Generalization of $\rho-$Attractive Elements in Modular Function Spaces

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Abstract: In this paper, we introduce two new classes of mappings called $\rho-\alpha-$ and $\rho-\alpha-k-$non-spreading mappings to broaden the idea of $\rho-$ attractive elements in modular function spaces (MFS). In the MFS that are put up, we also demonstrate several approximation results and existence results. Illustration examples are provided to clarify the results.

Keywords:- Attractive points, modular spaces, non-spreading mappings, $\rho-$ attractive elements, $k-$non-spreading mappings, $k-\alpha-$non-spreading mappings.

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1 Introduction

In this section, firstly we will introduce a literature review of the most relevant work done in $\rho-$ attractive elements in MFS. Afterward, we will provide the theoretical background listing the most related topics to our work. Lastly, we provide in detail information on our mathematical definitions and theorems applied.

Literature review

In [20] authors developed the following notion of attractive points of nonlinear mapping in Hilbert spaces:

Let X be a nonempty subset of a Hilbert space H and $T: X \to H$. Then the set of attractive points $A(T)$ is given by,

$$A(T) = \{a \in H : \|Tx - a\| \leq \|x - a\| \ \forall\, x \in X\}$$

They provided evidence for the idea that there are attractive points in a Hilbert space for the so-called hybrid mappings. With the exception and closedness, they continued to demonstrate a weak Mann-type convergence theorem.

Research on attractive points gained momentum as a result of the hypothesis provided by [3]. Different mapping classes were combined. Non-spreading mappings are considered to be a new class of mappings proposed by, [18].

A mapping $T: X \to X$ is said to be non-spreading mapping if for any $a, b \in X$. Then,

$$\|Ta - Tb\|^2 \leq \frac{1}{2}(\|a - Tb\|^2 + \|Ta - b\|^2) .$$

Using the Hausdorff metric, [19], developed the category of $k-$non-spreading multivalued mappings based on generalized non-spreading mappings. In CAT(0) spaces, [1], investigated the convergence theorems and attractive points for normally generalized hybrid mappings (a non-linear generalization of a Hilbert space known as "Hadamard spaces").

The strong and weak convergence theorem of the Ishikawa iteration for an $(\alpha, \beta)-$generalized hybrid mapping in a uniformly convex Banach space was confirmed by [4], in 2015 as well. For normally generalized hybrid mappings in CAT(0) spaces, [2], developed an attractive point theorem. Recently, many mathematicians have developed an interest in fixed point theory in MFS. The first proposed the concept of MFS was introduced in, [7], and it was furthermore generalized in, [8]. Continuing in the same direction, [9], [10], worked on fixed point theory in the field of MFS where he proved Banach contraction principle in that space. An introduction of some fixed points for generalized contraction mappings in MFS was carried out, [11].

The proof of results of approximating fixed points in MFS was proposed for the first time by [12], after that, a multivalued $\rho-$quasi nonexpansive mappings in MFS was handled, [13]. Cyclic Kannan maps in MFS were investigated by [14], where sufficient conditions for the existence and uniqueness of fixed points were given. Detailed discussions on modular spaces are also provided in, [5], [15], [16].

In 2021, [6], introduced the notion of $\rho-$attractive elements in MFS, they also established a class of mappings called $\rho-k-$non-spreading mappings and verified the existence results and some approximation results in the setup of MFS.

The efforts mentioned above encourage us to broaden the idea of attractive elements in the context of MFS. This paper's main goal is to define classes of $\rho-\alpha-$ and $\rho-\alpha-k-$non-spreading mappings. This will enable us to demonstrate both the existence and approximation results for attractive elements in MFS, various numerical examples will be used to support our findings.

Theoretical background

Now we will review some fundamental concepts and definitions related to our topic, before introducing the definitions, it is worth mentioning that we will use the symbols as defined in Table 1:

Table 1. symbols.

Symbol	Meaning
Ω	nonempty set
Σ	nontrivial $\sigma-$algebra of subsets of Ω
P	a nontrivial $\delta-$ring of subsets of Ω
$\mathcal{E}$	linear space of all simple functions with supports from P
$\mathcal{M}_\infty$	space of all extended measurable functions

Note that:

1) P is closed with respect to forming of countable intersections, and finite unions and differences.
   i.e., suppose that $E \cap A \in P$ for any $E \in P$ and $A \in P$. Assume that there exists an increasing sequence of sets $K_n \in P$ such that $\Omega = \cup K_n$.
2) A measurable function $f : \Omega \to [-\infty,\infty]$ such that there exists a sequence $\{g_n\} \subset E$, $|g_n| \leq |f|$ *and* $g_n(\omega) \to f(\omega)\ \forall\ \omega \in \Omega$.

Definition 1.1, [3].
Let $\rho : \mathcal{M}_\infty \to [0,\infty]$ be an even, convex, and nontrivial function. We say that ρ is a regular convex function pseudomodular if:

a) $\rho(0) = 0$;
b) ρ is monotone, i.e., $|f(\omega)| \leq |g(\omega)|$ for any $\omega \in \Omega \Longrightarrow \rho(f) \leq \rho(g)$, where $f, g \in \mathcal{M}_\infty$;
c) ρ is orthogonally subadditive, i.e., $\rho(f\chi_{A\cup B}) \leq \rho(f\chi_A) + \rho(f\chi_B)\ \forall\ A,B \in \Sigma$ such that $A \cap B \neq \phi$, $f \in \mathcal{M}_\infty$; χ_A denotes the characteristic function of the set A.
d) ρ has Fatou property, that is, $|f_n(\omega)| \uparrow |f(\omega)| \forall \omega \in \Omega \Longrightarrow \rho(f_n) \uparrow \rho(f)$, where $f \in \mathcal{M}_\infty$;
e) ρ is order continuous in $\mathcal{E}$, i.e., $g_n \in \mathcal{E}$, and $|g_n(\omega)| \downarrow 0 \Longrightarrow \rho(g_n) \downarrow 0$.

Definition 1.2, [2].
A set $A \in \Sigma$ is $\rho-$null if $\rho(g1_A) = 0\ \ \forall\ \ g \in \mathcal{E}$.

A property holds $\rho-$almost everywhere ($\rho-a.e.$) if the set $\{\omega \in \Omega : \rho(\omega)\ does\ not\ hold\}$ is $\rho-$null.

We identify any pair of measurable sets whose symmetric difference is $\rho-$null as well as any pair of measurable functions differing only on a $\rho-$null set. For this, we define $\mathcal{M} = \{f \in \mathcal{M}_\infty : |f(\omega)| < \infty\ \rho-a.e.\}$ where each $f \in \mathcal{M}$ is an equivalence class of functions equal $\rho-a.e.$, rather than an individual function.

Definition 1.3, [2].

Let ρ be a regular convex function pseudomodular. Then, we say that ρ is a regular convex function modular if $\rho(f) = 0 \Longrightarrow f = 0\ \rho - a.e.$

The class of all nonzero regular convex function modular defined on Ω is denoted by $\mathcal{R}$.

Definition 1.4, [7].

Let ρ be a convex function modular. Then the modular function space $\mathcal{L}_\rho$ is defined as:

$$\mathcal{L}_\rho = \{f \in \mathcal{M}_\infty : \rho(\lambda f) \longrightarrow 0\ as\ \lambda \longrightarrow 0\}.$$

Generally, the modular ρ is not subadditive so it is not like a norm.

