The Prime Graphs $PG_3(R)$ and $PG_4(R)$ over a Ring R

PINKU SARKAR, KUNTALA PATRA
Department of mathematics, Gauhati university, Guwahati, 781014, assam, INDIA

Abstract: - Let G be a simple graph. L(G) is the Laplacian matrix of G. We de fine a simple undirected graph $PG_3(R)$ whose vertices are all the elements of the ring R and two distinct vertices a, b are adjacent if and only if $a.b = 0$ *or* $b.a = 0$ *or* $a + b = 0$ *or* $a + b$ *is a unit element of* R. Also, we define a si mple undirected graph $PG_4(R)$ whose vertices are all the elements of the ring R and two distinct vertices a, b are adjacent if and only if $a.b = 0$ *or* $b.a = 0$ *or* $a + b = 0$. In this paper we discuss degree of the vertices $PG_3(R)$, $PG_4(R)$ for $R = Z_n$ where, Z_n is the group of integer modulo n. Also, discuss planarity of the graph $PG_3(R)$, $PG_4(R)$ for $R = Z_n$. Here we introduced Laplacian of the graphs $PG_3(R)$, $PG_4(R)$ for $R = Z_p$ and $R = Z_p \times Z_p$ where, p is prime and we find their girth, algebraic connectivity, clique number and discuss Eulerian property.

Key-Words: - Laplacian matrix, prime graph, planarity, girth.

Received: July 7, 2022. Revised: January 7, 2023. Accepted: February 6, 2023. Published: March 2, 2023.

1 Introduction

In 1973 Fiedler worked o n Algebraic connectivit y of graphs, [6]. In 1994 Merris introduced m any properties of the Laplacian matrix of graphs, [11]. In 1995 He discussed r elations between the spectrum of the Laplacian and properties of graphs, [12]. The first branch of algebraic graph theor y involves the study of graphs in connection with linear algebra. In particular, it studies the spectru m of the adjacency matrix or the Laplacian matrix of a graph. The Laplacian matrix of a graph G which is denoted by L(G) is si mply the m atrix D(G) − A(G) where, $D(G)$ is degree matrix and A(G) is adjacency matrix of the graph G whose (i, j)- entry is equal to 1 if vertices i, j are adjacent and 0 otherwise. In this paper we introduce the Laplacian matrix of $PG_3(R)$, $PG_4(R)$ for $R = Z_p$ and $R = Z_p \times Z_p$. Also, we discuss some properties of the graphs. Kishor F. Pawar and Sandeep S. Joshi defined a sim ple undirected graph $\boldsymbol{PG_1(R)}$ whose vertices are all the elements of the ring R and two distinct vertices a, b are adjacent if and only if $a.b = 0$ *or* $b.a = 0$ *or* $a + b$ *is a unit element of* R, [4]. He also defined the graph $\boldsymbol{PG_2(R)}$ whose vertices are all the elements of the ring R and two distinct vertices a, b are adjacent if and only if $a.b = 0$ *or* $b.a = 0$ *or* $a + b$ is a zero-div isor (including zero) of t he ring R, [15]. Satyanarayana defined the prime graph whose vertices are all ele ments of the ring R and two distinct vertices a, b are adjacent if and only if $aRb = 0$ *or* $bRa = 0$. This grap h is den oted by $\boldsymbol{PG(R)}$ [14]. In this paper m odified the adjacency condition of these graphs we introduce two new simple undirected graphs one is $PG_3(R)$, whose vertices are all the elements of the ring R and two distinct vertices a, b are adjacent if and only if $a.b = 0$ *or* $b.a = 0$ *or* $a + b = 0$ *or* $a + b$ *is a unit element of* R and other is $PG_4(R)$ whose vertices are all the elements of the ring R and two distinct vertices a, b are adjacent if and only if $a.b = 0$ *or* $b.a = 0$ *or* $a + b = 0$.

2 Preliminary Definitions

Definition 2.1: Let G be a graph of n vertices. The Laplacian matrix of the graph G is denoted by $L(G)$ is defined as $L(G) = D(G) - A(G)$ where $D(G)$ is the degree matrix of the graph G and $A(G)$ is the adjacency matrix of G. Then i-jth entry of the $n \times n$ Laplacian matrix $L(G)$ are given by

$$L_{ij} = \begin{cases} deg\ v_i \quad if\ i = j \\ -1\ \ if\ i \neq j\ and\ v_i\ is\ adjacent\ to\ v_j \\ 0\ \ otherwise \end{cases}$$

Definition 2.2: let R be a ring. A non-zero element a of R is called a zero-div isor if there is a non-zero element b in R such that $a.b = 0$ or $b.a = 0$. The set of zero-divisors in a ring R is denoted by $Z(R)$, [4].

Definition 2.3: The elements which are not zero-divisors are called units. The set of all units in a ring R is denoted by U(R), [4].

Definition 2.4: The number of edges incident to a vertex v is called the degree of the vertex v, and it is denoted by $d(v)$.

Definition 2.5: Let G be a graph, The minimum number of vertices(lines) whose removal makes the graph G disconnected is called vertex-connectivity (line-connectivity) of the graph G. Vertex connectivity of G is denoted by $v(G)$.

Definition 2.6: The girth of a graph G is the shortest cycle in the graph G, which is denoted by $gr(G)$. If the graph G contains no cycle, then the girth of the graph G is equal to infinity.

Definition 2.7: The second smallest eigenvalue of L(G) is called algebraic connectivity of G. algebraic connectivity is denoted by $a(G)$. If L(G) has eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n = 0$ (where $n = |V(G)|$), then λ_{n-1} is called algebraic connectivity, [5].

Definition 2.8: A connected graph G is called **Eulerian** if there exists a walk with no repeated edges which includes all edges of the graph G.

Definition 2.9: A graph that can be drawn in the plane without any edge crossing is called a planar graph. If any graph contains a non-planar subgraph, then the graph is non-planar.

Definition 2.10: In a graph G the maximal complete subgraph is called a clique. The number of vertices in a clique is called clique number.

Definition 2.11: For a ring R a simple undirected graph G=(V,E) is said to be prime graph of R which is denoted by $PG_1(R)$ if all elements of R are taken as vertices of the graph and two distinct vertices a and b are adjacent if either (i) $a.b = 0$ or $b.a = 0$ or (ii) $a + b$ *is an unit element of* R, [4].

Definition 2.12: Let R be a ring. A graph G= (V, E) is said to be the prime graph of R (denoted by $PG(R)$) if vertices of the graph G are all elements of the ring R and two distinct vertices are adjacent if and only if $aRb = 0$ or $bRa = 0$, [14].

Definition 2.13: For a ring R a simple undirected graph G= (V, E) is said to be graph $\Gamma_2(R)$ if all the non-zero elements of R as vertices and two distinct vertices a and b are adjacent if and only if either $a.\ b = 0$ or $b.\ a = 0$ or $a + b$ is a zero-divisor (including zero), [12].

Theorem 2.14: Algebraic connectivity of a graph G is $a(G) = n$ if and only if $G = K_n$.

Theorem 2.15: A connected graph G is Eulerian if and only if all the vertices of the graph G are of even degree.

Theorem 2.16: Every non-zero vertices of the graph $PG_1(Z_p)$ are unit element of Z_p and $\deg(u) = \phi(p) - 1$ for any odd prime

Theorem 2.17: Every planar graph has a vertex of degree at most five.

3 Main Results

Definition 3.1: A simple undirected graph $PG_3(R)$ whose vertices are all the elements of the ring R and two distinct vertices a, b are adjacent if and only if $a.b = 0$ or $b.a = 0$ or $a + b = 0$ or $a + b$ *is a unit element of* R.

