Numerical Investigation and Factor Analysis of the Spatial-Temporal Multi-Species Competition Problem

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Abstract: - This work considers the spatial-temporal multi-species competition model. A mathematical model is described by a coupled system of nonlinear diffusion reaction equations. We use a finite volume approximation with semi-implicit time approximation for the numerical solution of the model with corresponding boundary and initial conditions. To understand the effect of the diffusion to solution in one and two-dimensional formulations, we present numerical results for several cases of the parameters related to the survival scenarios. We control all non-diffusion parameters, including reproductive growth rate, competition rate, and initial condition of population density of competing species, and compare the dynamic and equilibrium under regular diffusion rate and small diffusion rate; we found that competing species with small diffusion rate can reach a higher equilibrium over the whole geographic domain, but requires more time steps. The random initial conditions' effect on the time to reach equilibrium is investigated. We control other parameters and examine the impact of the initial condition of the species population; we found that regardless of the values of initial conditions in the system, competing species populations will arrive at an equilibrium point. The influence of diffusion on the survival scenarios is presented. We control other parameters and examine the effect of diffusion of species; we found that when the ratio of diffusion rates passes some thresholds, the survival status will change. In real-world problems, values of the parameters are usually unknown yet vary in some range. To evaluate the impact of parameters on the system stability, we simulate a spatial-temporal model with random parameters and perform factor analysis for two and three-species competition models. From the perspective of the numerical experiment, we release control for all parameters and perform factor analysis on simulation results. We found that the initial population condition has a minimum effect on the final population, which aligns with the outcome of our controlled numerical experiment on the initial condition. Diffusion is the dominant factor when diffusion rates are on the same scale as other parameters. This dominant factor aligns with our controlled numerical experiment on diffusion rate, where the change in diffusion rate leads to different survival statuses of species. However, when diffusion rates are 1/10 on the scale of other parameters, reproductive growth rates and competition rates become the dominant factors.

Key-Words: - Spatial-temporal model, Multi-species competition model, System of diffusion-reaction equations, Numerical investigation, Impact of parameters, Factor analysis

Received: September 19, 2021. Revised: September 6, 2022. Accepted: October 9, 2022. Published: November 4, 2022.

1 Introduction

Understanding the stability of ecosystems is of fundamental importance to ecology, [1],[2],[3]. Fundamental mathematical models of such systems are described by a coupled system of ordinary differential equations (ODEs). The Lotka–Volterra Competition model (LVC) is a basic model that describes the dynamics of the species population competing for some shared resource. The LVC model has been applied in many areas, including biological systems, industry, and economics, [4],[5],[6],[7],[8]. For example, the model can simulate the marsh ecosystems for the wetlands at the Nueces River mouth, [9].

The LVC model is based on the logistic population model :

$$\frac{du}{dt} = ru(1-u),$$

where u is the size of the population at a given time t and r > 0 is the per-capita growth rate.

For the general case of multi-species competition, we have

$$\frac{du^{(k)}}{dt} = r_k u^{(k)} (1 - u^{(k)}) - \sum_{l \neq k} \alpha_{kl} u^{(k)} u^{(l)},$$

$$\forall k = 1, \dots, M,$$

where $u^{(k)}$ is the population of the *k*-th species, r_k is the *k*-th population growth rate, α_{kl} is the interaction coefficient due to competition $(u^{(l)} \text{ compete with } u^{(k)})$, and *M* is the number of species (equations).

In such systems, a temporal dynamic model can represent and describe the behaviors of the entire system. However, real ecosystems interact in different locations, and spatial structure is essential and has an enormous impact on the final equilibrium state, [10],[11]. Systems of the partial differential (PDEs) used equations are to describe spatial-temporal systems. A mathematical model is described by a coupled system of unsteady nonlinear reaction-diffusion equations. For the multi-species interaction, we have

$$\frac{\partial u^{(k)}}{\partial t} - \nabla \cdot \left(\varepsilon_k \nabla u^{(k)}\right)$$
$$= r_k u^{(k)} \left(1 - u^{(k)}\right) - \sum_{l \neq k} \alpha_{kl} u^{(k)} u^{(l)},$$

where ε_k the diffusion coefficient. This model incorporates spatial structure by adding diffusion terms in equations and considering the system of equations in the domain $\Omega \subset \mathbb{R}^d$, where *d* is the spatial dimension.

Previous research has put much effort into modifying the Lotka-Volterra competition model (LVC). The parameters r_k , α_{kl} , and ε_k can be modeled as functions instead of constants. Diffusion ε_k , more specifically, traveling waves, and their minimal-speed selection mechanisms in the LV model can be studied by applying the upper-lower solution technique on the cooperative system, [12]. The crowding effect of diffusion ε_k can also be modeled within the LVC model, [13]. Advection rate can be introduced to the LVC model as a supplement of diffusion to bring much richer phenomena, [14],[15]. Carrying capacity can be replaced by function instead of constant and can vary between species; connection topology can be modeled as a 'loop,' 'star,' 'chain,' or 'full' connection when there are more than two species in the system, [16]. In addition to making parameters endogenous, some other efforts have also been made to modify the LVC model to simulate real-world problems. For example, simulation adding small immigration into the prey or predator population can stabilize the system, [17]. The random fluctuating LV environment can also be modeled within the LV show how switching between model to environments can make survival harder, [18]. Stochastic noises can be introduced to the LV population model and are presented to play an essential role in the permanence and characterization of the system, [19].

Multiple methods are used for solving the LVC model, such as Haar wavelet (HW), Adams-Bashforth-Moulton (ABM) methods, [20], and finite element method. After simulation, multiple methods have been used to analyze the simulated species interaction data. Real-world data can be used to evaluate the multi-species interaction model, [21]. For example, the population dynamics model of individual reefs can be compared with data on coral reefs in Pilbara, [22]. Numerical models and machine learning can be combined to identify the factors influencing Alexandrium catenella blooms, [23].

In this work, we consider a spatial-temporal multi- species competition model in one- and twodimensional formulations. For the numerical solution of the model with corresponding boundary and initial conditions, we construct a finite volume approximation with а semi-implicit time approximation. When two or more species compete for the same limited food source or in some way influence each other's growth, one or several of the species usually becomes extinct. To understand the diffusion effect of the solution, we perform a numerical investigation for several cases of the parameters related to the survival scenarios. The influence of the diffusion and initial conditions on the survival scenarios is presented for two and threespecies competition models. It turns out that equilibrium depends only on system parameters (birth rate, competition, predation, etc.) and does not depend on the initial conditions. The time it takes for the system to be in the steady state depends on how far initial populations of species are located from the equilibrium point in the phase space of the model. After understanding the effect of parameters and initial conditions on the test problems, we simulate the spatial-temporal model with random input parameters, i.e., releasing the hold for all parameters. Simulations with different combinations of parameters and initial conditions support the hypothesis that such systems reach equilibrium sooner or later. At last, we apply factor analysis to

evaluate the impact of parameters on the system stability.

The main novelty of this work is in performing factor analysis methods for PDE-based data sets. For the PDE-based mathematical models, the finding of an analytical solution is limited by special cases and cannot be performed for the general spatial-temporal model and consider complex diffusion and domain structure and effect of the various boundary conditions, [24],[25]. In this paper, we construct a discrete system using the Finite Volume Method with semi-implicit time approximation. Simulation of the ecological system with a PDE-based discrete system allows us to (1) take into the spatial distribution of the species concerning boundary conditions and (2) use the limited information about ecosystem parameters by a given estimated range instead of acquiring specific value which is usually unknown for the real systems. The presented approach allows us to identify which ecosystem parameters are dominant, only given the range of each parameter. For this purpose, we use a factor analysis for (a) understanding the system and (b) predicting the system. From the goal of understanding the system, the factor analysis approach allows us to understand the relative importance of parameters when information about specific parameter values is limited. For example, given ranges of parameters in a food web containing multiple species, we can identify what parameters (e.g., birth rate. competition rate, or diffusion rate) or what species is the" critical factor" that influences the final solution of the population dynamics. It can be applied not only to two and three-species competing systems but also to systems corresponding to more giant food web (state-of-the-art simulations in terms of the number of species, [26], [27], [28] simulated system of four species). Currently, in the ecology and environmental field, factor analysis is applied more to real-world data, [29],[30]. Still, we have shown that it can also be used for PDE-based simulation and can be applied to more complex ecosystem simulation when computational power increase. We can also apply this simulation approach with the factor analysis method to study catastrophic events in future works. The extinction property has been studied analytically, [31], [32] and numerically, [33],[34], and the presented method can bring about a novel perspective to study the topic numerically. From the goal of predicting the system, the factor analysis approach can be used for dimension reduction of input data of the predictive model. We can remove less dominant factors to speed up the prediction process or replace the input of the predictive model with the main components. Some other feature selection methods can also serve the purpose of dimension reduction [35]. Multiple predictive models can be applied, afterward, such as beta regression, [36], explainable prediction, [37], and neural networks, [38]. In future works, we will concentrate on the following directions: (1) a realworld range of parameters for a given application; (2) apply the result of factor analysis to build machine learning predictive models, and predict final population density, as well as time to reach equilibrium; and (3) use the model to predict the equilibrium population density and simulate future catastrophic event (given the assumption that the system is at equilibrium then catastrophe strokes).