Therefore, the modular space $\mathcal{L}_\rho$can be fitted with an F −norm defined by:

$$\|f\|_\rho = inf\left\{\gamma > 0 : \rho\left(\frac{f}{\gamma}\right) \leq \gamma\right\}.$$

If ρ is a convex modular. Then,

$$\|f\|_\rho = inf\left\{\gamma > 0 : \rho\left(\frac{f}{\gamma}\right) \leq 1\right\}.$$

defines a norm on the modular space $\mathcal{L}_\rho$, and is called the Luxemburg norm.

Definition 1.5, [10].

Let $\mathcal{L}_\rho$ be a modular space. Then:

a) The sequence $\{f_n\} \subset \mathcal{L}_\rho$ is said to be ρ −convergent to $f \in \mathcal{L}_\rho$ if $\rho(f_n - f) \to 0\ as\ n \to \infty$;

b) The sequence $\{f_n\} \subset \mathcal{L}_\rho$ is said to be ρ −Cauchy if $\rho(f_n - f_m) \to 0\ as\ n, m \to \infty$;

c) We say that $\mathcal{L}_\rho$ is ρ −complete if and only if any ρ −Cauchy sequence in $\mathcal{L}_\rho\ is\ \rho$ −convergent.

Definition 1.6, [7].

A subset E of $\mathcal{L}_\rho$ is called:

a) ρ −closed if the ρ −limit of a ρ-convergent sequence of E always belongs to E;

b) ρ −compact if every sequence in E has a ρ −convergent subsequence in E;

c) ρ −bounded if $\delta_\rho(E) = sup\{\rho(f - g) : f, g \in E\} < \infty$;

d) The ρ −distance between $f\ and\ E$ is defined as:

$$d_\rho(f, E) = inf\{\rho(f - j) : j \in E\}.$$

The nomenclature defined for ρ is similar to metric spaces but ρ does not satisfy triangle inequality. Hence, if a sequence in $\mathcal{L}_\rho$ is ρ −convergent it does not imply ρ −Cauchy. This is only true if and only if ρ satisfies Δ_2 −condition.

Definition 1.7, [17].

The modular function ρ is said to satisfy the Δ_2 −condition if $\rho(2f_n) \to 0$ as n approaches ∞, whenever $\rho(f_n) \to 0$ as n approaches ∞.

The modular ρ satisfies some uniform convexity type properties.

Definition 1.8, [16].

Let $\rho \in \mathcal{R}$:

a) For $r > 0, \epsilon > 0$. Define,

$$D_1(r, \epsilon) = \{(f, h) : f, h \in \mathcal{L}_\rho, \rho(f) \leq r, \rho(h) \leq r, \rho(f - h) \geq \epsilon r\}.$$

Let

$$\delta_1(r, \epsilon) = inf\left\{1 - \left(\frac{1}{r^\rho}\right)\left(\frac{f+h}{2}\right) : (f, h) \in D_1(r, \epsilon)\right\},\ if\ D_1(r, \epsilon) \neq \phi,$$

and $\delta_1(r, \epsilon) = 1$ if $D_1(r, \epsilon) = \phi$. We say that ρ satisfies $(UC1)$ if for every $r > 0, \epsilon > 0$, $\delta_1(r, \epsilon) > 0$.

a) Note that for every $r > 0$, $D_1(r, \epsilon) \neq \phi$ for every $\epsilon > 0$ small enough.

We say that ρ satisfies $(UUC1)$ if for every $s \geq 0$, $\epsilon > 0$, there exists $\eta_1(s, \epsilon) > 0$ depending only upon s and ϵ such that $\delta_1(r, \epsilon) > \eta_1(s, \epsilon) > 0$ for any $r > s$.

b) We say that ρ satisfies $(UUC2)$ if for every $s \geq 0, \epsilon > 0$, there exists $\eta_2(s, \epsilon) > 0$ depending upon s and ϵ such that $\delta_2(r, \epsilon) > \eta_2(s, \epsilon) > 0$ for any $r > s$. Note that $(UC1) \Longrightarrow$

$(UUC1)$ and $(UCC1) \Longrightarrow (UUC2)$. If $\rho \in \mathcal{R}$ satisfies Δ_2 −condition, then $(UUC2)$ is equivalent to $(UUC1)$.

Definition 1.9, [6].
We will say that ρ is uniformly continuous if for every $\epsilon > 0$ and $R > 0$, there exists $\delta > 0$ such that:

$$|\rho(g) - \rho(g + h)| < \epsilon \ if \ \rho(h) \leq \delta, \quad \rho(g) \leq R.$$

A sequence $\{t_n\} \subset (0,1)$ is called bounded away from 0 if there exists $a > 0$ such that $t_n \geq a$ for every $n \in \mathbb{N}$. Similarly, $\{t_n\} \subset (0,1)$ is called bounded away from 1 if there exists $b < 1$ such that $t_n \geq b$ for every $n \in \mathbb{N}$. The following lemma helps study the convergence of fixed points as well as attractive elements in the $(UUC1)$ MFS.

Lemma 1.1, [6].
Let $\rho \in \mathcal{R}$ satisfy $(UUC1)$ and let $\{t_n\} \subset (0,1)$ be bounded away from 0 and 1. If $\exists\, R \geq 0$ such that:

$$\limsup_n \rho(f_n) \leq R,$$

$$\limsup_n \rho(g_n) \leq R, \quad and$$

$$\lim_{n\to\infty} \rho(t_n f_n + (1 - t_n) g_n) = R,$$

then

$$\lim_{n\to\infty} \rho(f_n - g_n) = 0.$$

The following theorem is necessary because MFS do not satisfy the triangle inequality.

Theorem 1.1, [6].
Let $\rho \in \mathcal{R}$ satisfy Δ_2 −condition. Let $\{f_n\}$ and $\{g_n\}$ be two sequences in $\mathcal{L}_\rho$. Then:

$$\lim_{n\to\infty} \rho(g_n) = 0 \implies limsup_n\, \rho(f_n + g_n) = \lim_{n\to\infty} \rho(f_n)$$

and

$$\lim_{n\to\infty} \rho(g_n) = 0 \implies liminf_n\, \rho(f_n + g_n) = \lim_{n\to\infty} \rho(f_n)$$

Definition 1.10, [6].
Let $X \subset \mathcal{L}_\rho$ be convex and ρ −bounded. A function $\tau : X \to [0,\infty]$ is called a ρ −type if there exists a sequence $\{y_k\}$ of elements of X such that for any $x \in X$, $\tau(x) = limsup_k\ \rho(y_k - x)$.

Now the following lemma establishes an important minimizing sequence property of uniformly convex MFS which is used to prove the existence of fixed points.

Lemma 1.2, [17].
Assume that $\rho \in \mathcal{R}$ is $(UUC1)$. Let X be a ρ −closed ρ −bounded convex nonempty subset of $\mathcal{L}_\rho$. Let τ be a ρ −type defined on X. Then any minimizing sequence in τ is ρ-convergent. Its ρ-limit is independent of the minimizing sequence.

The following lemma is a modification of the above that is used to prove the existence of attractive elements without the condition of closedness.

