Some properties of (Λ, α)-open sets

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Abstract: The purpose of the present paper is to introduce new classes of generalized (Λ, α)-open sets, namely $s(\Lambda, \alpha)$-open sets, $p(\Lambda, \alpha)$-open sets, $\alpha(\Lambda, \alpha)$-open sets, $\beta(\Lambda, \alpha)$-open sets and $b(\Lambda, \alpha)$-open sets. Moreover, some properties of $s(\Lambda, \alpha)$-open sets, $p(\Lambda, \alpha)$-open sets, $\alpha(\Lambda, \alpha)$-open sets, $\beta(\Lambda, \alpha)$-open sets and $b(\Lambda, \alpha)$-open sets are investigated.

Key–Words: (Λ, α)-closed set, (Λ, α)-open set,

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1 Introduction

In 1963, Levine [7] introduced and investigated the concepts of semi-open sets and semi-continuity in topological spaces. It is shown in [11] that semi-continuity is equivalent to quasicontinuity due to Marcus [8]. In 1997, Park et al. [13] introduced and studied the concept of δ-semi-open sets in topological spaces. In 2001, Lee et al. [6] investigated the further properties of δ-semi-open sets and related sets. On the other hand, Mashhour et al. [9] introduced the concepts of preopen sets and precontinuous functions. As generalizations of these concepts, Raychaudhuri and Mukherjee [10] defined δ-preopen sets and δ-almost continuous functions. Njåstad [12] introduced a new class of near open sets in a topological space, so called α-open sets. The class of α-open sets is contained in the class of semi-open and preopen sets and contains open sets. In 2002, Ganster et al. [4] introduced the concepts of pre-Λ-sets and pre-Λ-sets in a given topological space and investigated the topologies defined by these families of sets. In 2004, Georgiou [5] introduced and studied the notion of (Λ, δ)-closed sets and showed that (Λ, δ)-compactness and (Λ, δ)-connectedness are preserved by (Λ, δ)-continuous surjections. In 2007, Caldas et al. [3] introduced and investigated the concepts of Λ_α-sets and (Λ, α)-closed sets which are defined by utilizing the notions of α-open sets and α-closed sets. In [2], the present authors introduced and investigated the concept of (Λ, θ)-open sets in topological spaces. Quite recently, some properties of (Λ, sp)-open sets are studied in [1]. In this paper, we introduce new classes of sets called $s(\Lambda, \alpha)$-open sets, $p(\Lambda, \alpha)$-open sets, $\alpha(\Lambda, \alpha)$-open sets, $\beta(\Lambda, \alpha)$-open sets and $b(\Lambda, \alpha)$-open sets. The relationships between these concepts are considered. Moreover, some properties of $s(\Lambda, \alpha)$-open sets, $p(\Lambda, \alpha)$-open sets, $\alpha(\Lambda, \alpha)$-open sets, $\beta(\Lambda, \alpha)$-open sets and $b(\Lambda, \alpha)$-open sets are discussed.

2 Preliminaries

Throughout the paper, space (X, τ) (or simply X) always mean a topological space on which no separation axioms are assumed unless explicitly stated. Let A be a subset of a topological space (X, τ). The closure of A and the interior of A are denoted by $\text{Cl}(A)$ and $\text{Int}(A)$, respectively. A subset A of a topological space (X, τ) is said to be *α-open* [12] if $A \subseteq \text{Int}(\text{Cl}(\text{Int}(A)))$. The complement of an α-open set is called *α-closed*. The family of all α-open sets in a topological space (X, τ) is denoted by $\alpha(X, \tau)$. Let A be a subset of a topological space (X, τ). A subset

$\Lambda_\alpha(A)$ [3] is defined as follows:

$$\Lambda_\alpha(A) = \cap\{O \in \alpha(X,\tau) | A \subseteq O\}.$$

Lemma 1. [3] *For subsets A, B and $A_i (i \in I)$ of a topological space (X,τ), the following properties hold:*

(1) $A \subseteq \Lambda_\alpha(A)$.

(2) If $A \subseteq B$, *then* $\Lambda_\alpha(A) \subseteq \Lambda_\alpha(B)$.

(3) $\Lambda_\alpha(\Lambda_\alpha(A)) = \Lambda_\alpha(A)$.

(4) $\Lambda_\alpha(\cap\{A_i | i \in I\}) \subseteq \cap\{\Lambda_\alpha(A_i) | i \in I\}$.

(5) $\Lambda_\alpha(\cup\{A_i | i \in I\}) = \cup\{\Lambda_\alpha(A_i) | i \in I\}$.

A subset A of a topological space (X,τ) is called a Λ_α*-set* [3] if $A = \Lambda_\alpha(A)$.

Lemma 2. [3] *For subsets A and $A_i (i \in I)$ of a topological space (X,τ), the following properties hold:*

(1) $\Lambda_\alpha(A)$ *is a* Λ_α*-set.*

(2) If A is α-open, then A is a Λ_α-set.

(3) If A_i is a Λ_α-set for each $i \in I$, then $\cap_{i\in I} A_i$ is a Λ_α-set.

(4) If A_i is a Λ_α-set for each $i \in I$, then $\cup_{i\in I} A_i$ is a Λ_α-set.

A subset A of a topological space (X,τ) is called (Λ,α)*-closed* [3] if $A = T \cap C$, where T is a Λ_α-set and C is an α-closed set. The complement of a (Λ,α)-closed set is called (Λ,α)*-open*. The collection of all (Λ,α)-open (resp. (Λ,α)-closed) sets in a topological space (X,τ) is denoted by $\Lambda_\alpha O(X,\tau)$ (resp. $\Lambda_\alpha C(X,\tau)$). Let A be a subset of a topological space (X,τ). A point $x \in X$ is called a (Λ,α)*-cluster point* of A [3] if for every (Λ,α)-open set U of X containing x we have $A \cap U \neq \emptyset$. The set of all (Λ,α)-cluster points of A is called the (Λ,α)*-closure* of A and is denoted by $A^{(\Lambda,\alpha)}$.

Lemma 3. [3] *Let A and B be subsets of a topological space (X,τ). For the (Λ,α)-closure, the following properties hold:*

(1) $A \subseteq A^{(\Lambda,\alpha)}$ *and* $[A^{(\Lambda,\alpha)}]^{(\Lambda,\alpha)} = A^{(\Lambda,\alpha)}$.

(2) $A^{(\Lambda,\alpha)} = \cap\{F | A \subseteq F \text{ and } F \text{ is } (\Lambda,\alpha)\text{-closed}\}$.

(3) If $A \subseteq B$, *then* $A^{(\Lambda,\alpha)} \subseteq B^{(\Lambda,\alpha)}$.

(4) A is (Λ,α)-closed if and only if $A = A^{(\Lambda,\alpha)}$.

(5) $A^{(\Lambda,\alpha)}$ *is* (Λ,α)*-closed.*

Definition 4. *Let A be a subset of a topological space (X,τ). The union of all (Λ,α)-open sets contained in A is called the (Λ,α)-interior of A and is denoted by* $A_{(\Lambda,\alpha)}$.

Lemma 5. *Let A and B be subsets of a topological space (X,τ). For the (Λ,α)-interior, the following properties hold:*

(1) $A_{(\Lambda,\alpha)} \subseteq A$ *and* $[A_{(\Lambda,\alpha)}]_{(\Lambda,\alpha)} = A_{(\Lambda,\alpha)}$.

(2) If $A \subseteq B$, *then* $A_{(\Lambda,\alpha)} \subseteq B_{(\Lambda,\alpha)}$.

(3) A is (Λ,α)-open if and only if $A_{(\Lambda,\alpha)} = A$.

(4) $A_{(\Lambda,\alpha)}$ *is* (Λ,α)*-open.*

(5) $(X - A)^{(\Lambda,\alpha)} = X - A_{(\Lambda,\alpha)}$.

3 Some properties of (Λ,α)-open sets

In this section, we introduce new classes of sets called $s(\Lambda,\alpha)$-open sets, $p(\Lambda,\alpha)$-open sets, $\alpha(\Lambda,\alpha)$-open sets, $\beta(\Lambda,\alpha)$-open sets and $b(\Lambda,\alpha)$-open sets. We also investigate some of their fundamental properties.

Definition 6. *A subset A of a topological space (X,τ) is said to be:*

(i) $s(\Lambda,\alpha)$*-open if* $A \subseteq [A_{(\Lambda,\alpha)}]^{(\Lambda,\alpha)}$;

(ii) $p(\Lambda,\alpha)$*-open if* $A \subseteq [A^{(\Lambda,\alpha)}]_{(\Lambda,\alpha)}$;

(iii) $\alpha(\Lambda,\alpha)$*-open if* $A \subseteq [[A_{(\Lambda,\alpha)}]^{(\Lambda,\alpha)}]_{(\Lambda,\alpha)}$;

(iv) $\beta(\Lambda,\alpha)$*-open if* $A \subseteq [[A^{(\Lambda,\alpha)}]_{(\Lambda,\alpha)}]^{(\Lambda,\alpha)}$.