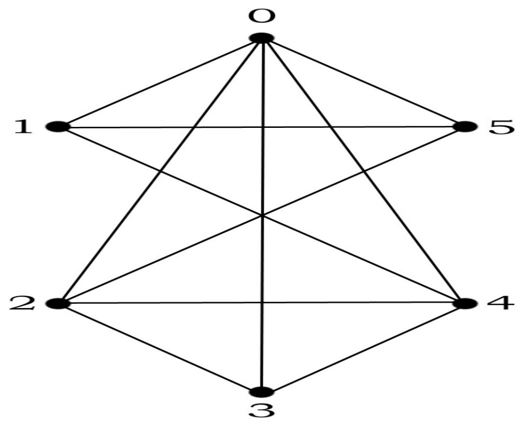
Definition 3.2: A simple undirected graph $PG_4(R)$ whose vertices are all the elements of the ring R and two distinct vertices a, b are adjacent if and only if $a.b = 0$ or $b.a = 0$ or $a + b = 0$.

Definition 3.3: Degree, Planarity, Eulerian property of $PG_3(Z_n)$

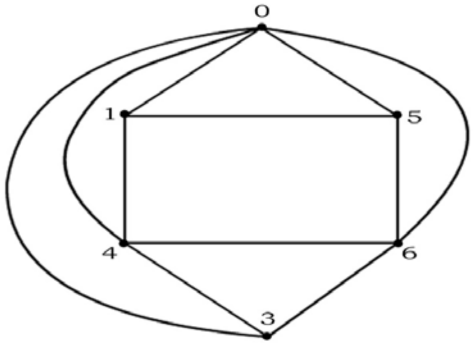
Theorem 3.3.1: The graph $PG_3(Z_n)$ is planar if and only if $n = 2, 3, 4$ and 6

Proof: For $n = 2$, the graph $PG_3(Z_n)$ is a complete graph K_2, so it is planar. The graph $PG_3(Z_3)$ is a complete graph K_3 which is also a planar graph. For

$n = 4$, the graph $PG_3(Z_n)$ is a complete graph K_4, so it is a planar graph.



$PG_3(\mathbb{Z}_6)$



$PG_3(\mathbb{Z}_6)$

Fig. 1: Graph $PG_3(Z_6)$

In the above graph structures, we can see the graph $PG_3(Z_6)$ can be drawn with no edge crossing. So, the graph $PG_3(Z_6)$ is a planar graph.

If n is prime, then the graph $PG_3(Z_n)$ is a complete graph K_n. If $n > 3$ (n is prime) the graph $PG_3(Z_n)$ always has a subgraph K_5, So the graph is not planar. If $n = 2^m$ where $m \geq 3$, a subgraph induced by $\{a, n-a, b, n-b, 0\}$ forms a complete graph K_5 where, a is an even element and b is an odd element of Z_n. So, the graph is not planar.

For $n = p^m$ where $m > 1$, A subgraph induced by $\{p, n-p, p^{m-1}, n-p^{m-1}, 0\}$ forms a complete graph K_5. So, the graph $PG_3(Z_n)$ for $n = p^m$ where $m > 1$ is not planar.

If $n = p_1^{n_1}.p_2^{n_2}.p_3^{n_3}\dots p_k^{n_k}$ where, $p_1, p_2, \cdots p_k$ are distinct prime and $n_i \in \mathbb{N}$ for $i = 1,2,\dots,k$ then the subgraph induced by $\{p, n-p, p_k^{n_k}, n-p_k^{n_k}, 0\}$ forms a complete subgraph K_5 where, $= p_1^{n_1}.p_2^{n_2}.p_3^{n_3}\dots p_{k-1}^{n_{k-1}}$.

Hence the result.

Theorem 3.3.2: Degree of any unit vertex of the graph $PG_3(Z_{p^2})$ is $\phi(\mathrm{n})$. Where; $\mathrm{n} = p^2$.

Proof: Let a be a unit element of $PG_3(Z_{p^2})$. There is $\phi(\mathrm{n})$ number of unit elements. For every unit element, there exists an element b such that $a+b$ is a unit element. Also, $a + a = 2a$ is a unit in $PG_3(Z_{p^2})$, but the graph is simple, so a is not adjacent with itself.

So, the number of b is $\phi(\mathrm{n}) - 1$. $\cdots$ (1)

$a.0 = 0 \Rightarrow a \sim 0$.

0 is already count in (1)

Also, there is a vertex $c = n - a$ for which $a + c = 0 \Rightarrow a \sim c$.

So, the number of adjacent vertices with any unit vertex is $\phi(\mathrm{n}) - 1 + 1$.

Therefore, the degree of any unit vertex of the graph $PG_3(Z_{p^2})$ is $\phi(\mathrm{n})$.

Theorem 3.3.3: Degree of any zero-divisor of the graph $PG_3(Z_{p^2})$ is $p^2 - 1$.

Proof: zero vertex is adjacent with every vertex (because, $0.a = 0$). So, the degree of zero vertex is $p^2 - 1$.

Non-zero zero-divisors of Z_{p^2} are multiples of p.

Any two non-zero zero-divisors a, b are adjacent because both are multiples of p, so $a.b$ is multiple of p^2 which is 0 in Z_{p^2}.

Let u be a unit element and z be any zero-divisor of Z_{p^2}. Since, $u + z$ is not multiple of p, so $u + z$ is unit. Therefore, any zero-divisor z is adjacent with every unit element.

Hence, any zero-divisor of Z_{p^2} is adjacent with every element of Z_{p^2}.

Degree of any zero-divisor of the graph $PG_3(Z_{p^2})$ is $p^2 - 1$.

Theorem 3.3.4: If p is odd prime, then the graphs $PG_3(Z_{p^2})$ and $PG_3(Z_p)$ are Eulerian.

Proof: Degree of any unit vertex of the graph $PG_3(Z_{p^2})$ is $\phi(p^2) = p(p-1)$ which is even for any prime p. Also, the degree of any zero-divisor of the graph $PG_3(Z_{p^2})$ is $p^2 - 1$, which is even for any odd prime p. Therefore, the graph $PG_3(Z_{p^2})$ is Eulerian if p is odd prime.

The graph $PG_3(Z_p)$ is a complete graph K_p. So, the degree of every vertex is $p-1$, which is even for any odd prime p. Hence, the graph $PG_3(Z_p)$ is Eulerian if p is odd prime.

Theorem 3.3.5: If z is a non-zero zero divisor and u is any unit element of $Z_{p.q}$ then

a) degree of z in the graph $PG_3(Z_{p.q})$ is $\phi(\text{pq}) + p$, where z is multiple of p.
b) degree of z in the graph $PG_3(Z_{p.q})$ is $\phi(\text{pq}) + q$, where z is multiple of q.
c) degree of u in the graph $PG_3(Z_{p.q})$ is $\phi(\text{pq}) - 1$.

Where p, q are distinct prime.

Proof: Any non-zero zero divisors of $Z_{p.q}$ are multiples of p and multiples of q.

a) Let z be a non-zero zero divisor which is multiple of p.

For any unit element u there exists $b \in Z_{p.q}$ such that $z + b = u$, there are $\phi(\text{pq})$ number of $b(\neq z)$ which are adjacent with z.

z is adjacent with such zero-divisors which are multiples of q. Because, $z.c \equiv 0\ (mod\ pq)$ where c is a multiple of q. There are $p-1$ numbers of zero-divisors which are multiples of q.

Also, it is adjacent with zero vertex.