Lemma 1.3, [6].
Assume that $\rho \in \mathcal{R}$ is $(UUC1)$. Let X be a ρ −bounded convex nonempty subset of $\mathcal{L}_\rho$. Let τ be a ρ −type defined on X. Then any minimizing sequence in τ is ρ −convergent in $\mathcal{L}_\rho$. Its ρ −limit is independent of the minimizing sequence.

Definition 1.11, [3].
Let $\rho \in \mathcal{R}$. The growth function of a modular function ρ denoted by ω_ρ is defined as :

$$\omega_\rho(x) = sup\left\{\frac{\rho(xf)}{\rho(f)}, 0 < \rho(f) < \infty\right\}, \quad \forall\, x \in [0,\infty)$$

Note that if $x \in [0,1]$, then $\omega_\rho(x) \leq 1$.

Definition 1.12, [3].
Let X be a nonempty subset of a Hilbert space H. Let $T: X \to X$ be a mapping. $x \in X$ is said to be a fixed point of T if $x = Tx$.
The set of all fixed points is denoted by $F(T)$.

Definition 1.13, [14].
Let X be a nonempty subset of a Hilbert space H. A mapping $T: X \to X$ is said to be:
(a) ρ −nonexpansive mapping if $\rho(Tx - Ty) \leq \rho(x - y), \forall x, y \in X$.
(b) ρ −quasi-nonexpansive mapping if $\rho(Tx - y) \leq \rho(x - y), \forall x \in X\ \&\ y \in F(T)$.

Theorem 1.2, [6].

Let $\mathcal{L}_\rho$ be complete, $\rho \in \mathcal{R}$ is $(UUC1)$ and uniformly continuous. Assume that X is a nonempty $\rho-$ bounded convex subset of $\mathcal{L}_\rho$. Let $T: X \to X$ be a $\rho - k -$non-spreading mapping with $k \in (0,0.5]$. Then T has a $\rho-$attractive point.

Theorem 1.3, [6].

Let $\mathcal{L}_\rho$ be complete, $\rho \in \mathcal{R}$ is $(UUC2)$ and uniformly continuous. Assume that X is a nonempty $\rho-$ bounded, $\rho-$ closed convex subset of $\mathcal{L}_\rho$. Let $T: X \to X$ be a $\rho - \alpha -$non-spreading mapping with $\alpha \in (0,1), k \in (0, \alpha]$. Then T has a fixed point.

2 Main Results

We start this section by giving the notions of mappings $\rho - \alpha -$and $\rho - \alpha - k -$non-spreading mappings. Then, we explain the concept of $\rho-$attractive elements and prove the existence and some of the convergent results.

Definition 2.1

Let $\rho \in \mathcal{R}$. And $T: X \to \mathcal{L}_\rho$. Then:

(1) T is $\rho - \alpha -$non-spreading mapping if for $\alpha \in (0,1)$ we have:

$$\rho^2(Tx - Ty) \le \alpha\rho^2(x - Ty) + (1-\alpha)\rho^2(Tx - y) \ \ \forall x, y \in X$$

(2) T is $\rho - \alpha - k -$non-spreading mapping if for $\alpha \in (0,1)$ *and* $k > 0$ we have:

$$\rho^2(Tx - Ty) \le k(\alpha\rho^2(x - Ty) + (1-\alpha)\rho^2(Tx - y)) \ \ \forall x, y \in X$$

Note that for $\alpha = \frac{1}{2}$, $k = 1$ we get a $\rho - \frac{1}{2} - 1 -$non-spreading mapping which is similar to $\rho - \frac{1}{2} -$non-spreading mapping with $F(T) \neq \emptyset$ is $\rho-$quasi-nonexpansive mapping.

In fact, if y is a fixed point of T, then from the previous definition taking $\propto = \frac{1}{2}$, $k = 1$ we get:

$$\rho^2(Tx - Ty) \le \frac{1}{2}\rho^2(x - Ty) + \frac{1}{2}\rho^2(Tx - y)$$
$$\Longrightarrow 2\rho^2(Tx - Ty) \le \rho^2(x - Ty) + \rho^2(Tx - y)$$
$$\Longrightarrow 2\rho^2(Tx - y) \le \rho^2(x - y) + \rho^2(Tx - y) \ \textit{since } y \textit{ is a fixed point of } T$$
$$\Longrightarrow \rho^2(Tx - y) \le \rho^2(x - y).$$

Definition 2.2 (ρ −attractive element)

Let ρ be a convex modular function, X be a nonempty subset of $\mathcal{L}_\rho$, and $T: X \to X$ be a mapping. Then a function $y \in \mathcal{L}_\rho$ is called $\rho-$attractive element of T if $\forall x \in X$ we have $\rho(Tx - y) \le \rho(x - y)$.

The set of all $\rho-$attractive elements of T is denoted by $A_\rho(T)$.

Now, before we prove the existence of $\rho-$attractive element of T, we start with the following two lemmas.

Lemma 2.1

Let $\rho \in \mathcal{R}$ be uniformly continuous. Let $X \in \mathcal{L}_\rho$(nonempty) and $T: X \to \mathcal{L}_\rho$, with $A_\rho(T) \neq \emptyset$. Then $A_\rho(T)$ is closed.

Remark:

To prove this, we have to show that for any $\{y_n\} \subset A_\rho(T)$, such that $\lim_{n\to\infty} \rho(y_n - y) = 0$, then $y \in A_\rho(T)$.

Proof:

Let $x \in X \implies \rho(Tx - y) \le \rho\big((Tx - y_n) - (y - y_n)\big)$,

Since ρ is uniformly continuous and taking $\lim_{n\to\infty} \rho(Tx - y) \le \rho\big((Tx - y_n) - (y - y_n)\big) \implies \rho(Tx - y) \le \lim_{n\to\infty} \rho(Tx - y_n) \le \lim_{n\to\infty} \rho(x - y_n) = \rho(x - y). i.e\ y \in A_\rho(T)$. Hence $A_\rho(T)$ *is closed*. ■

The equation above showcase clearly that an attractive point is not necessarily a fixed point from their definitions. In this point, it is worth mentioning that if the mapping is $\rho-$quasi-nonexpansive mapping then the $\rho-$attractive elements which are in X must be fixed points of T.

Lemma 2.2

Let $\rho \in \mathcal{R}$ be uniformly continuous, $X \in \mathcal{L}_\rho$ (nonempty), and $T: X \to \mathcal{L}_\rho$ be an $\rho-$quasi-nonexpansive mapping. Then $A_\rho(T) \cap X = F(T)$.

Proof:

$(\Longrightarrow)$ $\textit{Let } y \in A_\rho(T) \cap X. \textit{ Then } y \in A_\rho(T). \textit{ So } \rho(Tx - y) \le \rho(x - y)\ \forall\, x \in X.$

$Now\ for\ a\ special\ case\ \ Let\ x=y$

$$\Longrightarrow\ \rho(Ty-y)\le\rho(y-y)=0$$
$$\Longrightarrow\rho(Ty-y)=0\Longrightarrow Ty=y$$
$$\Longrightarrow y\in F(T).$$

Conversely,

$(\Longleftarrow)$ $Let\ z\in F(T).\ Since\ T\ \ is\ \ \rho$-quasi-nonexpansive mapping, then for $x\in X$, $we\ have\ \ \rho(Tx-z)\le\rho(x-z).\ \ So,\ z\in A_\rho(T).\ Therefore\ \ z\in A_\rho(T)\ \cap X.$ ■

Now, we have to prove the existence of $\rho-$attractive point for $\rho-\alpha-$non-spreading mapping for $\alpha\in(0,1)$.