The paper is organized as follows. Section 2 describes the mathematical model with fine-scale approximation using the finite volume method and a semi-implicit scheme for time approximation. Section 3 presents numerical results for two and three-species competition models in one and twodimensional formulations. In Section 4, we simulate a spatial-temporal model with random parameters and perform factor analysis to evaluate the impact of parameters on the system stability. The paper ends with a conclusion.

2 Mathematical Models with Approximation by Space and Time

A mathematical model is the simplified representation of the complex real-world objects or systems used to understand the system's complex interactions to predict possible outcomes of different changes. In mathematical biology and ecology, the mathematical model fundamental is the Lotka-Volterra model, which describes the temporal dynamic of the species population in various ecosystem models. However, real ecosystems interact in different locations, and spatial structure is essential and dramatically impacts the population.

In this work, we consider the spatial-temporal multi-species competition problem in one and twodimensional domains. Let $\Omega \subset R^d$ be the computational domain, where for d = 1, we have a one- dimensional case, and for d = 2, we obtain two- dimensional problem. The population of species in Ω is described by a coupled system of nonlinear diffusion - reaction equation.

$$\begin{aligned} &\frac{\partial u^{(k)}}{\partial t} - \nabla \cdot \left(\varepsilon_k \nabla u^{(k)} \right) \\ &= r_k u^{(k)} (1 - u^{(k)}) - \sum_{l \neq k} \alpha_{kl} u^{(k)} u^{(l)}, \end{aligned}$$

The system of equations is considered with initial conditions

$$u^{(k)} = u_0^{(k)}, \quad x \in \Omega, \quad t = 0,$$
 (2)
and boundary conditions

 $u^{(k)} = 0, \quad x \in \partial \Omega, \quad t > 0.$

(3)In this work, we consider the following special cases:

• Two-species competition

$$\frac{\partial u^{(1)}}{\partial t} - \nabla \cdot (\varepsilon_1 \nabla u^{(1)}) = r_1 u^{(1)} (1 - u^{(1)}) - \alpha_{12} u^{(1)} u^{(2)},$$

$$\frac{\partial u^{(2)}}{\partial t} - \nabla \cdot (\varepsilon_2 \nabla u^{(2)}) = r_2 u^{(2)} (1 - u^{(2)}) - \alpha_{21} u^{(1)} u^{(2)},$$
• Three-species competition

$$\frac{\partial u^{(1)}}{\partial t} - \nabla \cdot (\varepsilon_1 \nabla u^{(1)}) = r_1 u^{(1)} (1 - u^{(1)}) - \alpha_{12} u^{(1)} u^{(2)} - \alpha_{13} u^{(1)} u^{(3)},$$

$$\frac{\partial u^{(2)}}{\partial t} - \nabla \cdot (\varepsilon_2 \nabla u^{(2)}) = r_2 u^{(2)} (1 - u^{(2)}) - \alpha_{21} u^{(1)} u^{(2)} - \alpha_{23} u^{(3)} u^{(2)},$$

$$\frac{\partial u^{(3)}}{\partial t} - \nabla \cdot (\varepsilon_3 \nabla u^{(3)}) = r_3 u^{(3)} (1 - u^{(3)}) - \alpha_{31} u^{(1)} u^{(3)} - \alpha_{32} u^{(2)} u^{(3)},$$

Analytical solution of such spatial-temporal models is possible only for some simplified cases. A numerical simulation of the model required the construction of an accurate approximation for spatial variation (diffusive operators) and time approximation. Three common space approximation techniques exist: The Finite Difference Method, the Finite Element Method, and the Finite Volume Method. The main advantage of the Finite Element Method is the accurate representation of the solution in the complex computational domains by constructing unstructured grids. For simplified geometries, the Finite Volume Method is the common choice for space approximation because it provides a conservative approximation.

Let $\Omega = [0, L]^d$ (d = 1, 2) be the computational domain, where we have an interval for the onedimensional case and a square domain for the twodimensional problem. For the numerical solution of the model (1) with boundary conditions (2) and initial conditions (3), we construct a computational

mesh and use a Finite Volume Method for approximation.

Let $\mathcal{T}_{f_{h}}$ be the structured grid for domain Ω

$$\mathcal{T}_{\hbar} = \bigcup_{i=1}^{N_c} K_i$$

where K_i be the square cell and N_c is the number of cells, [41]. Here $K_i = [(i-1)h, ih]$ for onedimensional case, where h = L/N. For the twodimensional case, we have $K_i = [(l-1)h, lh] \times$ [(j-1)h, jh] with l, j = 1, ..., N and h = L/N, where i = j * N + l is the global cell indexing and $N_x = N_y = N$ is the number of nodes in x and y direction. Here we have $N_c = N$ for the onedimensional case and $N_c = N \times N$ for the twodimensional case.



Fig. 1: Illustration of the computational grid for one and two dimensional cases. Ki is the cell and Eij is the interface between to cells K_i and K_i

To write an approximation by space using the Finite Volume Method, we integrate equation (1) over the cell volume and obtain the following semidiscrete form:

$$\int_{K_i} \frac{\partial u^{(k)}}{\partial t} dx - \int_{K_i} \nabla \cdot \left(\varepsilon_k \nabla u^{(k)}\right) dx$$
$$= \int_{K_i} \left(r_k u^{(k)} (1 - u^{(k)}) - \sum_{l \neq k} \alpha_{kl} u^{(k)} u^{(l)} \right) dx ,$$
(4)

In the Finite Volume Method}, we set

$$\frac{1}{|K_i|}\int_{K_i}u^{(k)}dx=u_i^{(k)},$$

where $|K_i|$ is cell volume and $u_i^{(k)}$ is the cell average value of the function $u^{(k)}$ on cell K_i .

For the diffusion operator approximation, we use a classic two-point flux approximation

$$-\int_{K_{i}} \nabla \cdot (\varepsilon_{k} \nabla u^{(k)}) dx = \int_{\partial K_{i}} \varepsilon_{k} \nabla u^{(k)} \cdot n \, dx$$
$$\approx \sum_{j} T_{k,ij} \left(u_{i}^{(k)} - u_{j}^{(k)} \right)$$

with

$$T_{k,ij} = \varepsilon_k \left| e_{ij} \right| / d_{ij},$$

where d_{ij} is the distance between cell center points x_i and x_j , $|E_{ij}|$ is the length of the interface between two cells K_i and K_j . Note that, for a structured uniform grid, we have $d_{ij} = h$, $|E_{ij}| = 1$ for the one-dimensional case and $|E_{ij}| = h$ for the two-dimensional problem.

Therefore, we have

$$\frac{\partial u_i^{(k)}}{\partial t} |K_i| + \sum_j T_{k,ij} \left(u_i^{(k)} - u_j^{(k)} \right)$$

= $r_k u_i^{(k)} \left(1 - u_i^{(k)} \right) |K_i| - \sum_{l \neq k} \alpha_{kl} u_i^{(k)} u_i^{(l)} |K_i|,$
(5)

For the time derivative, we use a backward Euler approximation

$$\frac{\partial u_i^{(k)}}{\partial t} \approx \frac{u_i^{(k)} - \breve{u}_i^{(k)}}{\tau}$$

where τ is the time step size and $\breve{u}_i^{(k)}$ is the solution from the previous time layer.