The family of all $s(\Lambda,\alpha)$-open (resp. $p(\Lambda,\alpha)$-open, $\alpha(\Lambda,\alpha)$-open, $\beta(\Lambda,\alpha)$-open) sets in a topological space (X,τ) is denoted by $s\Lambda_\alpha O(X,\tau)$ (resp. $p\Lambda_\alpha O(X,\tau)$, $\alpha\Lambda_\alpha O(X,\tau)$, $\beta\Lambda_\alpha O(X,\tau)$).

The complement of a $s(\Lambda,\alpha)$-open (resp. $p(\Lambda,\alpha)$-open, $\alpha(\Lambda,\alpha)$-open, $\beta(\Lambda,\alpha)$-open) set is called $s(\Lambda,\alpha)$*-closed* (resp. $p(\Lambda,\alpha)$*-closed*, $\alpha(\Lambda,\alpha)$*-closed*, $\beta(\Lambda,\alpha)$*-closed*). The family of all $s(\Lambda,\alpha)$-closed (resp. $p(\Lambda,\alpha)$-closed, $\alpha(\Lambda,\alpha)$-closed, $\beta(\Lambda,\alpha)$-closed) sets in a topological space (X,τ) is denoted by $s\Lambda_\alpha C(X,\tau)$ (resp. $p\Lambda_\alpha C(X,\tau)$, $\alpha\Lambda_\alpha C(X,\tau)$, $\beta\Lambda_\alpha O(X,\tau)$).

Proposition 7. *For a topological space (X,τ), the following properties hold:*

(1) $\Lambda_\alpha O(X,\tau) \subseteq \alpha\Lambda_\alpha O(X,\tau) \subseteq s\Lambda_\alpha O(X,\tau) \subseteq \beta\Lambda_\alpha O(X,\tau)$.

(2) $\alpha\Lambda_\alpha O(X,\tau) \subseteq p\Lambda_\alpha O(X,\tau) \subseteq \beta\Lambda_\alpha O(X,\tau)$.

(3) $\alpha\Lambda_\alpha O(X,\tau) = s\Lambda_\alpha O(X,\tau) \cap p\Lambda_\alpha O(X,\tau)$.

Proof. (1) Let $V \in \Lambda_\alpha O(X,\tau)$. Then, $V = V_{(\Lambda,\alpha)} \subseteq [[V_{(\Lambda,\alpha)}]^{(\Lambda,\alpha)}]_{(\Lambda,\alpha)} \subseteq [V^{(\Lambda,\alpha)}]_{(\Lambda,\alpha)} \subseteq [[V^{(\Lambda,\alpha)}]_{(\Lambda,\alpha)}]^{(\Lambda,\alpha)}$. This shows that $\Lambda_\alpha O(X,\tau) \subseteq \alpha\Lambda_\alpha O(X,\tau) \subseteq s\Lambda_\alpha O(X,\tau) \subseteq \beta\Lambda_\alpha O(X,\tau)$.

(2) Let $V \in \alpha\Lambda_\alpha O(X,\tau)$. Then, we have $V \subseteq [V^{(\Lambda,\alpha)}]_{(\Lambda,\alpha)} \subseteq [[V^{(\Lambda,\alpha)}]_{(\Lambda,\alpha)}]^{(\Lambda,\alpha)}$. Thus, $\alpha\Lambda_\alpha O(X,\tau) \subseteq p\Lambda_\alpha O(X,\tau) \subseteq \beta\Lambda_\alpha O(X,\tau)$.

(3) Let $V \in s\Lambda_\alpha O(X,\tau) \cap p\Lambda_\alpha O(X,\tau)$. Then, $V \in s\Lambda_\alpha O(X,\tau)$ and $V \in p\Lambda_\alpha O(X,\tau)$. Therefore, $V \subseteq [V_{(\Lambda,\alpha)}]^{(\Lambda,\alpha)}$ and $V \subseteq [V^{(\Lambda,\alpha)}]_{(\Lambda,\alpha)}$. Thus, $V \subseteq [V^{(\Lambda,\alpha)}]_{(\Lambda,\alpha)} \subseteq [[V_{(\Lambda,\alpha)}]^{(\Lambda,\alpha)}]_{(\Lambda,\alpha)}$. This shows that $V \in \alpha\Lambda_\alpha O(X,\tau)$ and hence

$$s\Lambda_\alpha O(X,\tau) \cap p\Lambda_\alpha O(X,\tau) \subseteq \alpha\Lambda_\alpha O(X,\tau).$$

On the other hand, by (1) and (2), $\alpha\Lambda_\alpha O(X,\tau) \subseteq s\Lambda_\alpha O(X,\tau) \cap p\Lambda_\alpha O(X,\tau)$. Thus, $\alpha\Lambda_\alpha O(X,\tau) = s\Lambda_\alpha O(X,\tau) \cap p\Lambda_\alpha O(X,\tau)$. □

Definition 8. *A subset A of a topological space (X,τ) is called $r(\Lambda,\alpha)$-open if $A = [A^{(\Lambda,\alpha)}]_{(\Lambda,\alpha)}$. The complement of a $r(\Lambda,\alpha)$-open set is called $r(\Lambda,\alpha)$-closed.*

The family of all $r(\Lambda,\alpha)$-open (resp. $r(\Lambda,\alpha)$-closed) sets in a topological space (X,τ) is denoted by $r\Lambda_\alpha O(X,\tau)$ (resp. $r\Lambda_\alpha C(X,\tau)$).

Proposition 9. *For a subset A of a topological space (X,τ), the following properties hold:*

(1) A is $r(\Lambda,\alpha)$-open if and only if $A = F_{(\Lambda,\alpha)}$ for some (Λ,α)-closed set F.

(2) A is $r(\Lambda,\alpha)$-closed if and only if $A = U^{(\Lambda,\alpha)}$ for some (Λ,α)-open set U.

Proposition 10. *For a subset A of a topological space (X,τ), the following properties hold:*

(1) A is $s(\Lambda,\alpha)$-closed if and only if $[A^{(\Lambda,\alpha)}]_{(\Lambda,\alpha)} \subseteq A$.

(2) A is $p(\Lambda,\alpha)$-closed if and only if $[A_{(\Lambda,\alpha)}]^{(\Lambda,\alpha)} \subseteq A$.

(3) A is $\alpha(\Lambda,\alpha)$-closed if and only if $[[A^{(\Lambda,\alpha)}]_{(\Lambda,\alpha)}]^{(\Lambda,\alpha)} \subseteq A$.

(4) A is $\beta(\Lambda,\alpha)$-closed if and only if $[[A_{(\Lambda,\alpha)}]^{(\Lambda,\alpha)}]_{(\Lambda,\alpha)} \subseteq A$.

Lemma 11. *For a subset A of a topological space (X,τ), the following properties hold:*

(1) $[[[A_{(\Lambda,\alpha)}]^{(\Lambda,\alpha)}]_{(\Lambda,\alpha)}]^{(\Lambda,\alpha)} = [A_{(\Lambda,\alpha)}]^{(\Lambda,\alpha)}$.

(2) $[[[A^{(\Lambda,\alpha)}]_{(\Lambda,\alpha)}]^{(\Lambda,\alpha)}]_{(\Lambda,\alpha)} = [A^{(\Lambda,\alpha)}]_{(\Lambda,\alpha)}$.

Proposition 12. *For a subset A of a topological space (X,τ), the following properties are equivalent:*

(1) A is $r(\Lambda,\alpha)$-open.

(2) A is (Λ,α)-open and $s(\Lambda,\alpha)$-closed.

(3) A is $\alpha(\Lambda,\alpha)$-open and $s(\Lambda,\alpha)$-closed.

(4) A is $p(\Lambda,\alpha)$-open and $s(\Lambda,\alpha)$-closed.

(5) A is (Λ,α)-open and $\beta(\Lambda,\alpha)$-closed.

(6) A is $\alpha(\Lambda,\alpha)$-open and $\beta(\Lambda,\alpha)$-closed.

Proof. (1) ⇒ (2) ⇒ (3) ⇒ (4): Obvious.

(4) ⇒ (5): Let A be (Λ,α)-open and $s(\Lambda,\alpha)$-closed. Then, $A \subseteq [A^{(\Lambda,\alpha)}]_{(\Lambda,\alpha)}$ and $[A^{(\Lambda,\alpha)}]_{(\Lambda,\alpha)} \subseteq A$. Therefore, $A = [A^{(\Lambda,\alpha)}]_{(\Lambda,\alpha)}$. Thus, A is $r(\Lambda,\alpha)$-open and hence A is (Λ,α)-open. Since A is $s(\Lambda,\alpha)$-closed, A is $\beta(\Lambda,\alpha)$-closed. This shows that A is (Λ,α)-open and $\beta(\Lambda,\alpha)$-closed.