Theorem 2.1

Let $\mathcal{L}_\rho$ be complete, $\rho\in\mathcal{R}$ is $(UUC1)$ and uniformly continuous. Assume that X is a nonempty $\rho-$ bounded convex subset of $\mathcal{L}_\rho$. Let $T: X\to X$ be a $\rho-\alpha-$non-spreading mapping with $\alpha\in(0,1)$. Then T has a $\rho-$attractive point.

Proof:

Let $\{x_0\}\subset X$. Define the $\rho-$ type, $\tau: X\to[0,\infty)$ by

$$\tau(x)=limsup_n\,\rho(x-T^n(x_0))$$

By Lemma 1.3. ∃ a minimizing sequence (say $\{y_n\}$) of τ, s.t.

$$\tau(y_n)=inf_{x\in X}\tau(x).$$

But $\{T^n(x_0)\}\subset X$ and X is $\rho-$bounded (Definition 1.6), we get:

$$\tau(x)\le\delta_\rho(X)<\infty\ \ \forall x\in X.$$

Also

$$\tau(Tx)=limsup_n\,\rho\big(Tx-T^n(x_0)\big).$$

Now, by Definition 2.1 (1), we have:

$$\rho^2(T^n(x_0)-Tx)\le\alpha\rho^2\big(Tx-T^{n-1}(x_0)\big)+(1-\alpha)\rho^2\big(x-T^n(x_0)\big).$$

Letting $n\to\infty$ we have:

$$limsup_n\,\rho^2(T^n(x_0)-Tx)\le\alpha\ limsup_n\rho^2(Tx-T^{n-1}(x_0))+(1-\alpha)\ limsup_n\,\rho^2(x-T^n(x_0)).$$

$$Therefore\quad\tau^2(Tx)\le\alpha\tau^2(Tx)+(1-\alpha)\tau^2(x).$$

$$Implies\quad\tau^2(Tx)\le\frac{1-\alpha}{1-\alpha}\tau^2(x).$$

$$So,\ \ \tau^2(Tx)\le\ \tau^2(x).$$

Thus

$\tau^2(Ty_n)\le\tau^2(y_n).\ \ Therefore$ $\{T(y_n)\}$ is also a minimizing sequence of τ.

Now, depending on Lemma 1.3 $\{y_n\}$ converges to some $y\in\mathcal{L}_\rho$ and for any other minimizing sequence converges to y, then $\lim_{n\to\infty}Ty_n=y$.

So, we have to show that y is the $\rho-$attractive point of T.

By Definition 2.1 (1) and uniformly continuous of ρ , we get:

$lim_{n\to\infty}\rho^2(Ty_n-Tx)\le\alpha lim_{n\to\infty}\rho^2(Tx-y_n)+(1-\alpha)lim_{n\to\infty}\rho^2(x-Ty_n).$

So,

$$\rho^2(y-Tx)\le\alpha\rho^2(Tx-y)+(1-\alpha)\rho^2(x-y).$$

Hence $(1-\alpha)\rho^2(y-Tx)\le(1-\alpha)\rho^2(x-y)$.

$$Thus\ \ \rho^2(Tx-y)\le\rho^2(x-y).$$

$T\square erefore\ y$ is a $\rho-$attractive point of T. ■

Consequently, we have to prove the existence of $\rho-$attractive point for $\rho-\alpha-k-$non-spreading mapping for $\alpha\in(0,1), k\in(0,\alpha]$.

Theorem 2.2

Let $\mathcal{L}_\rho$ be complete, $\rho\in\mathcal{R}$ is $(UUC1)$ and uniformly continuous. Assume that X is a nonempty $\rho-$ bounded convex subset of $\mathcal{L}_\rho$. Let $T: X\to X$ be a $\rho-\alpha-k-$non-spreading mapping with $\alpha\in(0,1), k\in(0,\alpha]$. Then T has a $\rho-$attractive point.

Proof:

Let $\{x_0\}\subset X$. Define the $\rho-$ type, $\tau: X\to[0,\infty)$ by

$$\tau(x)=limsup_n\,\rho(x-T^n(x_0))$$

By Lemma 1.3. ∃ a minimizing sequence (say $\{y_n\}$) of τ, s.t.

$$\tau(y_n)=inf_{x\in X}\tau(x).$$

Since $\{T^n(x_0)\}\subset X$ and X is $\rho-$ bounded (Definition 1.6) we get:

$$\tau(x)\le\delta_\rho(X)<\infty\ \ \forall x\in X.$$

Also

$$\tau(Tx)=limsup_n\,\rho\big(Tx-T^n(x_0)\big).$$

Now, by Definition 2.1 (2), we have:

$$\rho^2(T^n(x_0)-Tx)\le k\big(\alpha\rho^2\big(Tx-T^{n-1}(x_0)\big)+(1-\alpha)\rho^2\big(x-T^n(x_0)\big)\big)$$

Letting $n \to \infty$ we have:

$$limsup_n\, \rho^2(T^n(x_0) - Tx) \leq k\Big(\alpha\, limsup_n \rho^2\big(Tx - T^{n-1}(x_0)\big) + (1-\alpha)\, limsup_n\, \rho^2\big(x - T^n(x_0)\big)\Big).$$

$$Therefore\ \ \tau^2(Tx) \leq k\alpha\tau^2(Tx) + k(1-\alpha)\tau^2(x).$$

$$So, \qquad \tau^2(Tx) \leq \frac{(1-\alpha)k}{1-\alpha k}\tau^2(x).$$

Since $\frac{(1-\alpha)k}{1-\alpha k} < 1$, we get:

$$\tau^2(Tx) \leq \ \tau^2(x).$$

Thus,

$\tau^2(Ty_n) \leq \tau^2(y_n)$. $Therefore$ $\{T(y_n)\}$ is also a minimizing sequence of τ.

Now, depending on Lemma 1.3, $\{y_n\}$ converges to some $y \in \mathcal{L}_\rho$ and for any other minimizing sequence converges to y. Then $\lim_{n\to\infty} Ty_n = y$.

So, we have to show that y is the $\rho-$attractive point of T.

Since ρ is uniformly continuous, we have

$$lim_{n\to\infty}\rho^2(Ty_n - Tx) \leq k\big(\alpha lim_{n\to\infty}\rho^2(Tx - y_n) + (1-\alpha)lim_{n\to\infty}\rho^2(x - Ty_n)\big).$$

$$Thus\ \ \rho^2(y - Tx) \leq \alpha k\rho^2(Tx - y) + (1-\alpha)k\rho^2(x-y).$$

$$Hence\ \ (1-\alpha k)\rho^2(y - Tx) \leq (1-\alpha)k\rho^2(x-y).$$

$$So, \qquad \rho^2(Tx - y) \leq \rho^2(x-y).$$

Therefore y is a $\rho-$attractive point of T. ■

Note that Theorem 2.1 is a special case of Theorem 2.2 when ($k = 1$).