To remove large time step restrictions regarding the diffusion operator, we approximate the diffusion part using the solution from the current time level. However, we use an explicit approximation for the nonlinear reaction part to linearize the problem. Finally, we obtain the following discrete problem using a semi-implicit time approximation scheme, [42],[43],[44].

$$\frac{u_{i}^{(k)} - \breve{u}_{i}^{(k)}}{\tau} |K_{i}| + \sum_{j} T_{k,ij} \left(u_{i}^{(k)} - u_{j}^{(k)} \right)$$
$$= r_{k} \breve{u}_{i}^{(k)} \left(1 - \breve{u}_{i}^{(k)} \right) |K_{i}| - \sum_{l \neq k} \alpha_{kl} \breve{u}_{i}^{(k)} \,\breve{u}_{i}^{(l)} |K_{i}|,$$
(6)

Note that we obtain an uncoupled system of equations and can solve the equation for each component separately. Algorithm:

• Set initial conditions for each species k:

 $u_i^{(k)} = u_0^{(k)}, \quad \forall i = 1, ..., N_c, \quad k = 1, ..., M,$ where *M* is the number of species and N_c is the number of grid cells.

• Solve the linear system of equations on each time layer till all species solutions converge to the final equilibrium state

$$A^{\hat{k}}u^{(k)} = b^k,$$

where $u^{(k)} = \left(u_1^{(k)}, \dots, u_{N_c}^{(k)}\right)^T$ is the vector of solution with size N_c and

$$\begin{aligned} A^{k} &= \frac{1}{\tau} u_{i}^{(k)} |K_{i}| + \sum_{j} T_{k,ij} \left(u_{i}^{(k)} - u_{j}^{(k)} \right) \\ b^{k} &= \frac{\breve{u}_{i}^{(k)}}{\tau} |K_{i}| + r_{k} \breve{u}_{i}^{(k)} \left(1 - \breve{u}_{i}^{(k)} \right) |K_{i}| \\ &- \sum_{l \neq k} \alpha_{kl} \breve{u}_{i}^{(k)} \breve{u}_{i}^{(l)} |K_{i}|, \end{aligned}$$
for $k = 1, \dots, M.$

Here we have the right-hand side vector b^k that depends on the solution from the previous time layer. The matrix A^k is the positive definite and symmetric. Furthermore, we have a tridiagonal matrix for the one-dimensional case and a five-diagonal matrix for the two-dimension. Implementation is performed using Python programming language using a standard solver from the NumPy package, [45],[46]. In most results, we perform simulations till both populations reach equilibrium, $|\bar{u}^{(k)} - \bar{u}^{(k)}| < \epsilon$ with $\epsilon = 10^{-5}$ for each k.

3 Numerical Results

Before we make factor analysis for randomly generated parameters, we consider some special cases with parameters (growth rate "r", competition efficiency " α ", and diffusion rate " ϵ ") fixed as constant and examine results such that either one species survives, two species survive, or three species survive.

Let

α

$$r = (r_1, \dots, r_M), \\ \varepsilon = (\varepsilon_1, \dots, \varepsilon_M), \\ \alpha_{11} \quad \alpha_{12} \quad \dots \quad \alpha_{1M} \\ \alpha_{21} \quad \alpha_{22} \quad \dots \quad \alpha_{2M} \\ \dots \quad \dots \quad \dots \quad \dots \\ \alpha_{M1} \quad \alpha_{M2} \quad \dots \quad \alpha_{MM} \end{pmatrix}$$

We present numerical results for multi-species competition in domain Ω . We consider the following domain and boundary conditions configurations:

- $1D: \Omega = [0,1]$ with zero (fixed) boundary conditions on $\partial \Omega$.
- 2D(a) : $\Omega = [0,1]^2$ with zero (fixed) boundary conditions on $\partial \Omega$.
- 2D(b) : $\Omega = [0,1]^2$ with zero (fixed) boundary conditions on the left and right boundaries and zero flux (free) boundary conditions on the top and bottom boundaries

In simulations, we use a grid with 100 nodes for the one-dimensional problem and a 25 × 25 grid for the two-dimensional case. We simulate with $\tau =$ 1 and set initial conditions $u_0^{(1)} = u_0^{(2)} = 0.5$ for two-species and $u_0^{(1)} = u_0^{(2)} = u_0^{(3)} = 0.5$ for threespecies competition models. As for diffusion rate, we consider cases with regular diffusion $\varepsilon = D$ (between 0.01 and 0.1 and has the same scale as other parameters) and small diffusion $\varepsilon = D/10$.

To represent the result and compare the final equilibrium state, we calculate the average solution over computational domain Ω for each species

$$\bar{u}^{(k)}(t) = \frac{1}{|\Omega|} \int_{\Omega} u^{(k)}(x,t) \, dx,$$

where $|\Omega|$ is the volume of domain Ω . In most results, we perform simulations till both populations reach equilibrium, $|\bar{u}^{(k)} - \check{u}^{(k)}| < \epsilon$ with $\epsilon = 10^{-5}$ for each *k*.

First, we present results for the two-species interaction problem, where we simulate two sets of parameters related to the species survival: Case 1 (one species survive) and Case 2 (both species survive). We compare the dynamics of the average solution and final state for three configurations: 1D, 2D(a), and 2D(b). Furthermore, the investigation of the diffusion parameters scale to the final state is presented. Next, we consider the three-species competition model with three cases of the test parameters: Case 1 (one species survive), Case 2 (two species survive), and Case 3 (all species survive). Similar to the previous problems, we investigate the final state and dynamic for 1D, 2D(a), and 2D(b) for small and regular diffusion.

After that, we present results for random diffusion to investigate the effect of the species diffusion coefficient on the final equilibrium state, survival group, and time to reach an equilibrium state. Next, we consider the influence of the initial conditions on the equilibrium state and the time to reach it.

3.1 Numerical Results for Two-Species Competition Model

We consider a two-species competition model and simulate two cases of the parameters:

• Case 1 (one species survive) D = (0.035, 0.014), r = (0.074, 0.084), $\alpha = \begin{pmatrix} 0.0 & 0.074 \\ 0.013 & 0.0 \end{pmatrix}$ • Case 2 (both species survive) D = (0.016, 0.014), r = (0.083, 0.081), $\alpha = \begin{pmatrix} 0.0 & 0.053 \\ 0.049 & 0.0 \end{pmatrix}$ for difference on the difference of the second se

As for diffusion rate, we set regular diffusion $\varepsilon = D$ and ten times smaller diffusion $\varepsilon = D/10$.

First, we present the population density at the final equilibrium when all the parameters are fixed.

In Fig. 2, we plot the solution for one and twodimensional formulations at the final time. Note that for 2d, we plot the solution over the middle line (y = 0.5). We observed that the effect of boundary constraint (set at zero) is more severe in regular diffusion groups, causing a lower final population density than in small diffusion groups. With the same survival status (one species survives, or both species survive) and boundary conditions (1D, 2D(a), or 2D(b)), the regular diffusion group always arrives at a lower population density equilibrium, compared to the small diffusion group. In other words, when the diffusion is smaller, the species can reach a higher population density equilibrium.

Second, we present the dynamic of average population density over time steps when all the parameters are fixed. In Fig. 3, we present the dynamic of the solution average over time for two species systems. We observed that the time to reach equilibrium is different among different diffusion conditions (regular diffusion D, or small diffusion D/10), survival status (one species survives, or both species survive), and boundary conditions (1D, 2D(a), or 2D(b)). In general, 1D and 2D(b) give similar solutions, while 2D(a) gives a different solution. In general, when the diffusion is small, it usually takes more time to reach equilibrium.

Third, we present the population density over 2D(a) (imitation of the pond) and 2D(b) (imitation of the river) at final equilibrium when all the parameters are fixed. In Fig. 4 and Fig. 5, we present the solution for 2d problems at the final time in whole domain Ω for regular diffusion $\varepsilon = D$ and small diffusion $\varepsilon = D/10$, respectively. Like Fig. 2, the effect of boundary constraint (set at zero) is more severe in regular diffusion groups. If the diffusion is regular (Fig. 4), the final population is only dense in the middle, and the area of zero population density is wider at the boundaries. If the diffusion is small (Fig. 5), the final population is more spread across the whole spatial domain.

To sum up, in a two-species competing model, when we compare solutions of final population density from two fixed groups of parameters (diffusion rates/ diffusion rates*0.1 / growth rates/ competition rates) that leads to two survival status (both species survive/ only one species survive), we observed that species with diffusion rates that are 10 times smaller in scale than other parameters could reach a higher equilibrium population density in the whole spatial domain, and it usually takes the species with small diffusion rates more time steps to reach equilibrium. 3.2 Numerical Results for Three-species Competition Model 1D 2 For the three-species competition model, we consider three cases:



2D(b)



Case 2 (two species survive)

Fig. 2: Solution at the final time for regular diffusion $\varepsilon = D$ (solid line) and small diffusion $\varepsilon = D/10$ (dashed line).