(5) ⇒ (6): The proof is obvious.

(6) ⇒ (1): Let A be $\alpha(\Lambda,\alpha)$-open and $\beta(\Lambda,\alpha)$-closed. Then, $A \subseteq [[A_{(\Lambda,\alpha)}]^{(\Lambda,\alpha)}]_{(\Lambda,\alpha)}$ and $[[A_{(\Lambda,\alpha)}]^{(\Lambda,\alpha)}]_{(\Lambda,\alpha)} \subseteq A$. Thus, $A = [[A_{(\Lambda,\alpha)}]^{(\Lambda,\alpha)}]_{(\Lambda,\alpha)}$ and hence $A^{(\Lambda,\alpha)} = [[[A_{(\Lambda,\alpha)}]^{(\Lambda,\alpha)}]_{(\Lambda,\alpha)}]^{(\Lambda,\alpha)} = [A_{(\Lambda,\alpha)}]^{(\Lambda,\alpha)}$. By Lemma 11, $[A^{(\Lambda,\alpha)}]_{(\Lambda,\alpha)} = [[A_{(\Lambda,\alpha)}]^{(\Lambda,\alpha)}]_{(\Lambda,\alpha)} = A$. Therefore, A is $r(\Lambda,\alpha)$-open. □

Corollary 13. *For a subset A of a topological space (X,τ), the following properties are equivalent:*

(1) A is $r(\Lambda,\alpha)$-closed.

(2) A is (Λ,α)-closed and $s(\Lambda,\alpha)$-open.

(3) A is $\alpha(\Lambda,\alpha)$-closed and $s(\Lambda,\alpha)$-open.

(4) A is $p(\Lambda,\alpha)$-closed and $s(\Lambda,\alpha)$-open.

(5) A is (Λ,α)-closed and $\beta(\Lambda,\alpha)$-open.

(6) A is $\alpha(\Lambda,\alpha)$-closed and $\beta(\Lambda,\alpha)$-open.

Definition 14. *A subset A of a topological space (X,τ) is called (Λ,α)-clopen if A is both (Λ,α)-open and (Λ,α)-closed.*

Proposition 15. *For a subset A of a topological space (X,τ), the following properties are equivalent:*

(1) A is (Λ,α)-clopen.

(2) A is $r(\Lambda,\alpha)$-open and $r(\Lambda,\alpha)$-closed.

(3) A is (Λ,α)-open and $\alpha(\Lambda,\alpha)$-closed.

(4) A is (Λ,α)-open and $p(\Lambda,\alpha)$-closed.

(5) A is $\alpha(\Lambda,\alpha)$-open and $p(\Lambda,\alpha)$-closed.

(6) A is $\alpha(\Lambda,\alpha)$-open and (Λ,α)-closed.

(7) A is $p(\Lambda,\alpha)$-open and (Λ,α)-closed.

(8) A is $\beta(\Lambda,\alpha)$-open and $\alpha(\Lambda,\alpha)$-closed.

Proof. (1) $\Rightarrow$ (2): Let A be a (Λ,α)-clopen set. Then, we have $A = A_{(\Lambda,\alpha)} = A^{(\Lambda,\alpha)}$ and hence $A = [A_{(\Lambda,\alpha)}]^{(\Lambda,\alpha)} = [A^{(\Lambda,\alpha)}]_{(\Lambda,\alpha)}$. This shows that A is $r(\Lambda,\alpha)$-open. Thus, A is $r(\Lambda,\alpha)$-open and $r(\Lambda,\alpha)$-closed.

(2) $\Rightarrow$ (3): Let A be $r(\Lambda,\alpha)$-open and $r(\Lambda,\alpha)$-closed. Then, $A = [A_{(\Lambda,\alpha)}]^{(\Lambda,\alpha)} = [A^{(\Lambda,\alpha)}]_{(\Lambda,\alpha)}$. Thus, $A_{(\Lambda,\alpha)} = [[A^{(\Lambda,\alpha)}]_{(\Lambda,\alpha)}]_{(\Lambda,\alpha)} = [A^{(\Lambda,\alpha)}]_{(\Lambda,\alpha)} = A$ and hence

$$\begin{aligned}[][[A^{(\Lambda,\alpha)}]_{(\Lambda,\alpha)}]^{(\Lambda,\alpha)} &= [[A_{(\Lambda,\alpha)}]^{(\Lambda,\alpha)}]^{(\Lambda,\alpha)} \\ &= [A_{(\Lambda,\alpha)}]^{(\Lambda,\alpha)} = A.\end{aligned}$$

Consequently, we obtain A is (Λ,α)-open and $\alpha(\Lambda,\alpha)$-closed.

(3) $\Rightarrow$ (4): Suppose that A is (Λ,α)-open and $\alpha(\Lambda,\alpha)$-closed. Then, we have $A = A_{(\Lambda,\alpha)}$ and $[[A^{(\Lambda,\alpha)}]_{(\Lambda,\alpha)}]^{(\Lambda,\alpha)} \subseteq A$, by Lemma 11, $[A_{(\Lambda,\alpha)}]^{(\Lambda,\alpha)} = [[[A_{(\Lambda,\alpha)}]^{(\Lambda,\alpha)}]_{(\Lambda,\alpha)}]^{(\Lambda,\alpha)} = [[A^{(\Lambda,\alpha)}]_{(\Lambda,\alpha)}]^{(\Lambda,\alpha)} \subseteq A$. Thus, A is $p(\Lambda,\alpha)$-closed. This shows that A is (Λ,α)-open and $p(\Lambda,\alpha)$-closed.

(4) $\Rightarrow$ (5): Let A be (Λ,α)-open and $p(\Lambda,\alpha)$-closed. Then, $A = A_{(\Lambda,\alpha)}$ and $[A_{(\Lambda,\alpha)}]^{(\Lambda,\alpha)} \subseteq A$. Thus, $A = A_{(\Lambda,\alpha)} \subseteq [[A_{(\Lambda,\alpha)}]^{(\Lambda,\alpha)}]_{(\Lambda,\alpha)} \subseteq A_{(\Lambda,\alpha)}$ and hence $[[A_{(\Lambda,\alpha)}]^{(\Lambda,\alpha)}]_{(\Lambda,\alpha)} = A_{(\Lambda,\alpha)} = A$. Therefore, A is $\alpha(\Lambda,\alpha)$-open. Thus, A is $\alpha(\Lambda,\alpha)$-open and $p(\Lambda,\alpha)$-closed.

(5) $\Rightarrow$ (6): Let A be $\alpha(\Lambda,\alpha)$-open and $p(\Lambda,\alpha)$-closed. Then, we have $A \subseteq [[A_{(\Lambda,\alpha)}]^{(\Lambda,\alpha)}]_{(\Lambda,\alpha)}$ and $[[A_{(\Lambda,\alpha)}]^{(\Lambda,\alpha)}]_{(\Lambda,\alpha)} \subseteq A$. Thus, $A = [[A_{(\Lambda,\alpha)}]^{(\Lambda,\alpha)}]_{(\Lambda,\alpha)}$ and hence $A^{(\Lambda,\alpha)} = [[[A_{(\Lambda,\alpha)}]^{(\Lambda,\alpha)}]_{(\Lambda,\alpha)}]^{(\Lambda,\alpha)}$. By Lemma 11, we have $A^{(\Lambda,\alpha)} = [A_{(\Lambda,\alpha)}]^{(\Lambda,\alpha)}$. Since $[A_{(\Lambda,\alpha)}]^{(\Lambda,\alpha)} \subseteq A$, we have $A^{(\Lambda,\alpha)} \subseteq A$ and hence $A^{(\Lambda,\alpha)} = A$. Therefore, A is (Λ,α)-closed and $\alpha(\Lambda,\alpha)$-open.

(6) $\Rightarrow$ (7): Let A be $\alpha(\Lambda,\alpha)$-open and (Λ,α)-closed. Then, $A \subseteq [[A_{(\Lambda,\alpha)}]^{(\Lambda,\alpha)}]_{(\Lambda,\alpha)}$ and $A = A^{(\Lambda,\alpha)}$, by Lemma 11, $A \subseteq [[A_{(\Lambda,\alpha)}]^{(\Lambda,\alpha)}]_{(\Lambda,\alpha)} \subseteq [[[A^{(\Lambda,\alpha)}]_{(\Lambda,\alpha)}]^{(\Lambda,\alpha)}]_{(\Lambda,\alpha)} = [A^{(\Lambda,\alpha)}]_{(\Lambda,\alpha)}$. This shows that A is $p(\Lambda,\alpha)$-open. Thus, A is $p(\Lambda,\alpha)$-open and (Λ,α)-closed.