As a special case, if we take $\alpha = \frac{1}{2}$ and $m = \frac{k}{2}$, then Definition 2.1 (2) implies that

$$\rho^2(Tx - Ty) \leq m\big(\rho^2(x - Ty) + \rho^2(Tx - y)\big),$$

which is Definition 2.1. in, [6], ($\rho - m -$non-spreading mapping).

In our main result if we take $\alpha = \frac{1}{2}$ and $m = \frac{k}{2}$, then we obtain the results of Theorem 1.2 and 1.3, [6].

Corollary 2.1, [6].

Let $\mathcal{L}_\rho$ be complete, $\rho \in \mathcal{R}$ is $(UUC1)$ and uniformly continuous. Assume that X is a nonempty $\rho-$ bounded convex subset of $\mathcal{L}_\rho$. Let $T: X \to X$ be a $\rho - m -$non-spreading mapping with $m \in (0,0.5]$. Then T has a $\rho-$attractive point.

Corollary 2.2, [6].

Let $\mathcal{L}_\rho$ be complete, $\rho \in \mathcal{R}$ is $(UUC2)$ and uniformly continuous. Assume that X is a nonempty $\rho-$ bounded, $\rho-$ closed convex subset of $\mathcal{L}_\rho$. Let $T: X \to X$ be a $\rho - \alpha -$non-spreading mapping with $\alpha \in (0,1), m \in (0, \alpha]$. Then T has a fixed point.

Theorem 2.3

Let $\rho \in \mathcal{R}$ satisfy $(UUC2)$ and Δ_2 $-$condition. Let X is a nonempty convex subset of $\mathcal{L}_\rho$ and $T: X \to \mathcal{L}_\rho$ be a $\rho - \alpha -$non-spreading mapping with $\alpha \in (0,1), k \in (0, \alpha]$. Suppose $A_\rho(T)$ is nonempty, define the sequence $\{x_n\}$ as follows:

$$x_{n+1} = a_n Tx_n + (1-a_n)Ty_n,$$
$$y_n = b_n x_n + (1-b_n)Tx_n \tag{1}$$

with $0 < a_n, b_n < 1$. Then $\lim_{n\to\infty}\rho(f_n - h)$ exists for $z \in A_\rho(T)$ and $\lim_{n\to\infty}\rho(x_n - Tx_n) = 0$.

Proof:

Suppose that $z \in A_\rho(T)$, since ρ is convex, we get:

$$\rho(x_{n+1} - z) = \rho(a_n Tx_n + (1-a_n)Ty_n - z)$$
$$\leq \rho(a_n(Tx_n - z) + (1-a_n)(Ty_n - z))$$
$$\leq a_n\rho(Tx_n - z) + (1-a_n)\rho(Ty_n - z)$$
$$\leq a_n\rho(x_n - h) + (1-a_n)\rho(y_n - z) \tag{2}$$

Also, we have:

$$\rho(y_n - z) = \rho(b_n x_n + (1-b_n)Tx_n - z)$$
$$\leq \rho(b_n(x_n - z) + (1-b_n)(Tx_n - z))$$
$$\leq b_n\rho(x_n - z) + (1-b_n)\rho(x_n - z)$$
$$\leq \rho(x_n - z) \tag{3}$$

Therefore:

$$\rho(x_{n+1} - z) \leq \rho(x_n - z)$$

Now because of $\{x_n\}$ is $\rho-$ bounded and $\rho(x_n - z)$ is a nonincreasing sequence, we get that $\lim_{n\to\infty}\rho(x_n - z)$ exists for $z \in A_\rho(T)$.

However, we have to show that $\lim_{n\to\infty}\rho(x_n - Tx_n) = 0$.

Let $\lim_{n\to\infty}\rho(x_n - z) = K$ (4)

For $z \in A_\rho(T)$, we have $\rho(Tx_n - z) \le \rho(x_n - z)$.

So, $limsup_n\ \rho(Tx_n - z) \le limsup_n\ \rho(x_n - z)$ $implies\ \ limsup_n\ \rho(Tx_n - z) \le K$ (5)

Also, $\rho(Ty_n - z) \le \rho(y_n - z) \le \rho(x_n - z)$.

Which implies that $\rho(Ty_n - z) \le limsup_n\ \rho(Ty_n - z) \le K$ (6)

And $\rho(y_n - z) \le \rho(x_n - z) \to limsup_n\ \rho(y_n - z) \le K$ (7)

Thus,

$$
\begin{aligned}
K &= \lim_{n\to\infty}\rho(x_{n+1} - z)\\
&= \lim_{n\to\infty}\rho(a_nTx_n + (1-a_n)Ty_n - z)\\
&= \lim_{n\to\infty}\rho(a_n(Tx_n - z) + (1-a_n)(Ty_n - z))
\end{aligned}
\tag{8}
$$

By (5), (6), (8) and Lemma 1.1, we get:

$$\lim_{n\to\infty}\rho(Tx_n - Ty_n) = 0.$$

Now we have to prove $\lim_{n\to\infty}\rho(x_n - Tx_n) = 0$.

For $\varepsilon > 0$, then $\exists\ n_0 \in \mathbb{N}$ such that:

$$\rho(Tx_n - Ty_n) < \varepsilon, \quad \forall n \ge n_0$$

By the definition of growth function, we have:

$$
\begin{aligned}
\rho(a_n(Tx_n - Ty_n)) &\le \omega_\rho(a_n)\rho(Tx_n - Ty_n)\\
&\le \rho(Tx_n - Ty_n) \le \varepsilon
\end{aligned}
$$

Thus,

$$\lim_{n\to\infty}\rho(a_n(Tx_n - Ty_n)) = 0. \tag{9}$$

Now,

$$
\begin{aligned}
\rho(x_{n+1} - z) &= \rho(a_nTx_n + (1-a_n)Ty_n - z)\\
&= \rho(a_n(Tx_n - Ty_n) + (Ty_n - z)).
\end{aligned}
$$

So, by Theorem 1.1 and (9), we get:

$$
\begin{aligned}
&liminf_n\ \rho(x_{n+1} - z)\\
&= liminf_n\ \rho(a_n(Tx_n - Ty_n) + (Ty_n - z))\\
&= liminf_n\ \rho(Ty_n - z)
\end{aligned}
$$

Therefore,

$$liminf_n\ \rho(Ty_n - z) = K.$$

Now,

$liminf_n\ \rho(Ty_n - z) \le liminf_n\ \rho(y_n - z) \to K \le liminf_n\ \rho(y_n - z)$ (10)

By (7) and (10), we get:

$$\lim_{n\to\infty}\rho(y_n - z) = K$$

Consequently,

$$\lim_{n\to\infty}\rho(y_n - z) = \lim_{n\to\infty}\rho(b_n(x_n - z) + (1-b_n)(Tx_n - z)) = K \tag{11}$$

Hence, by (4), (5), (11), and Lemma 1.1, we obtain:

$$\lim_{n\to\infty}\rho(x_n - Tx_n) = 0 \qquad \blacksquare$$

Definition 2.3

Let X be a nonempty subset of $\mathcal{L}_\rho$. A mapping $T: X \to X$ is said to satisfy condition (J) if there exists a nondecreasing function $\gamma: [0,\infty) \to [0,\infty)$ with $\gamma(0) = 0$, $\gamma(j) > 0\ \forall j \in (0,\infty)$, such that $\rho(x - Tx) \ge \gamma\left(D_\rho\left(x, A_\rho(T)\right)\right)$ where $D_\rho\left(x, A_\rho(T)\right) = \inf\{\rho(x-y): y \in A_\rho(T)\}$.