Fig. 3: Dynamic of the average solution over the domain for regular diffusion $\varepsilon = D$ (solid line) and small diffusion $\varepsilon = D/10$ (dashed line).



Case 2 (two species survive)

Fig. 4: Solution at the final time for regular diffusion $\hat{\varepsilon} = D$. First picture: first species. Second picture: second species





Fig. 5: Solution at the final time for small diffusion $\varepsilon = D/10$. First picture: first species. Second picture: second species

- Case 1 (one species survive) D = (0.078, 0.087, 0.012), r = (0.050, 0.087, 0.041), $\alpha = \begin{pmatrix} 0.0 & 0.048 & 0.067 \\ 0.051 & 0.0 & 0.094 \\ 0.028 & 0.041 & 0.0 \end{pmatrix}$ • Case 2 (two species survive)
- D = (0.022, 0.021, 0.063),r = (0.086, 0.091, 0.066),

0.031 0.045 0.0 0.051 0.0 0.019 $\alpha =$ 0.058 0.085 0.0 Case 3 (all three species survive) D = (0.031, 0.027, 0.026),r = (0.0980.095, 0.078),0.0 0.055 0.057 0.095 0.0 0.031 $\alpha =$ 0.070 0.022 0.0

First, we present the population density at the final equilibrium when all the parameters are fixed

under the three-species competing model. In Fig. 6, we plot the solution for one and two-dimensional formulations at the final time. Similar to the previous results, we plot the solution over the middle line (y = 0.5) for the 2d formulation. We observe that the effect of boundary conditions is generally more severe in regular diffusion groups than in small diffusion groups. Like a two-species competing system, in a three-species competing system, when the diffusion rate is lower, species can generally reach a higher final population density both at the center of the spatial domain and near the boundary. This shows that the boundary constraint is lower for species with lower diffusion. More investigation of the diffusion to the final state will be presented in the next part of the results.

Then, we present the dynamic of average population density over time steps when all the parameters are fixed under a three-species competing model. In Fig. 7, we present the dynamic of the solution average over time for three species system. Like a two-species competing system, in a three-species competing system, generally, when the diffusion is smaller, it usually takes more time to reach equilibrium.

To sum up, in a three-species competing model, when we compare solutions of final population density from two fixed groups of parameters (diffusion rates/ diffusion rates*0.1 / growth rates/ competition rates) that leads to two survival status (both species survive/ only one species survive), we observed that, similar to two-species competing model, species with diffusion rates that are 10 times smaller in scale than other parameters can reach a higher equilibrium population density in whole spatial domain with very few exceptions, and it usually takes the species with small diffusion rates more time steps to reach equilibrium.

3.3 Effect of the Diffusion

Next, we consider the influence of diffusion on the equilibrium state. We set all other parameters (growth rate "r" and competition efficiency " α ") the same and examine the diffusion rate. We perform 1000 simulations for each case with random diffusion coefficients

$$0.01 < \varepsilon_i < 0.1.$$

We simulate with fixed initial conditions $u_0^{(k)} = 0.5$ till both populations reach equilibrium, $|\bar{u}^{(k)} - \check{u}^{(k)}| < \epsilon$ with $\epsilon = 10^{-5}$ for each *k*.

First, we consider two-species competition models in one and two-dimensional formulations. Groups are represented by which species survive. In the two-species competition model, we have the following:

- 10 first species survive,
- 01 second species survived,
- 11 both species survived and
- 00 no one survived.

In Fig. 8 we plot scatter plots of the diffusion rate of two species and colored survival status at the final time for one and two-dimensional problems, as well as corresponding time steps to reach equilibrium. We observed that diffusion rate has a huge impact on the final survival status of species and the time step needed to reach equilibrium.

Different combination of diffusion rates of two species leads to different final survival status. Under 1D or 2D(b) spatial boundary condition, when the diffusion rates of both species are above 0.07, no species will survive. When one of the species' diffusion rates is smaller than 0.07, the species with a larger diffusion rate will survive. However, if two species have approximately the same diffusion rates and are both less than 0.07, both species will survive. In comparison, under 2D(a) spatial boundary condition, the threshold is reduced from 0.07 to around 0.04.

Different combination of diffusion rates of two species also leads to different time steps needed to reach equilibrium. When the combination of both species' diffusion rates is on the border of varying survival groups, it takes more time steps to reach equilibrium. In comparison, it takes fewer time steps to reach equilibrium when the combination of diffusion rates of both species lies inside the survival group.

Next, we consider three-species competition models, where we have eight groups:

100 - first species survive,

- 010 second species survived,
- 001 third species survived,

011 - second and third survived,

101 - first and third survived,

110 - first and second survived,

- 111 all species survived and
- 000 no one survived.

In Fig. 9 we plot scatter plot of the diffusion rate of three species and colored survival status at the final time for one and two-dimensional spatial boundary conditions, as well as corresponding time steps to reach equilibrium. When we expand from two species to three species in the system, we observed that 1D or 2D(b) spatial boundary condition still gives similar results, while 2D(a)gives a different result. In general, when the diffusion rates of all three species are small, three species can all survive. When all species have high diffusion rates, no species can survive. When not all three species have low diffusion rates, the species with higher diffusion rates can survive. Similarly, the different combination of diffusion rates of the three species also leads to different time steps needed to reach equilibrium and maximum located on the border of varying survival groups.

3.4 Effect of the Initial Conditions

Previously we set the initial condition for both species to be 0.5 and then got a general sense of the final equilibrium difference. Next, we remove the control for the initial condition ($0.01 < u_0^{(k)} < 0.99$) and examine how changes in the initial population can impact the final equilibrium population density and time to reach it. With the birth rate, competition rate, and diffusion rate fixed, we change the initial population density for both species.

In Fig. 10 we show a scatter plot of the initial population density of two species and use a blue line to represent a dynamic from initial condition points (green color) to final equilibrium points of each simulation for one and two-dimensional spatial boundary conditions, as well as corresponding time steps to reach equilibrium. We observed that disregarding the value of the initial population of both species, the final equilibrium population will rest at a point of focus. The farther the combination point of initial population density is from the final equilibrium combination point, the more time steps it would take to reach the equilibrium. We also observed that the farther the initial population density point is from the final equilibrium population density point, the more time steps it needs to reach the equilibrium.

In Fig. 11, we plot scatter plots of the initial population density of three species and use a blue line to connect initial condition points to the final equilibrium points of each simulation and corresponding time steps to reach equilibrium.

When we expand from two species to three species in the system, we observed that 1D or 2D(b) spatial boundary condition still gives a similar result. In contrast, 2D(a) gives a different result. In general, regardless of the values of the initial condition of population density of three species, the final equilibrium of population density for all three species in the system will rest at a final focus point.

The farther a combination of the initial condition of population density is from the final equilibrium point, the more time steps it needs to take to reach that equilibrium point.

4 Factor Analysis

Factor analysis method has long been applied to the analysis of Population Dynamics, [47],[48] as well as Ecosystem topics, [49],[50]. Finally, we present factor analysis for the given model.

Previous literature about factor analysis suggests that as sample size increases, the standard error in factor loadings across repeated samples will decrease, [51]. Therefore, we perform 10,000 simulations for each case with random parameters to satisfy the sample size requirement.

We consider one and two-dimensional test problems and simulate the system with random coefficients ε_k , r_k , α_{kl} with $l \neq k$ and initial conditions $u_0^{(k)} = \text{const:}$

• Two-species:

$$\varepsilon_1, \varepsilon_2, \alpha_{12}, \alpha_{21} \text{ and } u_0 = \left[u_0^{(1)}, u_0^{(2)}\right].$$

• Three-species:

 $\begin{bmatrix} \varepsilon_1, \varepsilon_2, \varepsilon_3, \alpha_{12}, \alpha_{13}, \alpha_{21}, \alpha_{23}, \alpha_{31}, \alpha_{32} \text{ and } u_0 = \\ \begin{bmatrix} u_0^{(1)}, u_0^{(2)} u_0^{(3)} \end{bmatrix}.$

We perform 10,000 simulations for each case with random parameters

 $0.01 < r_k, D_k, \alpha_{kl}, < 0.1,$

and $0.01 < u_0^{(k)} < 0.99$. Two cases of the diffusion coefficient are considered: regular diffusion ($\varepsilon_k = D_k$) and small diffusion ($\varepsilon_k = D_k/10$).