(7) $\Rightarrow$ (8): Let A be $p(\Lambda,\alpha)$-open and (Λ,α)-closed. Then, we have $A \subseteq [A^{(\Lambda,\alpha)}]_{(\Lambda,\alpha)}$ and $A = A^{(\Lambda,\alpha)}$. Thus, $[[A^{(\Lambda,\alpha)}]_{(\Lambda,\alpha)}]^{(\Lambda,\alpha)} \subseteq A^{(\Lambda,\alpha)} = A$. Therefore, A is $p(\Lambda,\alpha)$-open and $\alpha(\Lambda,\alpha)$-closed.

(8) $\Rightarrow$ (1): Let A be $p(\Lambda,\alpha)$-open and $\alpha(\Lambda,\alpha)$-closed. Then, $A \subseteq [A^{(\Lambda,\alpha)}]_{(\Lambda,\alpha)}$ and $[[A^{(\Lambda,\alpha)}]_{(\Lambda,\alpha)}]^{(\Lambda,\alpha)} \subseteq A$. Therefore, $A^{(\Lambda,\alpha)} \subseteq [[A^{(\Lambda,\alpha)}]_{(\Lambda,\alpha)}]^{(\Lambda,\alpha)} \subseteq A$ and hence $A^{(\Lambda,\alpha)} \subseteq A$. This shows that $A = A^{(\Lambda,\alpha)}$. Thus, A is (Λ,α)-closed. Since $[[A^{(\Lambda,\alpha)}]_{(\Lambda,\alpha)}]^{(\Lambda,\alpha)} \subseteq A$, $[[[A^{(\Lambda,\alpha)}]_{(\Lambda,\alpha)}]^{(\Lambda,\alpha)}]_{(\Lambda,\alpha)} \subseteq A_{(\Lambda,\alpha)}$, by Lemma 11, we have $A \subseteq [A^{(\Lambda,\alpha)}]_{(\Lambda,\alpha)} \subseteq A_{(\Lambda,\alpha)}$ and hence $A \subseteq A_{(\Lambda,\alpha)}$. This implies that $A = A_{(\Lambda,\alpha)}$. Therefore, A is (Λ,α)-open. Consequently, we obtain A is (Λ,α)-clopen. □

Definition 16. *A subset A of a topological space (X,τ) is called $\alpha(\Lambda,\alpha)$-$\star$-open (resp. $\beta(\Lambda,\alpha)$-$\star$-open) if $A = [[A_{(\Lambda,\alpha)}]^{(\Lambda,\alpha)}]_{(\Lambda,\alpha)}$ (resp. $A = [[A^{(\Lambda,\alpha)}]_{(\Lambda,\alpha)}]^{(\Lambda,\alpha)}$).*

Proposition 17. *A subset A of a topological space (X,τ) is $r(\Lambda,\alpha)$-open if and only if A is $\alpha(\Lambda,\alpha)$-$\star$-open.*

Proof. Suppose that A is a $r(\Lambda,\alpha)$-open set. Then, $A = [A^{(\Lambda,\alpha)}]_{(\Lambda,\alpha)}$. Thus, A is (Λ,α)-open and hence $A = [[A_{(\Lambda,\alpha)}]^{(\Lambda,\alpha)}]_{(\Lambda,\alpha)}$. Therefore, A is $\alpha(\Lambda,\alpha)$-$\star$-open.

Conversely, suppose that A is a $\alpha(\Lambda,\alpha)$-$\star$-open set. Then, $A = [[A_{(\Lambda,\alpha)}]^{(\Lambda,\alpha)}]_{(\Lambda,\alpha)}$. By Lemma 11,

$$\begin{aligned}[][A^{(\Lambda,\alpha)}]_{(\Lambda,\alpha)} &= [[[[A_{(\Lambda,\alpha)}]^{(\Lambda,\alpha)}]_{(\Lambda,\alpha)}]^{(\Lambda,\alpha)}]_{(\Lambda,\alpha)} \\ &= [[A_{(\Lambda,\alpha)}]^{(\Lambda,\alpha)}]_{(\Lambda,\alpha)} = A.\end{aligned}$$

This shows that A is $r(\Lambda,\alpha)$-open. □

Proposition 18. *A subset A of a topological space (X,τ) is $r(\Lambda,\alpha)$-closed if and only if A is $\beta(\Lambda,\alpha)$-$\star$-open.*

Proof. Suppose that A is a $r(\Lambda,\alpha)$-closed set. Then, we have $A = [A_{(\Lambda,\alpha)}]^{(\Lambda,\alpha)}$ and hence A is (Λ,α)-closed. Thus, $A = [A_{(\Lambda,\alpha)}]^{(\Lambda,\alpha)} = [[A^{(\Lambda,\alpha)}]_{(\Lambda,\alpha)}]^{(\Lambda,\alpha)}$. Therefore, A is $\beta(\Lambda,\alpha)$-$\star$-open.

Conversely, suppose that A is a $\beta(\Lambda,\alpha)$-$\star$-open set. Then, $A = [[A^{(\Lambda,\alpha)}]_{(\Lambda,\alpha)}]^{(\Lambda,\alpha)}$ and by Lemma 11, $[A_{(\Lambda,\alpha)}]^{(\Lambda,\alpha)} = [[[[A^{(\Lambda,\alpha)}]_{(\Lambda,\alpha)}]^{(\Lambda,\alpha)}]_{(\Lambda,\alpha)}]^{(\Lambda,\alpha)} = [[A^{(\Lambda,\alpha)}]_{(\Lambda,\alpha)}]^{(\Lambda,\alpha)} = A$. Thus, A is $r(\Lambda,\alpha)$-closed. □

Proposition 19. *For a subset A of a topological space (X,τ), the following properties are equivalent:*

(1) A is $\beta(\Lambda,\alpha)$-$\star$-open.

(2) A is $\beta(\Lambda,\alpha)$-open and (Λ,α)-closed.

(3) A is $\beta(\Lambda,\alpha)$-open and $\alpha(\Lambda,\alpha)$-closed.

Proposition 20. *For a subset A of a topological space (X,τ), the following properties are equivalent:*

(1) A is $\alpha(\Lambda,\alpha)$-$\star$-open.

(2) A is (Λ,α)-open and $\beta(\Lambda,\alpha)$-closed.

(3) A is $\alpha(\Lambda,\alpha)$-open and $\beta(\Lambda,\alpha)$-closed.

Definition 21. *A subset A of a topological space (X,τ) is said to be $b(\Lambda,\alpha)$-open if $A \subseteq [A_{(\Lambda,\alpha)}]^{(\Lambda,\alpha)} \cup [A^{(\Lambda,\alpha)}]_{(\Lambda,\alpha)}$. The complement of a $b(\Lambda,\alpha)$-open set is said to be $b(\Lambda,\alpha)$-closed.*

The family of all $b(\Lambda,\alpha)$-open (resp. $b(\Lambda,\alpha)$-closed) sets in a topological space (X,τ) is denoted by $b\Lambda_\alpha O(X,\tau)$ (resp. $b\Lambda_\alpha C(X,\tau)$).

Remark 22. *It is easy to see that for a topological space (X,τ),*

$$s\Lambda_\alpha O(X,\tau) \cup p\Lambda_\alpha O(X,\tau) \subseteq b\Lambda_\alpha O(X,\tau) \subseteq \beta\Lambda_\alpha O(X,\tau).$$

Proposition 23. *Let A be a subset of a topological space (X,τ). If $A = B \cup C$, where B is a $s(\Lambda,\alpha)$-open set and C is a $p(\Lambda,\alpha)$-open set, then A is $b(\Lambda,\alpha)$-open.*

Corollary 24. *For a subset A of a topological space (X,τ), the following properties are equivalent:*

(1) A is $r(\Lambda,\alpha)$-open.

(2) A is (Λ,α)-open and $b(\Lambda,\alpha)$-closed.

(3) A is $\alpha(\Lambda,\alpha)$-open and $b(\Lambda,\alpha)$-closed.