The following example explains a mapping that satisfies the condition (J).

Example 2.1

Let □ (the set of real numbers) be the space modulared as $\rho(x) = |x|$. Let $X = \{x \in □_\rho: 0 < x < 1\}$, define $T: X \to X$ as $Tx = \frac{x}{2}$.

Clearly, T is $\rho - \frac{1}{4} - 1$ –non-spreading mapping.

Clear that $y \in □_\rho$ is an attractive point of T if $\rho(Tx - y) \le \rho(x - y)\ \forall x \in X$.

Suppose that $y \in A_\rho(T)$, then:

$$\left|\frac{x}{2} - y\right| \le |x - y| \tag{12}$$

$$\left|\frac{x}{2} - y\right|^2 \le |x - y|^2$$

$$\left|\frac{x}{2} - y\right|^2 - |x - y|^2 \le 0$$

$$\left(\frac{x}{2} - y + x - y\right)\left(\frac{x}{2} - y - x + y\right) \le 0$$

$$\left(\frac{3x}{2} - 2y\right)\left(\frac{-x}{2}\right) \le 0$$

Hence, we have $y \le \frac{3x}{2}$. Because y must satisfy equation (12) $\forall\ \ 0 < x < 1,\ \ y \le 0$. Therefore $A_\rho(T) = (-\infty, 0]$.

Now define a continuous nondecreasing function $\gamma: [0,\infty) \to [0,\infty)$by $\gamma(z) = \frac{z}{8}$. Then we get:

$$\gamma\left(D_\rho\left(x, A_\rho(T)\right)\right) = \gamma\left(D_\rho(x,(-\infty,0])\right) = \gamma(|x|) = \frac{|x|}{8} < \left|\frac{x}{2} - x\right|$$

Thus, $\rho(x - Tx) \geq \gamma\left(D_\rho\left(x, A_\rho(T)\right)\right)\ \forall x \in X$.

Theorem 2.4
Let $\rho \in \mathcal{R}$ satisfy $(UUC2)$, Δ_2 −condition, and ρ is uniformly continuous. Let X be a nonempty convex subset of $\mathcal{L}_\rho$ and $T: X \to X$ be a $\rho-\alpha$ −non-spreading mapping with $\alpha \in (0,1), k \in (0, \alpha]$. Assume $A_\rho(T) \neq \emptyset$ and T satisfies the condition (J). Let $\{x_n\}$ be a sequence defined as follows:
$$x_{n+1} = a_n Tx_n + (1 - a_n)Ty_n,$$
$$y_n = b_n x_n + (1 - b_n)Tx_n$$
with $0 < a_n, b_n < 1$. Then $\{x_n\}$ ρ −converges to ρ −attractive point of T.

Proof:
It's clear that $\rho(x_{n+1} - z) \leq \rho(x_n - z)$ and $\lim_{n\to\infty} \rho(x_n - Tx_n) = 0$. Then by condition (J) and Theorem 2.3, we get:

$$liminf\ \rho(x_n - Tx_n) \geq liminf_n\ \gamma\left(D_\rho\left(x_n, A_\rho(T)\right)\right),$$
$$0 \geq liminf_n\ \gamma\left(D_\rho\left(x_n, A_\rho(T)\right)\right).$$

So, $\lim_{n\to\infty} \gamma\left(D_\rho\left(x_n, A_\rho(T)\right)\right) = 0$.

Follow that $\lim_{n\to\infty} D_\rho\left(x_n, A_\rho(T)\right) = 0$, since $\gamma(0) = 0$.

Now we have to show that $\{x_n\}$ is ρ-cauchy.

Because of $\lim_{n\to\infty} D_\rho\left(x_n, A_\rho(T)\right) = 0$, let $\epsilon > 0$, then $\exists\ n_0$ such that for $n \geq n_0$:

$D_\rho\left(x_n, A_\rho(T)\right) < \frac{\epsilon}{2}$ and $\left\{\inf \rho(x_n - z) : z \in A_\rho(T)\right\} < \frac{\epsilon}{2}$

Then $\exists\ z^* \in A_\rho(T)$ such that $\rho\left(x_{n_0} - z^*\right) < \epsilon$.

Now for $n, m \geq n_0$, by convexity of ρ and since $\rho(x_n - z)$ is nonincreasing we get:

$$\rho\left(\frac{x_{n+m} - x_n}{2}\right) \leq \rho\left(\frac{(x_{n+m} - z) - (x_n - z)}{2}\right) \leq \frac{1}{2}\rho(x_{n+m} - z) + \frac{1}{2}\rho(x_n - z)$$
$$< \frac{1}{2}\rho\left(x_{n_0} - z^*\right) + \frac{1}{2}\rho\left(x_{n_0} - z^*\right) = \rho\left(x_{n_0} - z^*\right) < \epsilon$$

Hence, by Δ_2 −condition, $\{x_n\}$ is ρ −cauchy sequence. Since $\mathcal{L}_\rho$ is complete, $\{x_n\}$ is ρ −converge to some $y \in \mathcal{L}_\rho$.

Now let $\lim_{n\to\infty} \rho(x_n - y) = 0$. Then by convexity of ρ and Theorem 2.3 we get:
$$\lim_{n\to\infty} \rho(Tx_n - y) = 0$$
Moreover, by definition 2.1 (2) and uniform convexity of ρ we have the following:
$$\rho\ (Tx_n - Tx) \leq k(\alpha\rho\ (x_n - Tx) + (1-\alpha)\rho^2(Tx_n - x))$$
This implies:
$$\rho^2(y - Tx) \leq k(\alpha\rho^2(y - Tx) + (1-\alpha)\rho^2(y - x))$$
Thus,
$$\rho(y - Tx) \leq \frac{(1-\alpha)k}{1-\alpha k}\rho(y - x) \leq \rho(y - x)$$

Therefore,
$y \in A_\rho(T)$ and $\lim_{n\to\infty} \rho(x_n - y) = 0$. ■

Definition 2.4, [18].
Let X be a subset of $\mathcal{L}_\rho$. A mapping $T: X \to \mathcal{L}_\rho$ is said to be ρ −demicompact if it has the property that whenever $\{x_n\} \in X$ is ρ −bounded and the $\{x_n - Tx_n\}$ is ρ −converge, then $\exists\ \{x_{n_k}\}$ subsequence which is ρ −converge.

Theorem 2.5
Let $\rho \in \mathcal{R}$ satisfy $(UUC2)$ and Δ_2 −condition. In addition, ρ is uniformly continuous. Let X be a nonempty convex subset of $\mathcal{L}_\rho$ and $T: X \to X$ be a $\rho - \alpha$ −non-spreading mapping with $\alpha \in (0,1)$, $k \in (0, \alpha]$ and ρ −demicompact mapping with $A_\rho(T) \neq \emptyset$. Let $\{x_n\}$ be a sequence defined as follows:
$$x_{n+1} = a_n Tx_n + (1 - a_n)Ty_n,$$
$$y_n = b_n x_n + (1 - b_n)Tx_n$$
with $0 < a_n,\ b_n < 1$. Then $\{x_n\}$ ρ −converges to ρ −attractive point of T.