In the investigation, we use the influence of the following parameters to the system solution: ε_k , α_{kl} , and r_k for $l \neq k$ and k = 1,2 for two-species competition and k = 1,2,3 for three species competition.



Fig. 6: Solution at the final time for regular diffusion $\hat{\varepsilon} = D$ (solid line) and small diffusion $\hat{\varepsilon} = D/10$ (dashed line). Red color: first species. Blue color: second species. Green color: third species



Case 3 (three species survive)

Fig. 7: Solution at the final time for regular diffusion $\varepsilon = D$ (solid line) and small diffusion $\varepsilon = D/10$ (dashed line). Red color: first species. Blue color: second species. Green color: third species



Case 2 (two species survive)

Fig. 8: Numerical results for random diffusion. Two-species model. First picture: 00 (grey), 01 (blue), 10 (red) and 11 (green) groups. Second picture: number of time steps to reach the equilibrium



Case 3 (three species survive)

Fig. 9: Numerical results for random diffusion. Three-species model. First picture: 000 (light grey), 001 (grey), 010 (cyan), 011 (blue), 100 (salmon), 101 (red), 110 (lime) and 111 (green) groups. Second picture: number of time steps to reach the equilibrium



Case 2 (two species survive)

Fig. 10: Numerical results for random initial conditions. Two-species model. First picture: the dynamic of the solution average. Second picture: number of time steps to reach the equilibrium



Case 3 (three species survive)

Fig. 11: Numerical results for random initial conditions. Three-species model. First picture: the dynamic of the solution average. Second picture: number of time steps to reach the equilibrium

4.1 Factor Analysis for Two-Species Competition Model

To sum up the factor analysis for the final population and time steps in a two-species competing system, we observed that diffusion is more dominant only when diffusion rates are on the same scale with other parameters when diffusion rates are ten times small in scale than other parameters, growth rates/ competition rate/ initial population density becomes the dominant factor. A possible explanation for the change of the dominant factor when the diffusion rate changes is the existence of the boundary conditions (set at zero). The boundary effect is more severe when the diffusion rate is larger, making it a more critical factor.

4.1.1 Factor Analysis for Final Population Density in Two-Species Competition Model

In Fig. 12, we present correlation matrices as well as loading matrix for each factor analysis for all three cases 1D, 2D(a) and 2D(b) with regular and small diffusion, we only include all parameters and final population density. From the correlation matrix, we observed that under regular diffusion, final population density is strongly correlated with both diffusion rates and growth rates, while under small

Table 2, we present a summary for factor analysis of the time steps until equilibrium in the two-species competition model. We observed that diffusion rates are essential factors only when diffusion has the same scale as other parameters. Under small diffusion, in which diffusion is ten times smaller in scale than other parameters, the dominant factors are growth rates or initial population density.

4.2 Factor Analysis for Three-Species Competition Model

To sum up the factor analysis for the final population and time steps in a three-species competing system, we observed that these twofactor analyses differ. For factor analysis for the final population, we observed that diffusion is more dominant only when diffusion rates are on the same scale as other parameters. Growth rates are the dominant factor when diffusion rates are ten times smaller in scale than other parameters. For factor analysis for time steps to reach equilibrium, we observed that the importance of diffusion is similar, disregarding the scale of diffusion rates. The dominant factors are always competition rates/ growth rates/ initial population density. diffusion, final population density is mostly strongly correlated with growth rates.

In

Table 1, we present a summary for factor analysis of final population density in a two-species competition model. We observed that under regular diffusion, diffusion rates are the most dominant factors, while under small diffusion, growth rates and competition rates become the most dominant factors.

4.1.2 Factor Analysis for Time Steps to Reach Equilibrium in Two-Species Competition Model

In Fig. 13, we present the correlation matrix and loading matrix when we only include all parameters and the number of time steps to reach equilibrium. We observed that under both regular and small diffusion, the time steps to reach equilibrium do not have much correlation with all parameters. Under regular diffusion, we only observed weak correlations (0.08 - 0.35) between time steps and diffusion, we only observed weak correlations (0.23 - 0.28) between time steps and growth rates.

In

4.2.1 Factor Analysis for Final Population Density in Three-Species Competition Model

In Fig. 14, we present three-species competition system correlation matrices and a loading matrix for each factor analysis for all three cases 1D, 2D(a) and 2D(b) with regular and small diffusion. From the correlation matrix, we observed that under regular diffusion, final population density is strongly correlated with both diffusion rates and growth rates, while under small diffusion, final population density is mostly strongly correlated with growth rates and weakly correlated with diffusion rates and competition rates.

In Table 3, we present a summary for factor analysis of the final population in a three-species competition model. When the diffusion rate is regular, the dominant factors that cause variation are diffusion rates; in contrast, when the diffusion rates are low, the dominant factors become the growth rates of the three species.

4.2.2 Factor Analysis for Time Steps to Reach Equilibrium in Three-Species Competition Model

In Fig. 15, we present three-species competition system correlation matrices and a loading matrix for each factor analysis for all three cases 1D, 2D(a) and 2D(b) with regular and small diffusion, we

observed

weak

time steps and diffusion rates. Under small

correlations (0.10 - 0.13) between time steps and

only

we

only include the number of time steps to reach equilibrium and all other parameters. We observed that under both regular and small diffusion, the time steps to reach equilibrium correlate very little with all parameters. Under regular diffusion, we only observed weak correlations (0.23 - 0.25) between

1*D*

2D(a)

diffusion.

growth rates.

2D(b)



Fig. 12: Two-species competition model. The correlation matrix (first row) and loading (second row) only include parameters and final population density

In

Table 4, we present a summary for factor analysis of the time steps to reach equilibrium in the three-species competition model. We observed that the three-species competing system is different from the two-species competing system in terms of dominant factors influencing time steps. In a two-species competing system,

1D

2D(b)

when diffusion is regular, it is the dominant factor, and diffusion rates are unimportant when diffusion is small. However, In a three-species competing system, the importance of diffusion is similar, disregarding the scale of diffusion rates. The dominant factors are always competition rates/ growth rates/ initial population density.

2D(a)



Fig. 13 Two-species competition model. Correlation matrix (first row) and loading (second row) only including parameters and time steps to reach equilibrium

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Table I	Factor	Analysis	of the	tinal	nonu	lation	1n	two	snecies.	system
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Dimension	1D	2D(a)	2D(b)	1D	2D(a)	2D(b)
Diffusion	Regular	Regular	Regular	Small	Small	Small
Cum. Var.	66.83%	62.50%	66.43%	67.98%	68.33%	68.01%
Factor 1 Factor 2 Factor 3 Factor 4	Fin. Pop. 2 Diffusion 1 Fin. Pop. 2 Diffusion 2 Grow 2 Grow 1	Fin. Pop. 2 Diffusion 1 Fin. Pop. 2 Diffusion 2 Compete 2 Grow 1	Diffusion 2 Diffusion 1 Grow 2 Grow 1	Grow 1 Grow 2 Compete 1 Fin. Pop. Compete 2	Compete 2 Compete 1 Diffusion 2 Fin. Pop. Diffusion 1	Grow 1 Grow 2 Compete 1 Fin. Pop. 2 Compete 2

	Factor 5	Compete 2	Fin. Pop. 2 Grow 2	Fin. Pop. Compete 1	Diffusion 1	Diffusion 1	Diffusion 1
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Table 2. Factor Analysis of time steps in two species system

Dimension	1D	2D(a)	2D(b)	1D	2D(a)	2D(b)
Diffusion	Regular	Regular	Regular	Small	Small	Small
Cum. Var.	57.54%	61.85%	57.91%	60.21%	59.99%	60.77%
Factor 1 Factor 2	Diffusion 1 Diffusion 2	Diffusion 1 Time step n Diffusion 2	Time step n Diffusion 2 Compete 1	Init. Pop. 1 Grow 1	Grow 1 Time step n Grow 2	Grow 1 Init. Pop. 1
Factor 3 Factor 4 Factor 5	Compete 2 Init. Pop. 2 Time step n Diffusion 2	Grow 2 Compete 2 Time step n Grow 1	Time step n Grow 1 Init. Pop. 2 Diffusion 1	Time step n Grow 1 Grow 2 Compete 2	Compete 2 Init. Pop. 1 Time step n Grow 2	Compete 2 Time step n Grow 2 Time step n Grow 2