Therefore, the degree of z is $= \phi(\text{pq}) + (p-1) + 1 = \phi(\text{pq}) + p$.

b) Let z be a non-zero zero divisor which is multiple of q.

For any unit element u there exists $b \in Z_{p.q}$ such that $z + b = u$, there are $\phi(\text{pq})$ number of $b(\neq z)$ which are adjacent with z.

z is adjacent with such zero-divisors which are multiples of p. Because, $z.c \equiv 0\ (mod\ pq)$ where c is a multiple of p. There are $q-1$ numbers of zero-divisors which are multiples of p.

Also, it is adjacent with zero vertex.

Therefore, the degree of z is $= \phi(\text{pq}) + (q-1) + 1 = \phi(\text{pq}) + q$.

c) Let u be a unit element of $Z_{p.q}$.

For any unit element a there exists $b \in Z_{p.q}$ such that $u + b = a$, also, $u + u$ is a unit element in $Z_{p.q}$. So, there are $\phi(\text{pq}) - 1$ number of $b(\neq u)$ which are adjacent with u. Therefore, the degree of u in the graph $PG_3(Z_{p.q})$ is $\phi(\text{pq}) - 1$.

Where p, q are distinct prime.

Example 3.3.5.1: In the graph $PG_3(Z_{15})$, non-zero zero-divisors are 3, 5, 6, 9, 10, 12. And unit elements are 1, 2, 4, 7, 8, 11, 13, 14. Degrees of 3, 6, 9, 12 are $\phi(15) + 3 = 8 + 3 = 11$.

Degrees of 5, 10, 15 are $\phi(15) + 5 = 8 + 5 = 13$.

Degrees of 1, 2, 4, 7, 8, 11, 13, 14 are $\phi(15) - 1 = 8 - 1 = 7$.

3.4 Laplacian and Algebraic Connectivity of $\boldsymbol{PG_3(\mathbb{Z}_p)}$

Theorem 3.4.1: Laplacian of $PG_3(Z_p)$ is

$$L_{ij} = p - 1 \quad (i = j)$$

$$= -1 \quad (i \neq j)$$

Proof: $\bar{0}, \bar{1}, \bar{2}, \cdots \overline{p-1}$ are vertices of $PG_3(\mathbb{Z}_p)$. Here $\bar{0}$ is adjacent with all other vertices, because, $\bar{a}.\bar{0} = \bar{0}$ where, $\bar{a}$ is any vertex of the graph.

Therefore, $deg\ (\bar{0}) = p - 1$.

Let $\bar{a}$ is a non-zero element of $\mathbb{Z}_p$.

If we use the adjacency condition ‘ $\bar{a}.\bar{b} = \bar{0}$ or $\bar{b}.\bar{a} = \bar{0}$’

We get $\bar{a} \sim \bar{0}$ $\cdots$ (1)

If we use the adjacency condition ‘ $\bar{a} + \bar{b}$ *is unit element*’.

We get $\bar{a} \sim \bar{b}$ where, $\bar{b} \neq \overline{p-a}$ is any element of Z_p [$\bar{a} + \bar{b}$ is non-zero for $\bar{b} \neq \overline{p-a}$. In $\mathbb{Z}_p$ every non-zero element is a unit so $\bar{a} + \bar{b}$ is a unit for

$\bar{b} \neq \overline{p-a}$.] $\cdots$ (2)

If we use adjacency condition $\bar{a} + \bar{b} = \bar{0}$

We get $\bar{a} \sim \overline{p-a}$ $\cdots$ (3)

From (1) (2) and (3) $\bar{a}$ is adjacent with all elements of Z_p except itself (because the graph is simple).

Therefore, the degree of $\bar{a}$ is $= p-1$. [from (1), (2) and (3)]

Hence, the degree of every vertex of the graph is $p-1$.

So, the graph $PG_3(Z_p)$ forms a complete graph K_p.

By definition of Laplacian

$$L_{ij} = p-1 \quad (i=j)$$

$$= -1 \quad (i \neq j)$$

Theorem 3.4.2: If $G = PG_3(Z_p)$ where p is a prime, then algebraic connectivity $a(G) = p$.

Proof: The graph $PG_3(Z_p)$ forms a complete graph K_p. We know that (G) = n if and only if $G = K_n$. Therefore, algebraic connectivity of $G = PG_3(Z_p)$ is $a(G) = p$.

3.5 Laplacian of $PG_3(\mathbb{Z}_p \times \mathbb{Z}_p)$

Theorem 3.5.1: Laplacian of the graph $PG_3(Z_p \times Z_p)$ is

$$L_{ij} = p^2 - 1 \quad (i = j = 1)$$

$= (p-1)^2 + 2$ $(i = j \neq 1,\ u_i$ is zero divisor)

$= (p-1)^2$ $(i = j \neq 1,\ u_i$ is unit element)

$= -1$ if $v_i = (\bar{0}, \bar{a})$ and $v_j = (\bar{b}, \bar{0})$ where $\bar{a}, \bar{b}$ are non-zero elements of Z_p

or

$v_i = (\bar{a}, \bar{0})$ and $v_j = (\bar{0}, \bar{b})$ where $\bar{a}, \bar{b}$ are non-zero elements of Z_p

or

$v_i = (\bar{a}, \bar{0})$ and $v_j = (\bar{b}, \bar{c})$ where $\bar{b}, \bar{c}$ are non-zero elements of Z_p and $\bar{b} \neq \bar{p} - \bar{a}$

or

$v_i = (\bar{0}, \bar{a})$ and $v_j = (\bar{b}, \bar{c})$ where $\bar{b}, \bar{c}$ are non-zero elements of Z_p and $\bar{c} \neq \bar{p} - \bar{a}$

or

$v_i = (\bar{a}, \bar{0})$ and $v_j = (\bar{p} - \bar{a}, \bar{0})$ where, $\bar{a}$ is non-zero elements of Z_p

or

$v_i = (\bar{0}, \bar{a})$ and $v_j = (\bar{0}, \bar{p} - \bar{a})$ where, $\bar{a}$ is non-zero element of Z_p

Or

$v_i = (\bar{a}, \bar{b})$ and $v_j = (\bar{c}, \bar{d})$ where, $\bar{a}, \bar{b}, \bar{c}, \bar{d}$ are non-zero elements of Z_p and

$\bar{c} \neq \bar{p} - \bar{a}$ and $\bar{d} \neq \bar{p} - \bar{b}$

or

$v_i = (\bar{a}, \bar{b})$ and $v_j = (\bar{p} - \bar{a}, \bar{p} - \bar{b})$ where $\bar{b}, \bar{c}$ are non-zero elements of Z_p

$= 0$ otherwise

Where, p is an odd prime.

Proof:

All the elements of $Z_p \times Z_p$ are considered as vertices of the graph $PG_3(Z_p \times Z_p)$. Zero divisors of $Z_p \times Z_p$ are of the form $(\bar{a}, \bar{0})$ and $(\bar{0}, \bar{b})$,Where $\bar{a}, \bar{b}$ are elements of Z_p.Unit elements of $Z_p \times Z_p$ are of the form $(\bar{a}, \bar{b})$ where $\bar{a}, \bar{b}$ are non-zero elements of Z_p.

$(\bar{0}, \bar{0}) \sim (\bar{a}, \bar{b})$ $\forall\, \bar{a}, \bar{b} \in Z_p$ Since,$(\bar{0}, \bar{0}).(\bar{a}, \bar{b}) = (\bar{0}, \bar{0})$ $\forall\, \bar{a}, \bar{b} \in Z_p$.