Lemma 25. *Let A be a subset of a topological space (X,τ). If A is both $s(\Lambda,\alpha)$-closed and $\beta(\Lambda,\alpha)$-open, then A is $s(\Lambda,\alpha)$-open.*

Proof. Suppose that A is both $s(\Lambda,\alpha)$-closed and $\beta(\Lambda,\alpha)$-open. Since A is $s(\Lambda,\alpha)$-closed, $[A^{(\Lambda,\alpha)}]_{(\Lambda,\alpha)} \subseteq A$. Since A is $\beta(\Lambda,\alpha)$-open,

$$[A^{(\Lambda,\alpha)}]_{(\Lambda,\alpha)} \subseteq A \subseteq [[A^{(\Lambda,\alpha)}]_{(\Lambda,\alpha)}]^{(\Lambda,\alpha)}.$$

Thus, $[A^{(\Lambda,\alpha)}]_{(\Lambda,\alpha)} \subseteq A^{(\Lambda,\alpha)}$ and hence $[[A^{(\Lambda,\alpha)}]_{(\Lambda,\alpha)}]^{(\Lambda,\alpha)} \subseteq [A_{(\Lambda,\alpha)}]^{(\Lambda,\alpha)}$. Therefore, A is $s(\Lambda,\alpha)$-open. □

Proposition 26. *Let A be a subset of a topological space (X,τ). If A is $b(\Lambda,\alpha)$-open, then $A^{(\Lambda,\alpha)}$ is $r(\Lambda,\alpha)$-closed.*

Proof. Let A be $b(\Lambda,\alpha)$-open. Then, we have $A \subseteq [A_{(\Lambda,\alpha)}]^{(\Lambda,\alpha)} \cup [A^{(\Lambda,\alpha)}]_{(\Lambda,\alpha)}$. Thus,

$$\begin{aligned} A^{(\Lambda,\alpha)} &\subseteq [[A_{(\Lambda,\alpha)}]^{(\Lambda,\alpha)} \cup [A^{(\Lambda,\alpha)}]_{(\Lambda,\alpha)}]^{(\Lambda,\alpha)} \\ &\subseteq [[A_{(\Lambda,\alpha)}]^{(\Lambda,\alpha)}]^{(\Lambda,\alpha)} \cup [[A^{(\Lambda,\alpha)}]_{(\Lambda,\alpha)}]^{(\Lambda,\alpha)} \\ &= [[A^{(\Lambda,\alpha)}]_{(\Lambda,\alpha)}]^{(\Lambda,\alpha)} \subseteq A^{(\Lambda,\alpha)} \end{aligned}$$

and hence $A^{(\Lambda,\alpha)} = [[A^{(\Lambda,\alpha)}]_{(\Lambda,\alpha)}]^{(\Lambda,\alpha)}$. This shows that $A^{(\Lambda,\alpha)}$ is $r(\Lambda,\alpha)$-closed. □

Corollary 27. *For a subset A of a topological space (X,τ), the following hold:*

(1) If A is $s(\Lambda,\alpha)$-open, then $A^{(\Lambda,\alpha)}$ is $r(\Lambda,\alpha)$-closed.

(2) If A is $p(\Lambda,\alpha)$-open, then $A^{(\Lambda,\alpha)}$ is $r(\Lambda,\alpha)$-closed.

(3) If A is $\alpha(\Lambda,\alpha)$-open, then $A^{(\Lambda,\alpha)}$ is $r(\Lambda,\alpha)$-closed.

Proposition 28. *For a subset A of a topological space (X,τ), the following properties are equivalent:*

(1) $A \in \beta\Lambda_\alpha O(X,\tau)$.

(2) $A^{(\Lambda,\alpha)} \in r\Lambda_\alpha C(X,\tau)$.

(3) $A^{(\Lambda,\alpha)} \in \beta\Lambda_\alpha O(X,\tau)$.

(4) $A^{(\Lambda,\alpha)} \in s\Lambda_\alpha O(X,\tau)$.

(5) $A^{(\Lambda,\alpha)} \in b\Lambda_\alpha O(X,\tau)$.

Proof. (1) ⇒ (2): Let $A \in \beta\Lambda_\alpha O(X,\tau)$. Then, we have $A \subseteq [[A^{(\Lambda,\alpha)}]_{(\Lambda,\alpha)}]^{(\Lambda,\alpha)}$ and hence $A^{(\Lambda,\alpha)} \subseteq [[A^{(\Lambda,\alpha)}]_{(\Lambda,\alpha)}]^{(\Lambda,\alpha)} \subseteq A^{(\Lambda,\alpha)}$. Thus, $A^{(\Lambda,\alpha)} = [[A^{(\Lambda,\alpha)}]_{(\Lambda,\alpha)}]^{(\Lambda,\alpha)}$. Consequently, we obtain $A^{(\Lambda,\alpha)} \in r\Lambda_\alpha C(X,\tau)$.

(2) ⇒ (3): Let $A^{(\Lambda,\alpha)} \in r\Lambda_\alpha C(X,\tau)$. Then, $A^{(\Lambda,\alpha)} = [[A^{(\Lambda,\alpha)}]_{(\Lambda,\alpha)}]^{(\Lambda,\alpha)}$ and so $A^{(\Lambda,\alpha)} = [[A^{(\Lambda,\alpha)}]_{(\Lambda,\alpha)}]^{(\Lambda,\alpha)} = [[[A^{(\Lambda,\alpha)}]^{(\Lambda,\alpha)}]_{(\Lambda,\alpha)}]^{(\Lambda,\alpha)}$. Therefore, $A^{(\Lambda,\alpha)} \in \beta\Lambda_\alpha O(X,\tau)$.

(3) ⇒ (4): Let $A^{(\Lambda,\alpha)} \in \beta\Lambda_\alpha O(X,\tau)$. Then, we have $A^{(\Lambda,\alpha)} \subseteq [[[A^{(\Lambda,\alpha)}]^{(\Lambda,\alpha)}]_{(\Lambda,\alpha)}]^{(\Lambda,\alpha)}$. Therefore, $A^{(\Lambda,\alpha)} \subseteq [[[A^{(\Lambda,\alpha)}]^{(\Lambda,\alpha)}]_{(\Lambda,\alpha)}]^{(\Lambda,\alpha)} = [[A^{(\Lambda,\alpha)}]_{(\Lambda,\alpha)}]^{(\Lambda,\alpha)}$. Thus, $A^{(\Lambda,\alpha)} \in s\Lambda_\alpha O(X,\tau)$.

(4) ⇒ (5): The proof is obvious.

(5) $\Rightarrow$ (1): Let $A^{(\Lambda,\alpha)} \in b\Lambda_\alpha O(X,\tau)$. Then, we have

$$\begin{aligned} A &\subseteq A^{(\Lambda,\alpha)} \\ &\subseteq [[A^{(\Lambda,\alpha)}]^{(\Lambda,\alpha)}]_{(\Lambda,\alpha)} \cup [[A^{(\Lambda,\alpha)}]_{(\Lambda,\alpha)}]^{(\Lambda,\alpha)} \\ &= [A^{(\Lambda,\alpha)}]_{(\Lambda,\alpha)} \cup [[A^{(\Lambda,\alpha)}]_{(\Lambda,\alpha)}]^{(\Lambda,\alpha)} \\ &= [[A^{(\Lambda,\alpha)}]_{(\Lambda,\alpha)}]^{(\Lambda,\alpha)}. \end{aligned}$$

This shows that $A \in \beta\Lambda_\alpha O(X,\tau)$. □

Corollary 29. *For a subset A of a topological space (X,τ), the following properties are equivalent:*

(1) $A \in \beta\Lambda_\alpha C(X,\tau)$.

(2) $A_{(\Lambda,\alpha)} \in r\Lambda_\alpha O(X,\tau)$.

(3) $A_{(\Lambda,\alpha)} \in \beta\Lambda_\alpha C(X,\tau)$.

(4) $A_{(\Lambda,\alpha)} \in s\Lambda_\alpha C(X,\tau)$.

(5) $A_{(\Lambda,\alpha)} \in b\Lambda_\alpha C(X,\tau)$.

Definition 30. *A subset A of a topological space (X,τ) is called $rs(\Lambda,\alpha)$-open if there exists a $r(\Lambda,\alpha)$-open set U such that $U \subseteq A \subseteq U^{(\Lambda,\alpha)}$. The complement of a $rs(\Lambda,\alpha)$-open set is said to be $rs(\Lambda,\alpha)$-closed.*

The family of all $rs(\Lambda,\alpha)$-open (resp. $rs(\Lambda,\alpha)$-closed) sets in a topological space (X,τ) is denoted by $rs\Lambda_\alpha O(X,\tau)$ (resp. $rs\Lambda_\alpha C(X,\tau)$).

Proposition 31. *For a subset A of a topological space (X,τ), the following properties are equivalent:*

(1) A is $rs(\Lambda,\alpha)$-open.

(2) A is $s(\Lambda,\alpha)$-open and $s(\Lambda,\alpha)$-closed.

(3) A is $b(\Lambda,\alpha)$-open and $s(\Lambda,\alpha)$-closed.

(4) A is $\beta(\Lambda,\alpha)$-open and $s(\Lambda,\alpha)$-closed.

(5) A is $s(\Lambda,\alpha)$-open and $b(\Lambda,\alpha)$-closed.

(6) A is $s(\Lambda,\alpha)$-open and $\beta(\Lambda,\alpha)$-closed.

Proof. (1) $\Rightarrow$ (2): Let U be a $r(\Lambda,\alpha)$-open set such that $U \subseteq A \subseteq U^{(\Lambda,\alpha)}$. Then, $U \subseteq A_{(\Lambda,\alpha)}$ and hence $A \subseteq U^{(\Lambda,\alpha)} \subseteq [A^{(\Lambda,\alpha)}]_{(\Lambda,\alpha)}$. Therefore, A is $s(\Lambda,\alpha)$-open. On the other hand, since $U^{(\Lambda,\alpha)} = A^{(\Lambda,\alpha)}$ and U is $r(\Lambda,\alpha)$-open, $[A^{(\Lambda,\alpha)}]_{(\Lambda,\alpha)} = [U^{(\Lambda,\alpha)}]_{(\Lambda,\alpha)} = U \subseteq A$. Thus, A is $s(\Lambda,\alpha)$-closed.