Table 3. Factor Analysis of the final population in three species system

Dimension	1D	2D(a)	2D(b)	1D	2D(a)	2D(b)
Diffusion	Regular	Regular	Regular	Small	Small	Small
Cum. Var.	58.45%	60.40%	58.59%	59.10%	58.92%	59.10%
Factor 1	Init. Pop. 3	Diffusion 1	Compete 3	Grow 2	Grow 2	Init. Pop. 3
Factor 2	Grow 2	Compete 3	Init. Pop. 3	Compete 3	Init. Pop. 3	Compete 3
Factor 3	Init. Pop. 1	Init. Pop. 2	Grow 2	Diffusion 3	Compete 3	Init. Pop. 1
Factor 4	Compete 3	Grow 3	Init. Pop. 1	Init. Pop. 1	Init. Pop. 1	Time step n Grow 3
Factor 5	Diffusion 1	Diffusion 3	Diffusion 1	Time step n Grow 3	Time step n Grow 3	Grow 2
Factor 6	Time step n Diffusion 3	Diffusion 2	Time step n Diffusion 3	Init. Pop. 3	Diffusion 3	Diffusion 3
Factor 7	Compete 2	Compete 2	Compete 2	Time step n Compete 2	Compete 1	Compete 1
Factor 8	Diffusion 2	Compete 2	Diffusion 2	Compete 1	Compete 2	Time step n Compete 2
Factor 9	Compete 1	Grow 2	Compete 1	Diffusion 2	Compete 1	Compete 1

Table 4. Factor Analysis of time steps in three species system

Dimension	1D	2D (a)	2D(b)	1D	2D(a)	2D(b)
Diffusion	Regular	Regular	Regular	Small	Small	Small
Cum. Var.	58.45%	60.40%	58.59%	59.10%	58.92%	59.10%
Factor 1	Init. Pop. 3	Diffusion 1	Compete 3	Grow 2	Grow 2	Grow 2
Factor 2	Grow 2	Compete 3	Init. Pop. 3	Init. Pop. 3	Init. Pop. 3	Init. Pop. 3
Factor 3	Init. Pop. 1	Diffusion 1	Grow 2	Compete 3	Compete 3	Compete 3
Factor 4	Compete 3	Init. Pop. 2	Init. Pop. 1	Diffusion 3	Init. Pop. 1	Init. Pop. 1
Factor 5	Diffusion 1	Grow 3	Diffusion 1	Init. Pop. 1	Time step n Grow 3	Time step n Grow 3
Factor 6	Time step n Diffusion 3	Diffusion 2	Time step n Diffusion 3	Time step n Grow 3	Diffusion 3	Diffusion 3
Factor 7	Compete 2	Compete 2	Compete 2	Time step n Compete 2	Compete 1	Time step n Compete 2
Factor 8	Diffusion 2	Compete 2	Diffusion 2	Compete 1	Compete 2	Compete 1
Factor 9	Compete 1	Grow 2	Compete 1	Diffusion 2	Compete 1	Compete 1

1*D*

2D(a)

2D(b)



Small diffusion

Fig. 14: Three-species competition model. Correlation matrix (first row) and loading (second row) only including all parameters and final population density

2D(b)

1D

2-600

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C -110 A 42

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Q-100 000 338 441

S-REEVELENS AND ADDRESS

7-000.000

2-101000000

g------



2D(a)



Small diffusion

Fig. 15: Three-species competition model. Correlation matrix (first row) and loading (second row) only including all parameters and time steps to reach equilibrium

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4.3 Statistical Summary for Survival Status in All Simulations

For a two-species competing system, we released control for all variables, including diffusion rates, growth and competition efficiency, and initial

Table 5. We observed that 1D and 2D(b) spatial boundary conditions give the similar proportion of survival groups, while 2D(a) gives more different proportions.

When there is regular diffusion, for 1D and 2D(b) spatial boundary conditions, there is around 30% of simulations that result in either 00(no species survives), 01 (species two survives), or 10 (species one survives). n comparison, only around 10% of simulations result in both species surviving or 2D(a) spatial boundary conditions, together. 68.88% of simulations result in 00 (both species go extinct), while around 14.87% and 14.89% result in 01 (species two survives) or 10 (species one survives), and very rare, about 1.36% of the simulation result in 11 (both species survive). To conclude, when there is regular diffusion, it's generally harder to arrive at a final solution where both species survive; it's even harder, almost impossible, for both species to survive when we are approximating a pond (2D(a) scenario).

When there is small diffusion, all three spatial boundary conditions (1D, 2D(a), and 2D(b)) give a similar proportion of survival status. t is most likely that both species survive together (around 50% of simulations). However, it is almost impossible that both species will go extinct. Furthermore, it is equally likely that only one species survives (the proportions of simulation are both around 20 - 25%).

For the three-species competing system, we released control for all variables, including diffusion rates, growth and competition efficiency, and initial population density, ran 10,000 simulations for all 1D, 2D(a), and 2D(b) spatial boundary conditions, and summarized the proportion of different survival status groups, as is in $\Sigma \phi \alpha \lambda \mu \alpha!$ To $\alpha \rho \chi \epsilon i \sigma$ προέλευσης της αναφοράς δεν βρέθηκε. we observed that 1D and 2D(b) spatial boundary conditions give the similar proportion of survival groups, while 2D(a) shows more different proportion. When there is regular diffusion, the probability of one species surviving (001, 010, and 100) is similar to one another (around 10% for 1D or 2D(b) scenario, and approximately 12% for 2D(a) scenario), while the probability of two species survive (011, 110 and 101) is similar to one another (around 12% for 1D or 2D(b) scenario, and population density, and ran 10,000 simulations for all 1D, 2D(a), and 2D(b) spatial boundary conditions, and summarized the proportion of different survival status groups, as is in

about 1% for 2D(a) scenario). In general, when there are three species, it is almost impossible for all three species to survive together, and no species go extinct; what's worse, for 2D(a) scenario, which approximates the lake, around 58% of the simulation result in 000 (no species survive). When there is small diffusion, all three spatial boundary conditions (1D, 2D(a), and 2D(b)) give a similar proportion of survival status. It's equally likely that two or all species survive (the proportion for survival group 011/110/101/111 are all around 20%). It's equally likely that only one species survives (the proportion for survival group 001/010/100 are all around 6 - 8%), and it's almost impossible that all three species will go extinct.

5 Discussion

• Numerical results with fixed values of parameters that cause different survival status

Species with small diffusion rates (10 times smaller in scale than other parameters) can reach a higher equilibrium population density in the whole spatial domain with very few exceptions. It usually takes the species with small diffusion rates more time steps to reach equilibrium.

• Effect of diffusion examined by numerically releasing the control for diffusion rates

Different combinations of diffusion rates align with varying statuses of survival. Groups of the combination of diffusion rates in scatter plot overlap with groups of varying survival statuses. This overlap can be interpreted as diffusion rates strongly affecting the survival status. In a two-species competing system, under 1D or 2D(b) spatial boundary condition, when the diffusions of both species are above 0.07, no species will survive; under 2D(a) spatial boundary condition, the threshold is reduced from 0.07 to around 0.04. In three-species competing species, we observed a similar effect of diffusion on survival status as in a two-species system. Furthermore, species with a combination of diffusion rates on the border of different survival groups take more time steps to reach equilibrium than those whose combination is inside the groups.

• Effect of the initial condition examined by numerically releasing the control for initial population density

For both two-species competing model and the three-species competing model, regardless of the values of the initial condition of population density of species, the final equilibrium of population density for all three species in the system will rest at a final focus point. The farther a combination of the initial condition of population density is from the final equilibrium point, the more time steps it takes to reach that equilibrium point.