Clearly, the vertex $(\bar{0}, \bar{0})$ is adjacent with all other vertices except itself [since the graph is simple].

$$\therefore degree\, (\bar{0}, \bar{0}) = p^2 - 1. \quad \cdots \quad (1)$$

For $(\bar{a}, \bar{0}) \in Z_p \times Z_p$, where $\bar{a}$ is non-zero element of Z_p;

$(\bar{a}, \bar{0}) \sim (\bar{0}, \bar{b})$ where $\bar{b}$ is any element of Z_p. [Since, $(\bar{a}, \bar{0}).(\bar{0}, \bar{b}) = (\bar{0}, \bar{0})$]

There are p numbers of $(\bar{0}, \bar{b})$ in $Z_p \times Z_p$ which are adjacent with $(\bar{a}, \bar{0})$, where $\bar{b}$ is any element of Z_p.

$$\cdots \quad (i)$$

$(\bar{a}, \bar{0}) \sim (\bar{b}, \bar{c})$ where $\bar{b}$, $\bar{c}$ are non-zero elements of $Z_p$and $\bar{b} \neq \bar{p} - \bar{a}$.

[Because, $(\bar{a}, \bar{0}) + (\bar{b}, \bar{c}) = (\bar{a} + \bar{b}, \bar{c})$ is unit element of $Z_p \times Z_p$ since for $p \neq 2$ $\bar{a} + \bar{a} = 2\bar{a}$ is non-zero and $(\bar{a} + \bar{b})$ is non-zero for $\bar{b} \neq \bar{p} - \bar{a}$]

There are $p - 1$ number of elements in $Z_p \times Z_p$ are of the form $(\bar{p} - \bar{a}, \bar{c})$. Where $\bar{a}, \bar{c}$ are non-zero elements of Z_p. And for non-zero $\bar{b}, \bar{c}$ there are $(p-1)^2$ no. of $(\bar{b}, \bar{c})$ in $Z_p \times Z_p$.

So, number of $(\bar{b}, \bar{c})$ in the graph which are adjacent with $(\bar{a}, \bar{0})$ is $= (p-1)^2 - (p-1) \quad \cdots \quad (ii)$

Also, $(\bar{a}, \bar{0}) \sim (\bar{p} - \bar{a}, \bar{0})$ [Since $(\bar{a}, \bar{0}) + (\bar{p} - \bar{a}, \bar{0}) = (\bar{0}, \bar{0})$] $\cdots \quad (iii)$

$\therefore$ From (i), (ii) and (iii)

Number of adjacent vertices in $PG_3(Z_p \times Z_p)$ with the vertex $(\bar{a}, \bar{0})$ is

$$= p + (p-1)^2 - (p-1) + 1 = (p-1)^2 + 2 \quad \cdots \quad (2)$$

For any unit element $(\bar{a}, \bar{b})$ of $Z_p \times Z_p$ where $\bar{a}, \bar{b}$ are non-zero elements of Z_p;

$(\bar{a}, \bar{b})$ is not adjacent with $(\bar{p} - \bar{a}, \bar{u})$. Where, $\bar{u}$ is any element of Z_p and $\bar{u} \neq \bar{p} - \bar{b}$,

[Since, $(\bar{a}, b) + (\bar{p} - \bar{a}, \bar{u}) = (\bar{p}, \bar{b} + \bar{u}) = (\bar{0}, \bar{b} + \bar{u})$ is not unit element of $Z_p \times Z_p$ and not zero element for $\bar{u} \neq \bar{p} - \bar{b}$]

$$\cdots \quad (iv)$$

$(\bar{a}, \bar{b})$ is not adjacent with $(\bar{v}, \bar{p} - \bar{b})$. Where, $\bar{v}$ is any element of Z_p and $\bar{v} \neq \bar{p} - \bar{a}$,

[since $(\bar{a}, \bar{b}) + (\bar{v}, \bar{p} - \bar{b}) = (\bar{a} + \bar{v}, \bar{0})$ is not unit element of $Z_p \times Z_p$ and not zero element for $\bar{v} \neq \bar{p} - \bar{a}$]

$$\cdots \quad (v)$$

From (iv) there are $(p-1)$ number of $(\bar{p} - \bar{a}, \bar{u})$ in $Z_p \times Z_p$. Where, $\bar{u}$ is any element of Z_p and $\bar{u} \neq \bar{p} - \bar{b}$,

From (v) there are $(p-1)$ number of $(\bar{v}, \bar{p} - \bar{b})$ in $Z_p \times Z_p$.Where, $\bar{v}$ is any element of Z_p and $\bar{v} \neq \bar{p} - \bar{a}$,

And also, $(\bar{a}, \bar{b})$ is non-adjacent with itself (since, the graph is simple).

In $PG_3(Z_p \times Z_p)$ total number of non-adjacent vertices with the vertex $(\bar{a}, \bar{b})$ is $= (p-1) + (p-1) + 1 = 2p - 1$

Therefore, in $PG_3(Z_p \times Z_p)$ total number of adjacent vertices with the vertex $(\bar{a}, \bar{b})$ is

$= p^2 - (2p - 1) = (p - 1)^2$

So, degree of all units of $Z_p \times Z_p$ in $PG_3(Z_p \times Z_p)$ is $= (p - 1)^2$

By definition of Laplacian,

Laplacian of the graph $PG_3(Z_p \times Z_p)$ is

$L_{ij} = p^2 - 1 \quad (i = j = 1)$

$= (p - 1)^2 + 2 \quad (i = j \neq 1,\ u_i$ is zero divisor)

$= (p - 1)^2 \quad (i = j \neq 1,\ u_i$ is unit element)

$= -1 \quad if\ v_i = (\bar{0}, \bar{a})$ and $v_j = (\bar{b}, \bar{0})$ where $\bar{a}, \bar{b}$ are non-zero elements of Z_p

or

$v_i = (\bar{a}, \bar{0})$ and $v_j = (\bar{0}, \bar{b})$ where $\bar{a}, \bar{b}$ are non-zero elements of Z_p

or

$v_i = (\bar{a}, \bar{0})$ and $v_j = (\bar{b}, \bar{c})$ where $\bar{b}, \bar{c}$ are non-zero elements of Z_p and $\bar{b} \neq \bar{p} - \bar{a}$

or

$v_i = (\bar{0}, \bar{a})$ and $v_j = (\bar{b}, \bar{c})$ where $\bar{b}, \bar{c}$ are non-zero elements of Z_p and $\bar{c} \neq \bar{p} - \bar{a}$

or

$v_i = (\bar{a}, \bar{0})$ and $v_j = (\bar{p} - \bar{a}, \bar{0})$ where, $\bar{a}$ is non-zero element of Z_p

or

$v_i = (\bar{0}, \bar{a})$ and $v_j = (\bar{0}, \bar{p} - \bar{a})$ where, $\bar{a}$ is non-zero element of Z_p

Or

$v_i = (\bar{a}, \bar{b})$ and $v_j = (\bar{c}, \bar{d})$ where, $\bar{a}, \bar{b}, \bar{c}, \bar{d}$ are non-zero elements of Z_p and

$\bar{c} \neq \bar{p} - \bar{a}$ and $\bar{d} \neq \bar{p} - \bar{b}$

or

$v_i = (\bar{a}, \bar{b})$ and $v_j = (\bar{p} - \bar{a}, \bar{p} - \bar{b})$ where $\bar{b}, \bar{c}$ are non-zero elements of Z_p

$= 0$ otherwise

Where, p is an odd prime.

Theorem 3.5.2: The graph $PG_3(Z_p \times Z_p)$ is Eulerian, for any odd prime p.