Proof:
$\{x_n\}$ is a bounded sequence and $\lim_{n\to\infty} \rho(x_n - Tx_n) = 0$ by Theorem 2.3.

Also, $\exists$ a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and $y \in \mathcal{L}_\rho$ such that $\lim_{n\to\infty} \rho(x_{n_k} - y) = 0$ by definition 2.4.

Moreover, since ρ is uniformly continuous and $\lim_{n\to\infty} \rho(x_n - Tx_n) = 0$, we get:

$$\lim_{n\to\infty} \rho(Tx_{n_k} - y) = 0$$

Now, return to the definition 2.1 (2) and by the uniform continuity of ρ we have:

$$\lim_{n\to\infty} \rho^2\left(Tx_{n_k} - Tx\right) \le k \lim_{n\to\infty} (\alpha\rho^2\left(x_{n_k} - Tx\right) + (1-\alpha)\rho^2\left(Tx_{n_k} - x\right))$$

And so:

$$\rho^2(y - Tx) \le k(\alpha\rho^2(y - Tx) + (1-\alpha)\rho^2(y - x))$$

$$\rho(y - Tx) \le \frac{(1-\alpha)k}{1-\alpha k} \rho(y - x)$$

Therefore,

$$\rho(y - Tx) \le \rho(y - x)$$

since $\frac{(1-\alpha)k}{1-\alpha k} < 1$

Hence, $y \in A_\rho(T)$.

By Theorem (2.3) if $\lim_{n\to\infty} \rho(x_n - y) = 0 \; exists \; \forall\, y \in A_\rho(T)$, then $\lim_{n\to\infty} \rho(x_n - y) = 0$. ■

3 Numerical Results

Example 3.1

Let $\mathbb{R}$ (the set of real numbers) be the space modulared as $\rho(x) = |x|$. Let $X = \{x \in \mathcal{L}_\rho : 0 < x < 1\}$, define $T: X \to X$ as $Tx = \frac{x}{2}$.

X is a nonempty convex subset of $\mathbb{R}$ that satisfies $(UC1)$ conditions.

$\rho(x) = |x|$ is a uniformly continuous function and $(UUC2)$ holds.

$A_\rho(T)$ is nonempty.

$x_{n+1} = aTx_n + (1-a)Ty_n$, where,

$y_n = bx_n + (1-b)Tx_n$.

Choose $x_1 = 0.3$ using Matlab program we get the results in Table 2 below.

We see that the sequence $\{x_n\}$ converges to 0 and we can increase the speed of convergence by changing the values of $a \; and \; b$.

Note that when $a \; and \; b$ are both closed to zero then $\{x_n\}$ converge to 0 more rapidly.

Figure 1 below shows the differences between choosing $a \; and \; b$ in finding the sequence $\{x_n\}$.

Example 3.2

Let $\mathbb{R}$ (the set of real numbers) be the space modulared as $\rho(x) = |x|$. Let $X = \{x \in \mathcal{L}_\rho : 1 \le x < \infty\}$, define $T: X \to X$ as $Tx = \frac{4x-1}{5}$.

X is a nonempty convex subset of $\mathbb{R}$ that satisfies $(UC1)$ conditions.

$\rho(x) = |x|$ is a uniformly continuous function and $(UUC2)$ holds.

$A_\rho(T)$ is nonempty.

$x_{n+1} = aTx_n + (1-a)Ty_n$, where,

$y_n = bx_n + (1-b)Tx_n$.

Choose $x_1 = 6$ using Matlab program we get the results in Table 3 below.

We see that the sequence $\{x_n\}$ converges to 1 and we can increase the speed of convergence by changing the values of $a \; and \; b$.

Note that when $a \; and \; b$ are both closed to zero then $\{x_n\}$ converge to 0 more rapidly.

Figure 2 below shows the differences between choosing $a \; and \; b$ in finding the sequence $\{x_n\}$.

4 Conclusion and Future Work

In this paper, firstly, we introduced two new classes of mapping called $\rho-\alpha-$ and $\rho-\alpha-k-$non-spreading mappings. Specifically, these classes are of high importance as they are based on Modular Function Spaces (MFS). Moreover, in the following sections, we have proved the existence and uniqueness of $\rho-$ attractive elements for these classes. Furthermore, we have introduced various numerical examples to find the attractive elements based on our proven theorems. As for future works of our study, we are planning to consider recent studies of other mappings researches on Modular Function Spaces.

Table 2. The values of x_n when $x_1 = 0.3$ with different values of $a \; and \; b$ as follows:

n	x_n for $a=\frac{1}{2}$, $b=\frac{1}{2}$	x_n for $a=\frac{1}{4}$, $b=\frac{1}{4}$	x_n for $a=\frac{1}{4}$, $b=\frac{3}{4}$	x_n for $a=\frac{1}{10}$, $b=\frac{1}{10}$	x_n for $a=\frac{1}{10}$, $b=\frac{9}{10}$	x_n for $a=\frac{1}{100}$, $b=\frac{1}{100}$	x_n for $a=\frac{1}{100}$, $b=\frac{99}{100}$
1	0.3	0.3	0.3	0.3	0.3	0.3	0.3
2	0.13125	0.1078125	0.1359375	0.08925	0.14325	0.0764925	0.1492575
3	0.057421875	0.038745117	0.06159668	0.026551875	0.068401875	0.019503675	0.074259338
4	0.02512207	0.013924026	0.027910995	0.007899183	0.032661895	0.00497295	0.036945877
5	0.010990906	0.005003947	0.01264717	0.002350007	0.015596055	0.001267978	0.018381497
6	0.004808521	0.001798293	0.005730749	0.000699127	0.007447116	0.000323303	0.009145255
7	0.002103728	0.000646262	0.002596746	0.00020799	0.003555998	8.24341E-05	0.004549993
8	0.000920381	0.00023225	0.00117665	6.18771E-05	0.001697989	2.10186E-05	0.002263735
9	0.000402667	8.3465E-05	0.00053317	1.84084E-05	0.00081079	5.35923E-06	0.001126265
10	0.000176167	2.99952E-05	0.000241593	5.47651E-06	0.000387152	1.36647E-06	0.000560345
11	7.70729E-05	1.07795E-05	0.000109472	1.62926E-06	0.000184865	3.48415E-07	0.000278786

Table 3. The values of x_n when $x_1 = 6$ with different values of $a \; and \; b$ as follows:

n	$x_n \; for$ $a = \frac{1}{2}$, $b = \frac{1}{2}$	$x_n \; for$ $a = \frac{1}{10}$, $b = \frac{1}{10}$	$x_n \; for$ $a = \frac{1}{4}$, $b = \frac{3}{4}$	$x_n \; for$ $a = \frac{1}{10}$, $b = \frac{9}{10}$
1	6	6	6	6
2	4.8	4.352	4.85	4.928
3	3.888	3.247181	3.9645	4.085837
4	3.19488	2.50651	3.282665	3.424233
5	2.668109	2.009964	2.757652	2.904478
10	1.422953	1.136765	1.475758	1.56988
20	1.027191	1.002508	1.034857	1.051027
30	1.001748	1.000046	1.002554	1.004569
40	1.000112	1.000001	1.000187	1.000409
50	1.000007	1	1.000014	1.000037