Table 5.	Table 5. Survival Status for two species system									
	00	01	10	11						
	Regular diffusion									
1d	2796	3059	3064	1081						
	27.96%	30.59%	30.64%	10.81%						
2d(a)	6888	1487	1489	136						
	68.88%	14.87%	14.89%	1.36%						
2d(b)	3125	2904	3043	928						
	31.25%	29.04%	30.43%	9.28%						
	S	mall diffus	sion							
1d	0	2251	2279	5470						
	0.00%	22.51%	22.79%	54.70%						
2d(a)	26	2522	2533	4919						
	0.26%	25.22%	25.33%	49.19%						
2d(b)	0	2184	2192	5624						
	0.00%	21.84%	21.92%	56.24%						

Table 5. Survival Status for two species system

Table 6. Survival Status for three species system

Group	000	001	010	100	011	110	101	111
			Re	gular diffu	sion			
1d	1442	2026	2031	1987	806	822	730	156
	14.42%	20.26%	20.31%	19.87%	8.06%	8.22%	7.30%	1.56%
2d(a)	5800	1262	1319	1234	131	130	118	6
	58.00%	12.62%	13.19%	12.34%	1.31%	1.30%	1.18%	0.06%
2d(b)	1754	2012	2034	1985	712	728	651	124
	17.54%	20.12%	20.34%	19.85%	7.12%	7.28%	6.51%	1.24%
			Sı	nall diffus	ion			
1d	0	639	648	556	2031	2023	1943	2160
	0.00%	6.39%	6.48%	5.56%	20.31%	20.23%	19.43%	21.60%
2d(a)	3	858	835	773	1957	2001	1863	1710
	0.03%	8.58%	8.35%	7.73%	19.57%	20.01%	18.63%	17.10%
2d(b)	0	651	655	566	2030	2023	1941	2134
	0.00%	6.51%	6.55%	5.66%	20.30%	20.23%	19.41%	21.34%

• Factor Analysis after releasing control for all parameters

In a two-species competing model, diffusion rates are the more dominant factor for both final population density and time to reach equilibrium only when diffusion rates are on the same scale as other parameters. When diffusion rates are ten times small in scale than other parameters, growth rates/ competition rate or initial population density becomes the dominant factor. In a three-species competing system, for factor analysis for the final population, we observed that diffusion is more dominant only when diffusion rates are in the same scale with other parameters, when diffusion rates are ten times small in scale than other parameters, growth rates are the dominant factor; As of factor analysis for time steps to reach equilibrium, the importance of diffusion is similar disregard the scale of diffusion rates. The dominant factors are always competition rates/ growth rates/ initial population density.

6 Conclusion

A mathematical model of the spatial-temporal multispecies competition is considered. A discrete system is constructed using a finite volume approximation with a semi-implicit time approximation. The numerical results for two- and three-species models are presented for several exceptional cases of the parameters related to the survival status. We considered the one and two-dimensional model problems with two cases of the boundary conditions for the two-dimensional case. First, the effect of diffusion is investigated numerically. In special cases where parameters are fixed, we observed that the impact of boundary constraint is more severe in regular diffusion groups than in small diffusion groups, causing a lower population density both in the middle and near the boundary of the domain. We also observed that the dynamic of 1D is similar to 2D(b), while 2D(a) gives different dynamic. This suggests that we can use 1D to approximate 2D(b)to save computation time. Furthermore, from general cases where we release holds of diffusion while keeping other parameters fixed, we observed that different combinations of species' diffusion rates in the system lead to different final survival statuses of species. When the combination of both species' diffusion rates is on the border of varying survival groups, it takes more time for these groups of species to reach equilibrium. Second, the effect of the initial condition is investigated numerically. We release holds of initial conditions while keeping other parameters fixed. We observed similar patterns for both two-species competing systems and three-species competing systems. Take a twospecies competing system as an example; we observed that disregarding the value of the initial population of both species, the final equilibrium population will rest at a point of focus. The farther the combination point of initial population density is from the final equilibrium combination point, the more time steps it would take to reach the equilibrium.

Finally, the impact of parameters on the system stability is considered by simulating the spatialtemporal model with random input parameters. Factor analysis and statistical summary of the survival status of species in the system were performed. We observed that diffusion rates are the dominant factor. When diffusion rates are regular and on the same scale as other parameters. In contrast, when diffusion rates are small, which are ten times smaller in the scale of other parameters, growth and competition rates become the dominant factors. In a statistical summary of species survival status, we observed a similar pattern for both twospecies competing systems and three-species competing systems. For both systems, in each spatial boundary condition (1D, 2D(a), and2D(b), when the number of survived species is the same in the system (001, 010, 100 survival group in which only one species survive), the proportion of simulation is similar. However, the proportion of simulation for different survival groups varies when the diffusion rate is in different scale, this is the case for both competing systems. Take a two-species competing system as an example. When there is small diffusion, all three spatial boundary conditions (1D, 2D(a), and 2D(b)) give a similar proportion of survival status. It is most likely that both species survive together (49% - 56% of simulation), it is almost impossible that both species will go extinct (0% - 0.26%), and it is equally likely that only one species survives (both 01 and 10 survival groups take 21% - 26% of simulation). In contrast, when there is regular diffusion in two species competing systems, we observed that 1D and 2D(b) spatial boundary conditions give a different proportion of simulation for survival groups from 2D(a). Take a two-species competing system as an example; when there is regular diffusion, it is generally harder to arrive at a final solution where both species survive (around 10% of simulation); it is even harder, almost impossible (1.36% of simulation), for both species to survive when we are approximating a pond (2D(a) scenario).

In future works, we will concentrate on the following directions: (1) a real-world range of parameters for a given application; (2) apply the result of factor analysis to build machine learning predictive models, and predict final population density, as well as time to reach equilibrium; and (3) use the model to predict the equilibrium population density and simulate future catastrophic event (given the assumption that the system is at equilibrium then catastrophe strokes).

References:

- A. Okubo and S. A. Levin, Diffusion and Ecological Problems: Modern Perspectives, vol. 14. Springer, 2001. Accessed: Jul. 30, 2022. [Online]. Available: https://link.springer.com/book/10.1007/978-1-4757-4978-6
- [2] J. D. Murray, *Mathematical biology: I. An introduction*. Springer, 2002.
- [3] G. I. Marchuk, *Mathematical models in environmental problems*. Elsevier, 2011.
- [4] A. Marasco, A. Picucci, and A. Romano, "Market share dynamics using Lotka–Volterra models," *Technological forecasting and social change*, vol. 105, pp. 49–62, 2016.

- [5] W. Zhang and J. S. L. Lam, "Maritime cluster evolution based on symbiosis theory and Lotka–Volterra model," *Maritime Policy & Management*, vol. 40, no. 2, pp. 161–176, 2013.
- [6] W. Windarto and E. Eridani, "On modification and application of Lotka-Volterra competition model," in *Aip conference proceedings*, 2020, vol. 2268, p. 050007.
- [7] S.-Y. Wang, W.-M. Chen, and X.-L. Wu, "Competition analysis on industry populations based on a three-dimensional lotka–volterra model," *Discrete Dynamics in Nature and Society*, vol. 2021, 2021.
- [8] M. A. Khan, M. Azizah, S. Ullah, and others, "A fractional model for the dynamics of competition between commercial and rural banks in Indonesia," *Chaos, Solitons & Fractals*, vol. 122, pp. 32–46, 2019.
- [9] P. A. Montagna, A. L. Sadovski, S. A. King, K. K. Nelson, T. A. Palmer, and K. H. Dunton, "Modeling the effect of water level on the Nueces Delta marsh community," *Wetlands Ecol Manage*, vol. 25, no. 6, pp. 731–742, Dec. 2017, doi: 10.1007/s11273-017-9547-x.
- [10] Q. Chen, R. Han, F. Ye, and W. Li, "Spatiotemporal ecological models," *Ecological Informatics*, vol. 6, no. 1, pp. 37–43, Jan. 2011, doi: 10.1016/j.ecoinf.2010.07.006.
- [11] Y. R. Zelnik, J.-F. Arnoldi, and M. Loreau, "The Impact of Spatial and Temporal Dimensions of Disturbances on Ecosystem Stability," *Frontiers in Ecology and Evolution*, vol. 6, 2018, Accessed: Jul. 30, 2022. [Online]. Available: https://www.frontiersin.org/articles/10.3389/f evo.2018.00224
- [12] A. Alhasanat and C. Ou, "Minimal-speed selection of traveling waves to the Lotka– Volterra competition model," *Journal of Differential Equations*, vol. 266, no. 11, pp. 7357–7378, May 2019, doi: 10.1016/j.jde.2018.12.003.
- [13] M. K. A. Gavina *et al.*, "Multi-species coexistence in Lotka-Volterra competitive systems with crowding effects," *Sci Rep*, vol. 8, no. 1, p. 1198, Dec. 2018, doi: 10.1038/s41598-017-19044-9.
- [14] P. Zhou, "On a Lotka-Volterra competition system: diffusion vs advection," *Calc. Var.*, vol. 55, no. 6, p. 137, Oct. 2016, doi: 10.1007/s00526-016-1082-8.