Proof: $deg\ (\bar{0}, \bar{0}) = p^2 - 1$. For any odd prime p, $deg\ (\bar{0}, \bar{0}) = p^2 - 1$ is even. Degree of an y zero-divisor of $Z_p \times Z_p$ is $(p - 1)^2 + 2$, which is even for any odd prime p. Degree of any uni t element is $(p - 1)^2$, which is also even for any odd prime. Since $PG_3(Z_p \times Z_p)$ is connected and all vertices of $PG_3(Z_p \times Z_p)$ are of e ven degree. Therefore $PG_3(Z_p \times Z_p)$ is Eulerian, where p is any od d prime.

Theorem 3.5.3: Girth of the graph $PG_3(Z_p \times Z_p)$ is 3, Where p is any prime.

Proof: For an y odd prime p, unit ele ments $(\bar{a}, \bar{b})$, $(\bar{p} - \bar{a}, \bar{p} - \bar{b})$ and zero element $(\bar{0}, \bar{0})$ make a cycle of length 3 in $PG_3(Z_p \times Z_p)$. For $p = 2$ zero element and $(\bar{1}, \bar{0})$, $(\bar{0}, \bar{1})$ make a cycle of length 3. Therefore, girth of $PG_3(Z_p \times Z_p)$ is 3, where p is any prime.

Theorem 3.5.4 $PG_3(Z_p \times Z_p)$ is not a planar graph, where p is any odd prime.

Proof: For any odd prime p the graph $PG_3(Z_p \times Z_p)$ contains a subgraph K_5 with the vertices $(\bar{1}, \bar{0})$, $(\bar{2}, \bar{0})$, $(\bar{0}, \bar{1})$, $(\bar{0}, \bar{2})$, and $(\bar{0}, \bar{0})$, which is not a planar. Therefore, the graph $PG_3(Z_p \times Z_p)$ is not a planar graph for any odd prime p.

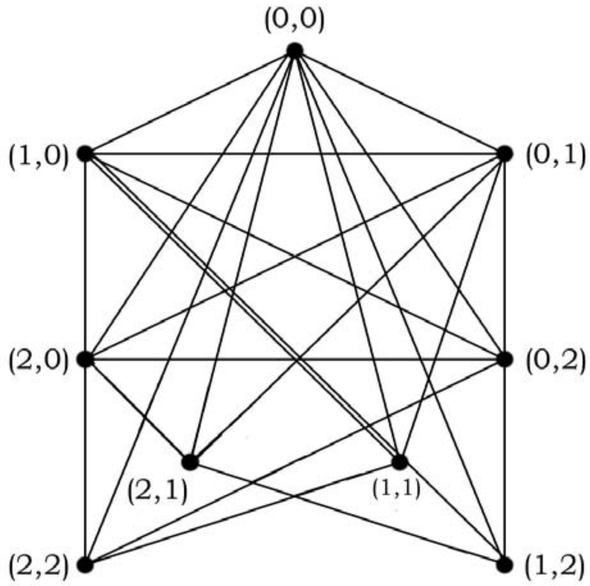


Fig. 2: $PG_3(Z_3 \times Z_3)$

3.6: Degree, planarity, girth of $PG_4(Z_n)$:

Theorem 3.6.1: Let $n = p_1^{n_1}.p_2^{n_2}.p_3^{n_3} \dots p_k^{n_k}$ where, $p_1, p_2, \cdots p_k$ are distinct prime and $n_i \in \mathbb{N}$ for $i = 1,2,\dots,k$; and a be any non-zero vertex of $PG_4(Z_n)$.

$(\boldsymbol{i})$ The degree of any vertex $a(\neq \frac{n}{2})$ of the graph $PG_4(Z_n)$ is $= \gcd(a,n) + 1$. Where; $a^2 \neq 0$ on Z_n.

$(\boldsymbol{ii})$ If $a^2 \equiv 0(mod\, n)$ then degree of $a = \gcd(a,n) - 1$.

$(\boldsymbol{iii})$ If n is even, then

The degree of $a = \frac{n}{2}$ in $PG_4(Z_n)$ is

$$= \frac{n}{2} - 1 \text{ if } a = \frac{n}{2} \text{ is even}$$

$$= \frac{n}{2} \text{ if } a = \frac{n}{2} \text{ is odd}$$

$(\boldsymbol{iv})$ If $n = a^2$ then degree of the vertex $a = a - 1$

$(\boldsymbol{v})$ If $\gcd(a,n) = 1$, then degree of a is 2.

Proof: $(\boldsymbol{i})$ Let $n = p_1^{n_1}.p_2^{n_2}.p_3^{n_3} \dots p_k^{n_k}$ where, $p_1, p_2, \cdots p_k$ are distinct primes and $n_i \in \mathbb{N}$ for $i = 1,2,\dots,k$. Also, let a $(a^2 \neq 0)$ be any vertex of $PG_4(Z_n)$ and $\gcd(a,n) = p_1^{r_1}.p_2^{r_2}.p_3^{r_3} \dots p_k^{r_k}$.

If b is multiple of $p_1^{n_1-r_1}.p_2^{n_2-r_2}.p_3^{n_3-r_3} \dots p_k^{n_k-r_k}$ then $a.b$ is multiple of n. Therefore $a.b = 0$ in Z_n. So, the vertex a is adjacent with b.

Number of b is the number of multiples of $p_1^{n_1-r_1}.p_2^{n_2-r_2}.p_3^{n_3-r_3} \dots p_k^{n_k-r_k}$ between 0 to n.

Number of multiples of $p_1^{n_1-r_1}.p_2^{n_2-r_2}.p_3^{n_3-r_3} \dots p_k^{n_k-r_k}$ between 0 to n is $= \frac{n}{p_1^{n_1-r_1}.p_2^{n_2-r_2}.p_3^{n_3-r_3} \dots p_k^{n_k-r_k}}$

$$= \frac{p_1^{n_1}.p_2^{n_2}.p_3^{n_3} \dots p_k^{n_k}}{p_1^{n_1-r_1}.p_2^{n_2-r_2}.p_3^{n_3-r_3} \dots p_k^{n_k-r_k}}$$

$$= p_1^{r_1}.p_2^{r_2}.p_3^{r_3} \dots p_k^{r_k} = \gcd(a,n).$$

Number of b which satisfies the adjacency condition $a.b = 0$ is $\gcd(a,n)$.

Also, a is adjacent with $n - a$ (using the adjacency condition $a + b = 0$).

So, the degree of a is $= \gcd(a,n) + 1$.

$(\boldsymbol{ii})$ If $a^2 \equiv 0(mod\ \mathrm{n})$ then $a.a = 0$. So, a satisfies the adjacency condition $a.b = 0$. But, the vertex a is not adjacent to itself because the graph is a simple graph. So, Number of b which satisfies the adjacency condition $a.b = 0$ is $\gcd(a,n) - 1$.

Also, a is adjacent with $n - a$ (using the adjacency condition $a + b = 0$). But, $a.(n-a) = 0$ in Z_n.

Therefore, the degree of a is $= \gcd(a,n) - 1$.

$(\boldsymbol{iii})$ If $a = \frac{n}{2}$ is even and $b = 0, 2, 4, \cdots, \frac{n}{2}, \cdots, n-2$. Then the adjacency condition $a.b = 0$ or $b.a = 0$ holds. But, $b \neq \frac{n}{2}$ (since, the graph is a simple graph). If we use the adjacency condition $a + b = 0$, then $\frac{n}{2}$ is adjacent to itself. But the graph is simple graph. So, the number of adjacent vertices with $\frac{n}{2}$ is $\frac{n}{2} - 1$.