(2) $\Rightarrow$ (3) and (3) $\Rightarrow$ (4): The proofs are obvious.

(4) $\Rightarrow$ (5): The proof is obvious.

(5) $\Rightarrow$ (6): This is obvious since $b\Lambda_\alpha O(X,\tau) \subseteq \beta\Lambda_\alpha O(X,\tau)$.

(6) $\Rightarrow$ (1): Since A is $s(\Lambda,\alpha)$-open and $\beta(\Lambda,\alpha)$-closed, A is $s(\Lambda,\alpha)$-closed. Thus, $[A^{(\Lambda,\alpha)}]_{(\Lambda,\alpha)} \subseteq A \subseteq [A_{(\Lambda,\alpha)}]^{(\Lambda,\alpha)} \subseteq [[A^{(\Lambda,\alpha)}]_{(\Lambda,\alpha)}]^{(\Lambda,\alpha)}$. Let $U = [A^{(\Lambda,\alpha)}]_{(\Lambda,\alpha)}$. Then, U is $r(\Lambda,\alpha)$-open and $U \subseteq A \subseteq U^{(\Lambda,\alpha)}$. Therefore, A is $rs(\Lambda,\alpha)$-open. □

Proposition 32. *Let (X,τ) be a topological space and $x \in X$. Then, $\{x\}$ is (Λ,α)-open if and only if $\{x\}$ is $s(\Lambda,\alpha)$-open.*

Proof. The necessity is clear. Suppose that $\{x\}$ is $s(\Lambda,\alpha)$-open. Then, $\{x\} \subseteq [\{x\}_{(\Lambda,\alpha)}]^{(\Lambda,\alpha)}$. Now, $\{x\}_{(\Lambda,\alpha)}$ is either $\{x\}$ or $\emptyset$. Since $\emptyset^{(\Lambda,\alpha)} = \emptyset$ and $\{x\} \subseteq [\{x\}_{(\Lambda,\alpha)}]^{(\Lambda,\alpha)}$, we have $\{x\}_{(\Lambda,\alpha)} \neq \emptyset$. Thus, $\{x\}_{(\Lambda,\alpha)} = \{x\}$ and hence $\{x\}$ is (Λ,α)-open. □

Lemma 33. *Let A be a subset of a topological space (X,τ). If $U \in \Lambda_\alpha O(X,\tau)$ and $U \cap A = \emptyset$, then $U \cap A^{(\Lambda,\alpha)} = \emptyset$.*

Proposition 34. *Let (X,τ) be a topological space and $x \in X$. Then, the following properties are equivalent:*

(1) $\{x\}$ is $p(\Lambda,\alpha)$-open.

(2) $\{x\}$ is $b(\Lambda,\alpha)$-open.

(3) $\{x\}$ is $\beta(\Lambda,\alpha)$-open.

Proof. (1) $\Rightarrow$ (2) and (2) $\Rightarrow$ (3): The proofs are obvious.

(3) $\Rightarrow$ (1): Let $\{x\}$ be $\beta(\Lambda,\alpha)$-open. Assume that $\{x\}$ is not $p(\Lambda,\alpha)$-open. Then, $\{x\} \not\subseteq [\{x\}^{(\Lambda,\alpha)}]_{(\Lambda,\alpha)}$ and so $\{x\} \cap [\{x\}^{(\Lambda,\alpha)}]_{(\Lambda,\alpha)} = \emptyset$. Since $[\{x\}^{(\Lambda,\alpha)}]_{(\Lambda,\alpha)}$ is (Λ,α)-open, by Lemma 33, $\{x\}^{(\Lambda,\alpha)} \cap [\{x\}^{(\Lambda,\alpha)}]_{(\Lambda,\alpha)} = \emptyset$ and hence $[\{x\}^{(\Lambda,\alpha)}]_{(\Lambda,\alpha)} = \emptyset$. Thus, $[[\{x\}^{(\Lambda,\alpha)}]_{(\Lambda,\alpha)}]^{(\Lambda,\alpha)} = \emptyset^{(\Lambda,\alpha)} = \emptyset$. This is a contradiction. □

Proposition 35. *Let (X,τ) be a topological space and $x \in X$. Then, $\{x\}$ is $p(\Lambda,\alpha)$-open or $\{x\}$ is $\alpha(\Lambda,\alpha)$-closed.*

Proof. Assume that $\{x\}$ is not $p(\Lambda,\alpha)$-open. Then, $\{x\} \not\subseteq [\{x\}^{(\Lambda,\alpha)}]_{(\Lambda,\alpha)}$ and so $\{x\} \cap [\{x\}^{(\Lambda,\alpha)}]_{(\Lambda,\alpha)} = \emptyset$. Since $[\{x\}^{(\Lambda,\alpha)}]_{(\Lambda,\alpha)}$ is (Λ,α)-open, by Lemma 33, $\{x\}^{(\Lambda,\alpha)} \cap [\{x\}^{(\Lambda,\alpha)}]_{(\Lambda,\alpha)} = \emptyset$ and hence $[\{x\}^{(\Lambda,\alpha)}]_{(\Lambda,\alpha)} = \emptyset$. Thus, $[[\{x\}^{(\Lambda,\alpha)}]_{(\Lambda,\alpha)}]^{(\Lambda,\alpha)} = \emptyset^{(\Lambda,\alpha)} = \emptyset$. This shows that $\{x\}$ is $\alpha(\Lambda,\alpha)$-closed. □

Proposition 36. *Let A be a subset of a topological space (X, τ). Then, A is $s(\Lambda, \alpha)$-open if and only if there exists a (Λ, α)-open set U such that $U \subseteq A \subseteq U^{(\Lambda,\alpha)}$.*

Proof. Suppose that A is $s(\Lambda, \alpha)$-open. Then, $A \subseteq [A_{(\Lambda,\alpha)}]^{(\Lambda,\alpha)}$. Let $U = A_{(\Lambda,\alpha)}$. Then, U is a (Λ, α)-open set such that $U \subseteq A \subseteq U^{(\Lambda,\alpha)}$.

Conversely, assume that there exists a (Λ, α)-open set U such that $U \subseteq A \subseteq U^{(\Lambda,\alpha)}$. Then, $U \subseteq A_{(\Lambda,\alpha)}$ and hence $U^{(\Lambda,\alpha)} \subseteq [A_{(\Lambda,\alpha)}]^{(\Lambda,\alpha)}$. Since $A \subseteq U^{(\Lambda,\alpha)}$, we have $A \subseteq [A_{(\Lambda,\alpha)}]^{(\Lambda,\alpha)}$. Thus, A is $s(\Lambda, \alpha)$-open. □

Proposition 37. *Let A be a subset of a topological space (X, τ). If there exists a $p(\Lambda, \alpha)$-open set U such that $U \subseteq A \subseteq U^{(\Lambda,\alpha)}$ then A is $\beta(\Lambda, \alpha)$-open.*

Proof. Since $U \subseteq A \subseteq U^{(\Lambda,\alpha)}$, we have $A^{(\Lambda,\alpha)} = U^{(\Lambda,\alpha)}$ and hence $[A^{(\Lambda,\alpha)}]_{(\Lambda,\alpha)} = [U^{(\Lambda,\alpha)}]_{(\Lambda,\alpha)}$. Since U is $p(\Lambda, \alpha)$-open, $U \subseteq [A^{(\Lambda,\alpha)}]_{(\Lambda,\alpha)}$. Thus, $A \subseteq [[A^{(\Lambda,\alpha)}]_{(\Lambda,\alpha)}]^{(\Lambda,\alpha)}$ and hence A is $\beta(\Lambda, \alpha)$-open. □

Theorem 38. *For a topological space (X, τ), the following properties are equivalent:*

(1) Every $s(\Lambda, \alpha)$-open set of X is $\alpha(\Lambda, \alpha)$-open.

(2) Every $s(\Lambda, \alpha)$-open set of X is $p(\Lambda, \alpha)$-open.

(3) Every $\beta(\Lambda, \alpha)$-open set of X is $p(\Lambda, \alpha)$-open.

(4) Every $b(\Lambda, \alpha)$-open set of X is $p(\Lambda, \alpha)$-open.

(5) Every $rs(\Lambda, \alpha)$-open set of X is $p(\Lambda, \alpha)$-open.

(6) Every $rs(\Lambda, \alpha)$-open set of X is $r(\Lambda, \alpha)$-open.

(7) Every $r(\Lambda, \alpha)$-closed set of X is $p(\Lambda, \alpha)$-open.