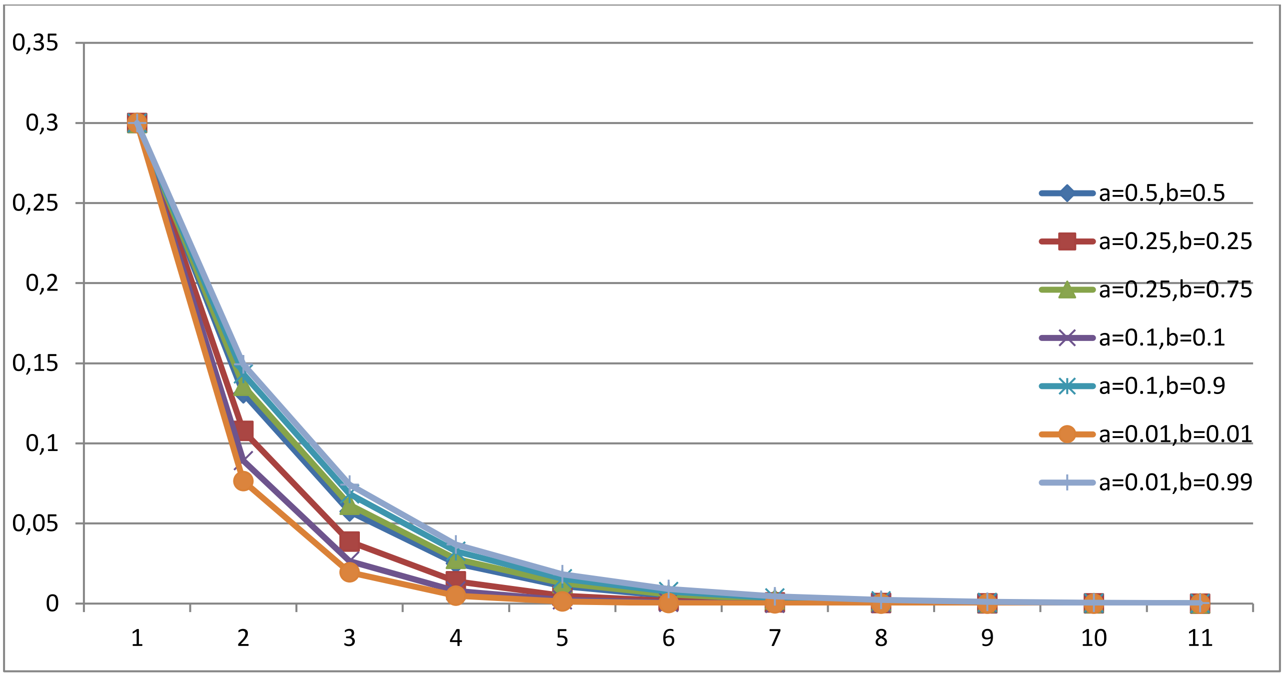

Fig. 1: Differences between choosing a and b in finding the sequence $\{x_n\}$ of example 3.1

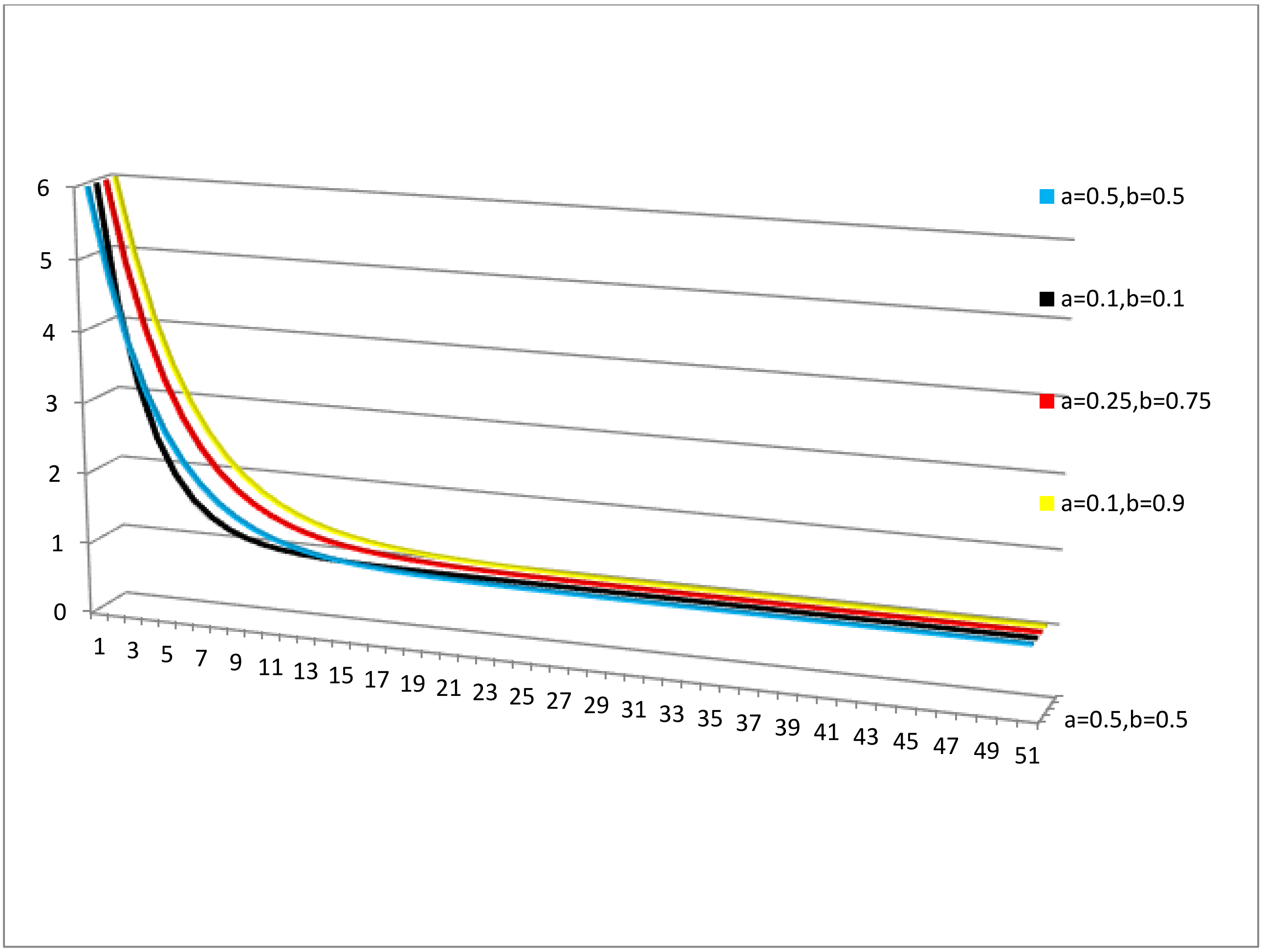

Fig. 2: Differences between choosing a and b in finding the sequence $\{x_n\}$ of example 3.2

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