X}$ forms a tree.

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