If $a = \frac{n}{2}$ is odd and $b = 0, 2, 4, \cdots, \frac{n}{2} - 1, \frac{n}{2} + 1, \cdots, n-2$. Then the adjacency condition $a.b = 0$ or $b.a = 0$ holds. If we use the adjacency condition $a+b=0$, then $\frac{n}{2}$ is adjacent to itself. But the graph is a simple graph. So, the number of adjacent vertices with $a=\frac{n}{2}$ is $\frac{n}{2}$.

$(\boldsymbol{iv})$ Let a be any vertex of the graph such that $n = a^2$ and $\gcd(a,n) = p_1^{r_1}.p_2^{r_2}.p_3^{r_3}\dots p_k^{r_k} = a$

Since, $n=a^2$, if b is multiple of a then a is adjacent with b because, here $a.b=0$ on Z_n. Number of $b =$ Number of multiples of a in Z_n $= \frac{n}{a} = \frac{a^2}{a} = a = \gcd(a,n)$. Also, a is adjacent with $n-a$ because, $a+(n-a)=0$ on Z_n. But, $n-a = a^2 - a = a(a-1)$ which is multiple of a. since the graph is a simple graph. So, the degree of a is $= \gcd(a,n) = a$.

$\boldsymbol{v}$) If $\gcd(a,n)=1$, then a is unit element of Z_n, which is adjacent with $n-a$ and zero element. Therefore, the degree of a is 2.

Theorem 3.6.2: The girth of the graph $PG_4(Z_n)$ is,

$girth\ PG_4(Z_n) = \infty$ if $n = 2,3,4,5$ or n is a prime

$= 3$ otherwise

Proof: The graph $PG_4(Z_2)$ is a complete graph K_2, So girth is infinite. $PG_4(Z_3)$, $PG_4(Z_4)$ are union of two copies of K_2 and union of three copies of K_2 with common vertex zero respectively. So, the girth of $PG_4(Z_3)$, $PG_4(Z_4)$ is infinite. $PG_4(Z_5)$ is a union of four copies of K_2 with common vertex zero. Therefore, the girth of $PG_4(Z_5)$ is infinite.

If $n>5$ and n is not prime then in the graph always exists a cycle of length three with the zero vertex and two non-zero zero divisors of Z_n which are adjacent. Therefore, the girth is 3.

Theorem 3.6.3: The graph $PG_4(Z_n)$ is Eulerian if and only if n is odd.

Proof: If n is odd then the degree of zero vertex of the graph $= n-1$ which is an even, and the degree of any non-zero vertex is either $\gcd(a,n)+1$ or $\gcd(a,n)-1$. Since, n is odd, $\gcd(a,n)$ is odd. So, $\gcd(a,n)+1$ and $\gcd(a,n)-1$ are even. Therefore, the graph $PG_4(Z_n)$ is Eulerian.

If n is even the degree of zero vertex of the graph $= n-1$ which is odd. So, the graph is not Eulerian if n is even.

Hence, the graph $PG_4(Z_n)$ is Eulerian if and only if n is odd.

Theorem 3.6.4: The graph $PG_4(Z_{pq})$ is not planar if $p,q>3$. Where, p,q are two distinct primes.

Proof: Zero-divisors of $PG_4(Z_{pq})$ are multiples of p and multiples of q. Set of non-zero zero-divisors form a complete bipartite graph $K_{p-1,q-1}$, whose one set of vertices is a set of multiples of p and the other is a set of multiples of q. In $PG_4(Z_{pq})$ there are $(q-1)$ number of multiples of p and $(p-1)$ number of multiples of q. So, if $p,q>3$ then the graph has a subgraph $K_{3,3}$ which is not planar. Therefore, the graph $PG_4(Z_{pq})$ is not a planar graph, if $p,q>3$.

Theorem 3.6.5: The graph $PG_4(Z_{p^2})$ is not planar if $p \geq 5$.

Proof: In the graph $PG_4(Z_{p^2})$ any zero-divisor is multiple of p. If a,b two distinct zero-divisors then $a.b$ has a factor p^2. So, $a.b=0$ in Z_{p^2}. Therefore, any two zero-divisors are adjacent in the graph. For $p \geq 5$, the number of zero-divisors is ≥ 5 and zero-divisors form a complete graph. So, if $p \geq 5$ the graph has a complete subgraph K_5. Hence the result.

3.7. Laplacian of $\boldsymbol{PG_4(\mathbb{Z}_p)}$

Theorem 3.7.1: Laplacian of $PG_4(Z_p)$ is

$L_{ij} = p-1 \quad (i=j=1)$

$= 2 \quad (i=j\neq 1)$

$= -1 \quad if\ v_i = \overline{0}$ and $v_j = \overline{a}$,

or

$if\ \ v_i = \ \bar{a}$ and $v_j = \bar{0}$,

or

$if\ \ v_i = \ \bar{a}$ and $v_j = \overline{p-a}$

or

$if\ \ v_i = \ \overline{p-a}$ and $v_j = \bar{a}$

$= \bar{0}$ Otherwise

where $\bar{a}$ is any non-zero vertex of $PG_4(Z_p)$ and p is any odd prime.

Proof: $\bar{0}, \bar{1}, \bar{2}, \cdots \overline{p-1}$ are vertices of $PG_4(\mathbb{Z}_p)$. Here the vertex $\bar{0}$ is adjacent with all other vertices.

Therefore, $deg\ (\bar{0}) = p-1$.

If $\bar{a}$ is non-zero element of $\mathbb{Z}_p$, then $\bar{a}$ is adjacent with $\bar{0}$ and $\overline{p-a}$ [since $\bar{a} + \overline{p-a} = \bar{0}$, using adjacency condition of the graph]

Therefore, the degree of any non-zero vertex $\bar{a}$ is 2

By definition of Laplacian

$L_{ij} = \ p-1 \ \ (i = j = 1)$

$= 2 \quad (i = j \neq 1)$

$= -1 \quad if\ \ v_i = \ \bar{0}$ and $v_j = \bar{a}$

Or

$if\ \ v_i = \ \bar{a}$ and $v_j = \bar{0}$

or

$if\ \ v_i = \ \bar{a}$ and $v_j = \overline{p-a}$

Or

$if\ \ v_i = \ \overline{p-a}$ and $v_j = \bar{a}$

$= \bar{0}$ Otherwise

Where, $\bar{a}$ is any non-zero vertex of $PG_4(Z_p)$ and p is any odd prime.

Theorem 3.7.2: If $G = PG_4(Z_p)$ where p is a prime (> 3)then $a(G) \leq 1$.

Proof: $\bar{0}, \bar{1}, \bar{2}, \cdots \overline{p-1}$ are vertices of $PG_4(\mathbb{Z}_p)$. The graph $PG_4(\mathbb{Z}_2)$ and $PG_4(\mathbb{Z}_3)$ are complete graphs K_2 and K_3 respectively. For $p > 3$ the graph $PG_4(Z_p)$ is union of $\frac{p-1}{2}$ copies of K_3 in which zero vertex is co mmon. For deletion of zero vertex the graph will be disconnected. So, vertex connectivit y $v(G) = 1$. We kn ow that if $G \neq K_n$ then $a(G) \leq v(G)$. Therefore, $a(G) \leq 1$.