- [15] X.-Q. Zhao and P. Zhou, "On a Lotka– Volterra competition model: the effects of advection and spatial variation," *Calc. Var.*, vol. 55, no. 4, p. 73, Jun. 2016, doi: 10.1007/s00526-016-1021-8.
- [16] V. Dakos, "Identifying best-indicator species for abrupt transitions in multispecies communities," *Ecological Indicators*, vol. 94, pp. 494–502, Nov. 2018, doi: 10.1016/j.ecolind.2017.10.024.
- [17] T. Tahara *et al.*, "Asymptotic stability of a modified Lotka-Volterra model with small immigrations," *Sci Rep*, vol. 8, no. 1, Art. no. 1, May 2018, doi: 10.1038/s41598-018-25436-2.
- [18] M. Benaïm and C. Lobry, "Lotka–Volterra with randomly fluctuating environments or 'how switching between beneficial environments can make survival harder,"" *The Annals of Applied Probability*, vol. 26, no. 6, pp. 3754–3785, Dec. 2016, doi: 10.1214/16-AAP1192.
- [19] M. Liu and M. Fan, "Permanence of Stochastic Lotka–Volterra Systems," J Nonlinear Sci, vol. 27, no. 2, pp. 425–452, Apr. 2017, doi: 10.1007/s00332-016-9337-2.
- [20] S. Kumar, R. Kumar, R. P. Agarwal, and B. Samet, "A study of fractional Lotka-Volterra population model using Haar wavelet and Adams-Bashforth-Moulton methods," *Mathematical Methods in the Applied Sciences*, vol. 43, no. 8, pp. 5564–5578, 2020, doi: 10.1002/mma.6297.
- [21] K. Devarajan, T. L. Morelli, and S. Tenan, "Multi- species occupancy models: review, roadmap, and recommendations," *Ecography*, vol. 43, no. 11, pp. 1612–1624, Nov. 2020, doi: 10.1111/ecog.04957.
- [22] F. Boschetti *et al.*, "Setting priorities for conservation at the interface between ocean circulation, connectivity, and population dynamics," *Ecol Appl*, vol. 30, no. 1, Jan. 2020, doi: 10.1002/eap.2011.
- [23] S.-S. Baek, Y. S. Kwon, J. Pyo, J. Choi, Y. O. Kim, and K. H. Cho, "Identification of influencing factors of A. catenella bloom using machine learning and numerical simulation," *Harmful Algae*, vol. 103, p. 102007, Mar. 2021, doi: 10.1016/j.hal.2021.102007.
- [24] B. C. T. Cabella, A. S. Martinez, and F. Ribeiro, "Full analytical solution and complete phase diagram analysis of the Verhulst-like two-species population

dynamics model," *arXiv preprint arXiv:1010.3361*, 2010.

- [25] K. Murty and D. Rao, "Approximate analytical solutions of general Lotka-Volterra equations," *Journal of mathematical analysis and applications*, vol. 122, no. 2, pp. 582– 588, 1987.
- [26] R. BHARDWAJ and S. DAS, "SYNCHRONIZATION OF CHAOTIC FOOD CHAIN WITH COMPETITIVE SPECIES," *Bull. Cal. Math. Soc*, vol. 111, no. 1, pp. 53–64, 2019.
- [27] J. O. Ojonubah and M. H. Mohd, "Impacts of asymmetric biotic interactions and environmental factors on the presenceabsence of multispecies.," *Pertanika Journal* of Science & Technology, vol. 28, no. 1, 2020.
- [28] Z. R. Miller, P. Lechón-Alonso, and S. Allesina, "No robust multispecies coexistence in a canonical model of plant–soil feedbacks," *Ecology Letters*, 2022.
- [29] P. S. Jamwal, M. Di Febbraro, M. L. Carranza, M. Savage, and A. Loy, "Global change on the roof of the world: Vulnerability of Himalayan otter species to land use and climate alterations," *Diversity and Distributions*, vol. 28, no. 8, pp. 1635–1649, 2022.
- [30] R. Engler, A. Guisan, and L. Rechsteiner, "An improved approach for predicting the distribution of rare and endangered species from occurrence and pseudo-absence data," *Journal of applied ecology*, vol. 41, no. 2, pp. 263–274, 2004.
- [31] L. Wang, "Study on asymptotic behavior of stochastic Lotka–Volterra system in a polluted environment," *Advances in Difference Equations*, vol. 2021, no. 1, pp. 1–18, 2021.
- [32] Z. Luo and X. Fan, "Optimal control for an age-dependent competitive species model in a polluted environment," *Applied Mathematics and Computation*, vol. 228, pp. 91–101, 2014.
- [33] F. Vadillo, "Comparing stochastic Lotka– Volterra predator-prey models," *Applied Mathematics and Computation*, vol. 360, pp. 181–189, 2019.
- [34] M. Liu, K. Wang, and Q. Wu, "Survival analysis of stochastic competitive models in a polluted environment and stochastic competitive exclusion principle," *Bulletin of mathematical biology*, vol. 73, no. 9, pp. 1969–2012, 2011.
- [35] R. Lou, Z. Lv, S. Dang, T. Su, and X. Li, "Application of machine learning in ocean

data," *Multimedia Systems*, Feb. 2021, doi: 10.1007/s00530-020-00733-x.

- [36] C. A. Johnson *et al.*, "Science to inform policy: linking population dynamics to habitat for a threatened species in Canada," *Journal of Applied Ecology*, vol. 57, no. 7, pp. 1314– 1327, 2020.
- [37] H.-C. Thorsen-Meyer *et al.*, "Dynamic and explainable machine learning prediction of mortality in patients in the intensive care unit: a retrospective study of high-frequency data in electronic patient records," *The Lancet Digital Health*, vol. 2, no. 4, pp. e179–e191, 2020.
- [38] M. R. Keshtkaran *et al.*, "A large-scale neural network training framework for generalized estimation of single-trial population dynamics," *BioRxiv*, pp. 2021–01, 2022.
- [39] Z. P. Chairez, "Spatial-temporal models of multi-species interaction to study impacts of catastrophic events," Texas A&M University-Corpus Christi, 2020.
- [40] M. Vasilyeva, A. Sadovski, and D. Palaniappan, "Multiscale solver for multicomponent reaction-diffusion systems in heterogeneous media," *arXiv preprint arXiv:2209.04495*, 2022.
- [41] A. A. Samarskii, *The theory of difference schemes*. CRC Press, 2001.
- [42] A. A. Samarskii and P. N. Vabishchevich, *Computational heat transfer*. 1995.
- [43] P. Vabishchevich, "Additive schemes (splitting schemes) for some systems of evolutionary equations," *Mathematics of Computation*, vol. 83, no. 290, pp. 2787– 2797, 2014.
- [44] N. Afanasyeva, P. N. Vabishchevich, and M. Vasilyeva, "Unconditionally stable schemes for non-stationary convection-diffusion equations," in *International conference on numerical analysis and its applications*, 2012, pp. 151–157.
- [45] G. Strang, *Linear algebra and its applications*. Belmont, CA: Thomson, Brooks/Cole, 2006.
- [46] C. R. Harris *et al.*, "Array programming with NumPy," *Nature*, vol. 585, no. 7825, pp. 357– 362, 2020.
- [47] G. C. Varley and G. R. Gradwell, "Recent Advances in Insect Population Dynamics," *Annual Review of Entomology*, vol. 15, no. 1, pp. 1–24, 1970, doi: 10.1146/annurev.en.15.010170.000245.
- [48] S. D. Albon, T. N. Coulson, D. Brown, F. E. Guinness, J. M. Pemberton, and T. H. Clutton-Brock, "Temporal changes in key

factors and key age groups influencing the population dynamics of female red deer," *Journal of Animal Ecology*, vol. 69, no. 6, pp. 1099–1110, 2000, doi: 10.1111/j.1365-2656.2000.00485.x.

- [49] N. Bernier and F. Gillet, "Structural relationships among vegetation, soil fauna and humus form in a subalpine forest ecosystem: a Hierarchical Multiple Factor Analysis (HMFA)," *Pedobiologia*, vol. 55, no. 6, pp. 321–334, Nov. 2012, doi: 10.1016/j.pedobi.2012.06.004.
- [50] P. Petitgas *et al.*, "Ecosystem spatial structure revealed by integrated survey data," *Progress in Oceanography*, vol. 166, pp. 189–198, Sep. 2018, doi: 10.1016/j.pocean.2017.09.012.
- [51] R. C. MacCallum, K. F. Widaman, S. Zhang, and S. Hong, "Sample size in factor analysis," *Psychological Methods*, vol. 4, no. 1, pp. 84– 99, 1999, doi: 10.1037/1082-989X.4.1.84.

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