(8) Every $r(\Lambda, \alpha)$-closed set of X is (Λ, α)-open.

Proof. (1) ⇒ (2): The proof is obvious.

(2) ⇒ (3): Let A be a $\beta(\Lambda, \alpha)$-open set. Then, $A \subseteq [[A^{(\Lambda,\alpha)}]_{(\Lambda,\alpha)}]^{(\Lambda,\alpha)}$. Let $B = [[A^{(\Lambda,\alpha)}]_{(\Lambda,\alpha)}]^{(\Lambda,\alpha)}$. Then, B is $r(\Lambda, \alpha)$-closed and so B is $s(\Lambda, \alpha)$-open. By (2), B is $p(\Lambda, \alpha)$-open and hence $A \subseteq B \subseteq [B^{(\Lambda,\alpha)}]_{(\Lambda,\alpha)} = B_{(\Lambda,\alpha)}$. Thus, $B \subseteq A^{(\Lambda,\alpha)}$. Therefore, $B_{(\Lambda,\alpha)} \subseteq [A^{(\Lambda,\alpha)}]_{(\Lambda,\alpha)}$. This shows that $A \subseteq [A^{(\Lambda,\alpha)}]_{(\Lambda,\alpha)}$. Consequently, we obtain A is $p(\Lambda, \alpha)$-open.

(3) ⇒ (4): The proof is obvious.

(4) ⇒ (5): Since $rs\Lambda_\alpha O(X, \tau) \subseteq s\Lambda_\alpha O(X, \tau)$ and $s\Lambda_\alpha O(X, \tau) \subseteq b\Lambda_\alpha O(X, \tau)$, we have $rs\Lambda_\alpha O(X, \tau) \subseteq b\Lambda_\alpha O(X, \tau)$ and by (4), $rs\Lambda_\alpha O(X, \tau) \subseteq p\Lambda_\alpha O(X, \tau)$.

(5) ⇒ (6): Since every $rs(\Lambda, \alpha)$-open set is $s(\Lambda, \alpha)$-closed, by (5), $rs(\Lambda, \alpha)$-open is both $s(\Lambda, \alpha)$-closed and $p(\Lambda, \alpha)$-open. Thus, every $rs(\Lambda, \alpha)$-open set is $r(\Lambda, \alpha)$-open by Proposition 12.

(6) ⇒ (7) and (7) ⇒ (8): The proofs are obvious.

(8) ⇒ (1): Let A be a $s(\Lambda, \alpha)$-open set. Thus, by Corollary 27, $A^{(\Lambda,\alpha)}$ is $r(\Lambda, \alpha)$-closed, by (8), $A^{(\Lambda,\alpha)}$ is (Λ, α)-open and hence $A^{(\Lambda,\alpha)} \subseteq [A^{(\Lambda,\alpha)}]_{(\Lambda,\alpha)}$. Therefore, A is $p(\Lambda, \alpha)$-open, by Proposition 7, A is $\alpha(\Lambda, \alpha)$-open. □

Corollary 39. *For a topological space (X, τ), the following properties are equivalent:*

(1) $\alpha\Lambda_\alpha O(X, \tau) = s\Lambda_\alpha O(X, \tau)$.

(2) Every $rs(\Lambda, \alpha)$-open set of X is $p(\Lambda, \alpha)$-closed.

(3) Every $rs(\Lambda, \alpha)$-open set of X is $r(\Lambda, \alpha)$-closed.

Definition 40. *A subset A of a topological space (X, τ) is said to be $p(\Lambda, \alpha)$-clopen if A is both $p(\Lambda, \alpha)$-open and $p(\Lambda, \alpha)$-closed.*

Corollary 41. *For a topological space (X, τ), the following properties are equivalent:*

(1) $\alpha\Lambda_\alpha O(X, \tau) = s\Lambda_\alpha O(X, \tau)$.

(2) Every $rs(\Lambda, \alpha)$-open set of X is $p(\Lambda, \alpha)$-clopen.

(3) Every $rs(\Lambda, \alpha)$-open set of X is (Λ, α)-clopen.

Proposition 42. *For a topological space (X, τ), the following properties are equivalent:*

(1) Every $p(\Lambda, \alpha)$-open set of X is $\alpha(\Lambda, \alpha)$-open.

(2) Every $p(\Lambda, \alpha)$-open set of X is $s(\Lambda, \alpha)$-open.

Definition 43. *Let A be a subset of a topological space (X, τ). A subset $\Lambda_{(\Lambda,\alpha)}(A)$ is defined as follows: $\Lambda_{(\Lambda,\alpha)}(A) = \cap\{U \in \Lambda_\alpha O(X, \tau) \mid A \subseteq U\}$.*

Lemma 44. *For subsets A, B of a topological space (X, τ), the following properties hold:*

(1) $A \subseteq \Lambda_{(\Lambda,\alpha)}(A)$.

(2) If $A \subseteq B$, then $\Lambda_{(\Lambda,\alpha)}(A) \subseteq \Lambda_{(\Lambda,\alpha)}(B)$.

(3) $\Lambda_{(\Lambda,\alpha)}[\Lambda_{(\Lambda,\alpha)}(A)] = \Lambda_{(\Lambda,\alpha)}(A)$.

(4) If A is (Λ, α)-open, $\Lambda_{(\Lambda,\alpha)}(A) = A$.

Lemma 45. *Let (X, τ) be a topological space and let $x, y \in X$. Then, $y \in \Lambda_{(\Lambda,\alpha)}(\{x\})$ if and only if $x \in \{y\}^{(\Lambda,\alpha)}$.*

Proof. Let $y \notin \Lambda_{(\Lambda,\alpha)}(\{x\})$. Then, there exists a (Λ,α)-open set V containing x such that $y \notin V$. Hence, $x \notin \{y\}^{(\Lambda,\alpha)}$. The converse is similarly shown. □

A subset N_x of a topological space (X,τ) is said to be (Λ,α)-*neighbourhood* of a point $x \in X$ if there exists a (Λ,α)-open set U such that $x \in U \subseteq N_x$.

Lemma 46. *A subset of a topological space (X,τ) is (Λ,α)-open in (X,τ) if and only if it is a (Λ,α)-neighbourhood of each of its points.*

Definition 47. *Let (X,τ) be a topological space and $x \in X$. A subset $\langle x\rangle_\alpha$ is defined as follows:*

$$\langle x\rangle_\alpha = \Lambda_{(\Lambda,\alpha)}(\{x\}) \cap \{x\}^{(\Lambda,\alpha)}.$$

Theorem 48. *Let (X,τ) be a topological space. Then, the following properties hold:*

(1) $\Lambda_{(\Lambda,\alpha)}(A) = \{x \in X \mid A \cap \{x\}^{(\Lambda,\alpha)} \neq \emptyset\}$ *for each subset A of X.*

(2) For each $x \in X$, $\Lambda_{(\Lambda,\alpha)}(\langle x\rangle_{sp}) = \Lambda_{(\Lambda,\alpha)}(\{x\})$.

(3) For each $x \in X$, $[\langle x\rangle_\alpha]^{(\Lambda,\alpha)} = \{x\}^{(\Lambda,\alpha)}$.

(4) If U is (Λ,α)-open in (X,τ) and $x \in U$, then $\langle x\rangle_\alpha \subseteq U$.

(5) If F is (Λ,α)-closed in (X,τ) and $x \in F$, then $\langle x\rangle_\alpha \subseteq F$.

Proof. (1) Suppose that $A \cap \{x\}^{(\Lambda,\alpha)} = \emptyset$. Then, we have $x \notin X - \{x\}^{(\Lambda,\alpha)}$ which is a (Λ,α)-open set containing A. Thus, $x \notin \Lambda_{(\Lambda,\alpha)}(A)$ and hence

$$\Lambda_{(\Lambda,\alpha)}(A) \subseteq \{x \in X \mid A \cap \{x\}^{(\Lambda,\alpha)} \neq \emptyset\}.$$

Next, let $x \in X$ such that $A \cap \{x\}^{(\Lambda,\alpha)} \neq \emptyset$ and suppose that $x \notin \Lambda_{(\Lambda,\alpha)}(A)$. There exists a (Λ,α)-open set U containing A and $x \notin U$. Let $y \in A \cap \{x\}^{(\Lambda,\alpha)}$. Thus, U is a (Λ,α)-neighbourhood of y which does not contain x. By this contradiction $x \in \Lambda_{(\Lambda,\alpha)}(A)$.