Example:

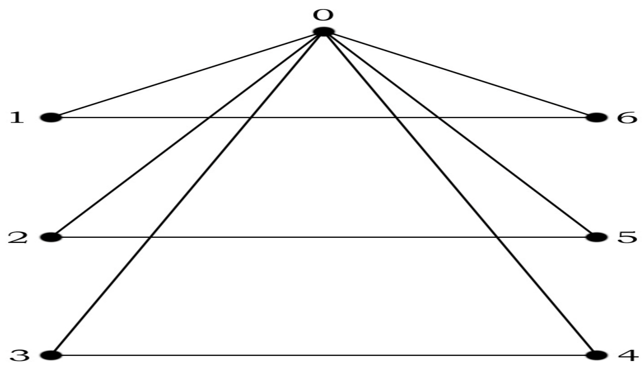


Fig. 3: $PG_4(Z_7)$

3.8. Laplacian of $PG_4(\mathbb{Z}_p \times \mathbb{Z}_p)$

Theorem 3.8.1: Laplacian of $PG_4(Z_p \times Z_p)$ is

$L_{ij} = \ p^2 - 1 \ \ (i = j = 1)$

$= p + 1 \quad (i = j \neq 1,\ u_i$ is zero divisor)

$= \ 2 \ \ (i = j \neq 1,\ u_i$ is unit element)

$= -1 \quad if\ \ v_i = \ (\bar{0}, \bar{a}\,)$ and $v_j = (\bar{b}, \bar{0}\,)$
where $\bar{a}$ is non-zero and $\bar{b}$ is any element of Z_p

Or

$v_i = \ (\bar{a}, \bar{0}\,)$ and $v_j = (\bar{0}, \bar{b}\,)$
where $\bar{a}$ is non-zero and $\bar{b}$ is any element of Z_p

or

$v_i = (\bar{a},\bar{0})$ and $v_j = (\bar{p}-\bar{a},\bar{0})$
where $\bar{a}$ is non-zero element of Z_p

or

$v_i = (\bar{0},\bar{a})$ and $v_j = (\bar{0},\bar{p}-\bar{a})$
where $\bar{a}$ is non-zero element of Z_p

or

$v_i = (\bar{a},\bar{b})$ and $v_j = (\bar{p}-\bar{a},\bar{p}-\bar{b})$

$= 0$ otherwise

Proof: All the elements of $Z_p \times Z_p$ are considered as vertices of the graph $PG_4(Z_p \times Z_p)$. Zero divisors of $Z_p \times Z_p$ are of the form $(\bar{a},\bar{0})$ and $(\bar{0},\bar{b})$, Where $\bar{a},\bar{b}$ are elements of Z_p .

Units of $Z_p \times Z_p$ are of the form $(\bar{a},\bar{b})$ where $\bar{a},\bar{b}$ are non-zero elements of Z_p.

$(\bar{0},\bar{0}).(\bar{a},\bar{b}) = (\bar{0},\bar{0})$ $\forall\, \bar{a},\bar{b} \in Z_p$ [from 1st adjacency condition of $PG_4(R)$]

Clearly, zero ele ment $(\bar{0},\bar{0})$ is adjacent wi th all other vertices

$\therefore deg\, (\bar{0},\bar{0}) = p^2 - 1$

Now for $(\bar{a},\bar{0}) \in Z_p \times Z_p$, where $\bar{a}$ is any non-zero element of Z_p.

$(\bar{a},\bar{0}) \sim (\bar{0},\bar{b})$ where $\bar{b}$ is any element of Z_p

[using adjacency condition of $PG_4(R)$

$(\bar{a},\bar{0}).(\bar{0},\bar{b}) = (\bar{0},\bar{0})$]

There are p number of $\bar{b}$ in Z_p.

$(\bar{a},\bar{0}) \sim (\bar{p}-\bar{a},\bar{0})$ where $\bar{a}$ is any non-zero element of Z_p.

[using 2nd adjacency condition of $PG_4(R)$

$(\bar{a},\bar{0}) + (\bar{p}-\bar{a},\bar{0}) = (\bar{0},\bar{0})$]

$\therefore$ Number of adjacent ve rtices with t he vertex $(\bar{a},\bar{0})$ is $= p+1$

Degree of $(\bar{a},\bar{0})$ in $PG_4(Z_p \times Z_p)$ is $= p+1$

Similarly, Degree of $(\bar{0},\bar{a})$ in $PG_4(Z_p \times Z_p)$ is $= p+1$

Therefore, Degree of all non-zero zero divisors of $Z_p \times Z_p$ in $PG_4(Z_p \times Z_p)$ is $= 2$

Now for any unit element $(\bar{a},\bar{b})$ of $Z_p \times Z_p$ where, $\bar{a},\bar{b}$ are non-zero elements of Z_p.

$(\bar{a},\bar{b}) \sim (\bar{0},\bar{0})$

$(\bar{a},\bar{b}) \sim (\bar{p}-\bar{a},\bar{p}-\bar{b})$

[Since, $(\bar{a},b) + (\bar{p}-\bar{a},\bar{p}-\bar{b}) = (\bar{0},\bar{0})$]

$\therefore$ The num ber of adjacent vertices with the vertex $(\bar{a},\bar{b})$ is $= 1+1 = 2$

The degree of $(\bar{a},\bar{b})$ in $PG_4(Z_p \times Z_p)$ is $= 2$

Degree of all units of $Z_p \times Z_p$ in $PG_4(Z_p \times Z_p)$ is $= 2$

By definition of Laplacian

$L_{ij} = p^2 - 1 \quad (i = j = 1)$

$= p+1 \quad (i = j \neq 1,\ u_i$ is non-zero zero divisor)

$= 2 \quad (i = j \neq 1,\ u_i$ is unit element)

$= -1 \quad if\ v_i = (\bar{0},\bar{a})$ and $v_j = (\bar{b},\bar{0})$
where $\bar{a}$ is non-zero and $\bar{b}$ is any element of Z_p

Or

$v_i = (\bar{a},\bar{0})$ and $v_j = (\bar{0},\bar{b})$
where $\bar{a}$ is non-zero and $\bar{b}$ is any element of Z_p

or

$v_i = (\bar{a}, \bar{0})$ and $v_j = (\bar{p} - \bar{a}, \bar{0})$ where $\bar{a}$ is non-zero element of Z_p

or

$v_i = (\bar{0}, \bar{a})$ and $v_j = (\bar{0}, \bar{p} - \bar{a})$ where $\bar{a}$ is non-zero element of Z_p

or

$v_i = (\bar{a}, \bar{b})$ and $v_j = (\bar{p} - \bar{a}, \bar{p} - \bar{b})$ where $\bar{a}$, $\bar{b}$ are non-zero elements of Z_p

$= 0$ otherwise

Theorem 3.8.2: $PG_4(Z_p \times Z_p)$ is Eulerian, where p is any odd prime.

Proof: $deg\,(\bar{0}, \bar{0}) = p^2 - 1$, for any odd prime $p^2 - 1$ is even. Degree of any zero-divisor of $Z_p \times Z_p$ is $p + 1$, which is even for any odd prime. Degree of any unit element is 2, which is even. Since $PG_4(Z_p \times Z_p)$ is connected and all vertices of $PG_3(Z_p \times Z_p)$ are of even degree. Therefore $PG_3(Z_p \times Z_p)$ is Eulerian, where p is any odd prime.

Theorem 3.8.3: Girth of $PG_4(Z_p \times Z_p)$ is 3, Where p is any prime.

Proof: For any odd prime, zero element and unit elements $(\bar{a}, \bar{b})$ and $(\bar{p} - \bar{a}, \bar{p} - \bar{b})$ make a cycle of length 3 in $PG_4(Z_p \times Z_p)$. For $p = 2$, zero element and $(\bar{1}, \bar{0})$, $(\bar{0}, \bar{1})$ makes a cycle of length 3. Therefore, girth of $PG_4(Z_p \times Z_p)$ is 3, where p is any prime.