(2) Let $x \in X$. Then,

$$\{x\} \subseteq \{x\}^{(\Lambda,\alpha)} \cap \Lambda_{(\Lambda,\alpha)}(\{x\}) = \langle x\rangle_\alpha,$$

by Lemma 44, $\Lambda_{(\Lambda,\alpha)}(\{x\}) \subseteq \Lambda_{(\Lambda,\alpha)}(\langle x\rangle_\alpha)$. Next, we show the opposite implication. Suppose that $y \notin \Lambda_{(\Lambda,\alpha)}(\{x\})$. Then, there exists a (Λ,α)-open set V such that $x \in V$ and $y \notin V$. Since $\langle x\rangle_\alpha \subseteq \Lambda_{(\Lambda,\alpha)}(\{x\}) \subseteq \Lambda_{(\Lambda,\alpha)}(V) = V$, we have $\Lambda_{(\Lambda,\alpha)}(\langle x\rangle_\alpha) \subseteq V$. Since $y \notin V$, $y \notin \Lambda_{(\Lambda,\alpha)}(\langle x\rangle_\alpha)$. This shows that $\Lambda_{(\Lambda,\alpha)}(\langle x\rangle_\alpha) \subseteq \Lambda_{(\Lambda,\alpha)}(\{x\})$ and hence $\Lambda_{(\Lambda,\alpha)}(\{x\}) = \Lambda_{(\Lambda,\alpha)}(\langle x\rangle_\alpha)$.

(3) By the definition of $\langle x\rangle_\alpha$, we have $\{x\} \subseteq \langle x\rangle_\alpha$ and $\{x\}^{(\Lambda,\alpha)} \subseteq (\langle x\rangle_\alpha)^{(\Lambda,\alpha)}$ by Lemma 3. On the other hand, we have $\langle x\rangle_\alpha \subseteq \{x\}^{(\Lambda,\alpha)}$ and $[\langle x\rangle_\alpha]^{(\Lambda,\alpha)} \subseteq [\{x\}^{(\Lambda,\alpha)}]^{(\Lambda,\alpha)} = \{x\}^{(\Lambda,\alpha)}$. Thus, $(\langle x\rangle_\alpha)^{(\Lambda,\alpha)} \subseteq \{x\}^{(\Lambda,\alpha)}$.

(4) Since $x \in U$ and U is a (Λ,α)-open set, we have $\Lambda_{(\Lambda,\alpha)}(\{x\}) \subseteq U$. Thus, $\langle x\rangle_\alpha \subseteq U$.

(5) Since $x \in F$ and F is a (Λ,α)-closed set, we have $\langle x\rangle_\alpha = \{x\}^{(\Lambda,\alpha)} \cap \Lambda_{(\Lambda,\alpha)}(\{x\}) \subseteq \{x\}^{(\Lambda,\alpha)} \subseteq F^{(\Lambda,\alpha)} = F$. □

Theorem 49. *The following properties are equivalent for any points x and y in a topological space (X,τ):*

(1) $\Lambda_{(\Lambda,\alpha)}(\{x\}) \neq \Lambda_{(\Lambda,\alpha)}(\{y\})$.

(2) $\{x\}^{(\Lambda,\alpha)} \neq \{y\}^{(\Lambda,\alpha)}$.

Proof. (1) $\Rightarrow$ (2): Suppose that $\Lambda_{(\Lambda,\alpha)}(\{x\}) \neq \Lambda_{(\Lambda,\alpha)}(\{y\})$. Then, there exists a point $z \in X$ such that $z \in \Lambda_{(\Lambda,\alpha)}(\{x\})$ and $z \notin \Lambda_{(\Lambda,\alpha)}(\{y\})$ or $z \in \Lambda_{(\Lambda,\alpha)}(\{y\})$ and $z \notin \Lambda_{(\Lambda,\alpha)}(\{x\})$. We prove only the first case being the second analogous. From $z \in \Lambda_{(\Lambda,\alpha)}(\{x\})$ it follows that $\{x\} \cap \{z\}^{(\Lambda,\alpha)} \neq \emptyset$ which implies $x \in \{z\}^{(\Lambda,\alpha)}$. By $z \notin \Lambda_{(\Lambda,\alpha)}(\{y\})$, we have $\{y\} \cap \{z\}^{(\Lambda,\alpha)} = \emptyset$. Since $x \in \{z\}^{(\Lambda,\alpha)}$, $\{x\}^{(\Lambda,\alpha)} \subseteq \{z\}^{(\Lambda,\alpha)}$ and $\{y\} \cap \{x\}^{(\Lambda,\alpha)} = \emptyset$. Therefore, it follows that $\{x\}^{(\Lambda,\alpha)} \neq \{y\}^{(\Lambda,\alpha)}$. Thus, $\Lambda_{(\Lambda,\alpha)}(\{x\}) \neq \Lambda_{(\Lambda,\alpha)}(\{y\})$ implies that $\{x\}^{(\Lambda,\alpha)} \neq \{y\}^{(\Lambda,\alpha)}$.

(2) $\Rightarrow$ (1): Suppose that $\{x\}^{(\Lambda,\alpha)} \neq \{y\}^{(\Lambda,\alpha)}$. Then, there exists a point $z \in X$ such that $z \in \{x\}^{(\Lambda,\alpha)}$ and $z \notin \{y\}^{(\Lambda,\alpha)}$ or $z \in \{y\}^{(\Lambda,\alpha)}$ and $z \notin \{x\}^{(\Lambda,\alpha)}$. We prove only the first case being the second analogous. It follows that there exists a (Λ,α)-open set containing z and therefore x but not y, namely, $y \notin \Lambda_{(\Lambda,\alpha)}(\{x\})$ and thus $\Lambda_{(\Lambda,\alpha)}(\{x\}) \neq \Lambda_{(\Lambda,\alpha)}(\{y\})$. □

Theorem 50. *Let (X,τ) be a topological space and $x, y \in X$. Then, the following properties hold:*

(1) $y \in \Lambda_{(\Lambda,\alpha)}(\{x\})$ *if and only if* $x \in \{y\}^{(\Lambda,\alpha)}$.

(2) $\Lambda_{(\Lambda,\alpha)}(\{x\}) = \Lambda_{(\Lambda,\alpha)}(\{y\})$ *if and only if* $\{x\}^{(\Lambda,\alpha)} = \{y\}^{(\Lambda,\alpha)}$.

Proof. (1) Let $x \notin \{y\}^{(\Lambda,\alpha)}$. Then, there exists a (Λ,α)-open set U such that $x \in U$ and $y \notin U$. Thus, $y \notin \Lambda_{(\Lambda,\alpha)}(\{x\})$. The converse is similarly shown.

(2) Let $\Lambda_{(\Lambda,\alpha)}(\{x\}) = \Lambda_{(\Lambda,\alpha)}(\{y\})$ for any $x, y \in X$. Since $x \in \Lambda_{(\Lambda,\alpha)}(\{x\})$, $x \in \Lambda_{(\Lambda,\alpha)}(\{y\})$ and by (1), we have $y \in \{x\}^{(\Lambda,\alpha)}$. By Lemma 3, $\{y\}^{(\Lambda,\alpha)} \subseteq \{x\}^{(\Lambda,\alpha)}$. Similarly, we have $\{x\}^{(\Lambda,\alpha)} \subseteq \{y\}^{(\Lambda,\alpha)}$ and hence $\{x\}^{(\Lambda,\alpha)} = \{y\}^{(\Lambda,\alpha)}$.

Conversely, suppose that $\{x\}^{(\Lambda,\alpha)} = \{y\}^{(\Lambda,\alpha)}$. Since $x \in \{x\}^{(\Lambda,\alpha)}$, we have $x \in \{y\}^{(\Lambda,\alpha)}$ and by (1), $y \in \Lambda_{(\Lambda,\alpha)}(\{x\})$. By Lemma 44, $\Lambda_{(\Lambda,\alpha)}(\{y\}) \subseteq \Lambda_{(\Lambda,\alpha)}(\Lambda_{(\Lambda,\alpha)}(\{x\})) = \Lambda_{(\Lambda,\alpha)}(\{x\})$. Similarly, we have $\Lambda_{(\Lambda,\alpha)}(\{x\}) \subseteq \Lambda_{(\Lambda,\alpha)}(\{y\})$ and hence $\Lambda_{(\Lambda,\alpha)}(\{x\}) = \Lambda_{(\Lambda,\alpha)}(\{y\})$. □

4 Conclusion

Open sets and closed sets are fundamental concepts for the study and investigation in topological spaces. This paper is concerned with the concepts of $s(\Lambda,\alpha)$-open sets, $p(\Lambda,\alpha)$-open sets, $\alpha(\Lambda,\alpha)$-open sets, $\beta(\Lambda,\alpha)$-open sets and $b(\Lambda,\alpha)$-open sets. The relationships between these concepts are established. Moreover, some properties of $s(\Lambda,\alpha)$-open sets, $p(\Lambda,\alpha)$-open sets, $\alpha(\Lambda,\alpha)$-open sets, $\beta(\Lambda,\alpha)$-open sets and $b(\Lambda,\alpha)$-open sets are obtained. The ideas and results of this paper may motivate further research.

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Conflict of Interest

The authors have no conflicts of interest to declare that are relevant to the content of this article.