Theorem 3.8.4: $PG_4(Z_p \times Z_p)$ is not a planar graph, where p is any odd prime.

Proof: The graph $PG_4(Z_2 \times Z_2)$ is union of K_3 and K_2 with a common vertex $(\bar{0}, \bar{0})$.So the graph is a planar for $p = 2$. If p is any odd prime then the graph $PG_4(Z_p \times Z_p)$ contains a sub graph K_5 with the vertices $(\bar{a}, \bar{0})$, $(\bar{0}, \bar{a})$, $(\overline{p - a}, \bar{0})$, $(\bar{0}, \overline{p - a})$, and $(\bar{0}, \bar{0})$, which is not a planar. Therefore $PG_4(Z_p \times Z_p)$ is not a planar graph for any odd prime p.

Theorem 3.8.5: Clique number of $PG_4(Z_p \times Z_p)$ is 5, Where p is any odd prime.

Proof: For any zero divisor $(\bar{a}, \bar{0})$ there exists a complete subgraph K_5 with vertices $(\bar{a}, \bar{0})$, $(\bar{0}, \bar{b})$, $(\overline{p - a}, \bar{0})$, $(\bar{0}, \overline{p - b})$, and $(\bar{0}, \bar{0})$. Similarly, for any zero divisor of the form $(\bar{0}, \bar{a})$ there exists a complete subgraph K_5 with vertices $(\bar{a}, \bar{0})$, $(\bar{0}, \bar{b})$, $(\overline{p - a}, \bar{0})$, $(\bar{0}, \overline{p - b})$, and $(\bar{0}, \bar{0})$. For any unit element $(\bar{a}, \bar{b})$ there exists a complete graph K_3 with vertices $(\bar{a}, \bar{b})$, $(\bar{0}, \bar{0})$, $(\overline{p - a}, \overline{p - b})$. Therefore, K_5 is the maximal complete subgraph in $PG_4(Z_p \times Z_p)$. Hence, Clique number of $PG_4(Z_p \times Z_p)$ is 5, Where p is any odd prime.

Example:

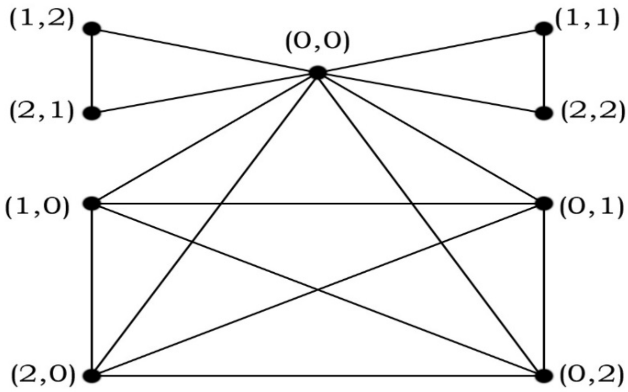


Fig. 4: $PG_4(Z_3 \times Z_3)$

4 Conclusion

The graph $PG_3(Z_n)$ is planar only for $n = 2, 3, 4$ and 6. Any zero-divisors of the graph is connected with every vertex of the graph except itself. The degree of the unit element of the graph $PG_3(Z_{p^2})$ is $\phi(p^2)$. The graph $PG_3(Z_{p^2})$, $PG_3(Z_p)$ and $PG_3(Z_p \times Z_p)$ are Eulerian for any odd prime p. On the other hand; the graph $PG_4(Z_n)$ is Eulerian if and only if n is odd. The graph $PG_4(Z_p \times Z_p)$ is also Eulerian for any odd prime p. Algebraic connectivity of the graph $PG_3(Z_p)$ is p but the algebraic connectivity of the $PG_4(Z_p)$ is less than or equal to 1. The graphs $PG_3(Z_p \times Z_p)$, $PG_4(Z_p \times Z_p)$ are not planar for any odd prime p. In the graph $PG_4(Z_n)$ there does not exist any cycle for $n = 2, 3, 4$ and 5 or n is prime.

5 Future Scope

We can work on the graphs $PG_3(R)$, $PG_4(R)$ for any non-commutative finite ring R. Chromatic number and dominating number of the graphs can be found out. Also, we can study Laplacian energy of the graphs $PG_3(R)$, $PG_4(R)$.

Acknowledgements:
We gratefully thank the referees for their suggestions and valuable comments. This work is financially supported by the Council of Scientific and Industrial Research (File no: 09/059(0068)/2019-EMR-I.

References:

[1] B. Mohar, *Laplace eigenvalues of graphs-a survey*, Discrete Mathematics, 109 (1992) 171-183.

[2] Chris Godsil, Gordon Royle, *Algebraic Graph Theory,* Springer-Verlag, Newyork Inc (2001).

[3] David S. Dummit, Richard M. Foote, *Abstract Algebra*, Second Edition. John Wiley & sons, Inc (1999).

[4] Kishor F. Pawar and Sandeep S. Joshi, *The Prime Graph $PG_1(R)$ of a Ring*, Palestine Journal of Mathematics, Vol. 6(1) (2017), 153-158.

[5] M. Fiedler, *Algebra connectivity of graphs*, Czechoslovak Mathematical Journal, 23(98) (1973) 298-305.

[6] M. Fiedler, *A property of eigenvectors of nonnegative symmetric matrices and its application to graph theory*, Czechoslovak Mathematical Journal, 25(98) (1975) 607-618.

[7] N. Alon, *Eigenvalues of expanders*, Combinatorica 6 (1986) 83-96.

[8] Nechirvan B. Ibrahim , *On the coneighbor eigenvalues and co-neighbour energy of a graph*, Gen. Lett. Math., 9(2) (2020), 67-73.

[9] R.B. Bapat, *The Laplacian matrix of a graph*, Math. Student 65 (1996) 214–223.

[10] R. Merris, *Laplacian matrices of graphs: a survey*, Linear Algebra and its Applications, 197/198 (1994) 143-176.

[11] R. Merris, *A survey of graph Laplacians*, Linear and Multilinear Algebra, 39(1995) 19-31.

[12] R. Sen Gupta, *The Graph $\Gamma_2(R)$ over a ring R*, Int. J. of Pure & Appl. Math. 86(6), 893-904 (2013).

[13] Robert Grone, *On the Geometric and Laplacian of a Graph*, Linear Algebra and its Applications, 150(1991) 167-178.

[14] Satyanarayana Bhavanari, Syam Prasad Kuncham and Nagaraju Dasari, *Prime Graph of a Ring*, Journal of Combinatorics, Information and system sciences, Vol. 35 (2010).

[15] Sandeep S. Joshi and Kishor F. Pawar, *On Prime Graph $PG_2(R)$ of a Ring*, International Journal of Mathematical

[16] K. J. Barman and K. Patra, *Line graph associative to order congruence graph of the commutative ring Z_p, p is prime.* Advances and applications in Discrete Mathematics. Volume 27, pp 284-294.

[17] Pinku Sarkar and Kuntala Patra, *study of some graphical parameters of some graph structure*, Advances and Applications in Discrete Mathematics, 32 (2022), 63-89.

Contribution of Individual Authors to the Creation of a Scientific Article (Ghostwriting Policy)

The authors equally contributed in the present research, at all stages from the formulation of the problem to the final findings and solution.

Sources of Funding for Research Presented in a Scientific Article or Scientific Article Itself

This work is financially supported by the Council of Scientific and Industrial Research (File no: 09/059(0068) /2019-EMR-I.

Conflict of Interest

The authors have no conflicts of interest to declare that are relevant to the content of this article.