A New Two-Parameter Lifetime Model with Statistical Properties and Applications

NAJWAN ALSADAT
Department of Quantitative Analysis, College of Business Administration, King Saud University, P.O. Box 2455, Riyadh, 11451,
SAUDI ARABIA

Abstract: - A new lifetime distribution called the truncated Cauchy power length-biased exponential (TCP-LBEX) distribution that extends the length-biased (LBEX) model is investigated. The statistical properties of the TCP-LBEX model including the quantile function, incomplete moment, moment, and entropy are derived. The method of maximum likelihood estimation was used to estimate the parameters of the TCP-LBEX. Monto Carlo simulations are used to assess the behavior of parameters. Finally, we demonstrate applications of two real-world data sets to show the flexibility and potentiality of the proposed model.

Key-Words: - Truncated Cauchy power family; Length-biased exponential; moments; Rényi Entropy; Maximum likelihood estimation.

Received: July 20, 2022. Revised: January 19, 2023. Accepted: February 15, 2023. Published: March 29, 2023.

1 Introduction

Some scientists have recently proposed strategies for incorporating probability models. This parameter addition phenomenon generates more classes of distributions, which are useful for modeling datasets in engineering science, biological science, economics, medicine, income, physics, and environmental sciences. Several G-class of distributions are the T-X class, [1], the odd Fréchet-G class, [2], Kumaraswamy-G class, [3], odd Dagum-G class, [4], weighted exponential-G class, [5], alpha power class, [6], weighted exponentiated-G class, [7], Marshall-Olkin alpha-power-G class, [8], truncated inverted Kumaraswamy-G class, [9], transmuted geometric-G class, [10], complementary generalized transmuted Poisson-G class, [11], the exponentiated odd log-logistic-G class, [12], Type-II half logistic-G class, [13], Topp-Leone-G class, [14], truncated Cauchy power Weibull-G class, [15], Lomax-G class, [16], Type-I half logistic Burr X-G class, [17], type I general exponential-G class, [18], sine Topp-Leone-G class, [19], generalized odd Weibull class, [20], a new power Topp-Leone-G class, [21], odd power Lindley-G class, [22], the transmuted transmuted-G class, [23], exponentiated version of the M class, [24], the transmuted Gompertz-G class, [25], transmuted odd Fréchet-G class, [26], transmuted odd Lindley-G class, [27], Topp Leone odd Lindley-G class, [28], Topp-Leone odd log-logistic class, [29], odd Perks-G class, [30] and Kumaraswamy transmuted-G class, [31], among others.

Recently, [32], discussed the TCP-G class of distributions. The distribution function (cdf) of the TCP-G class is

$$F(x;\ \mu,\omega)=\frac{4}{\pi}arctan(\,G(x;\ \omega)^{\mu}),x\in R, \quad (1)$$

where $\mu>0$ and the cdf of a baseline with parameter vector ω is denoted by $G(x;\ \omega)$. The corresponding probability density function (pdf) and hazard rate function (hrf) respectively are

$$f(x;\ \mu,\omega)=\frac{4\,\mu g(x;\ \omega)G(x;\ \omega)^{\mu-1}}{\pi[1+(G(x))^{2\,\mu}]}, \quad (2)$$

and

$$h(x;\ \mu,\omega)=\frac{f\ (x;\ \mu,\omega)}{\overline{F}(x;\ \mu,\omega)}$$
$$=\frac{4\,\mu g(x;\ \omega)G(x;\ \omega)^{\mu-1}}{\pi\,\eta^{2}[1+G(x;\ \omega)^{2\,\mu}]\left[1-\frac{4}{\pi}arctan\,G\,(x;\ \omega)^{\mu}\right]}. \quad (3)$$

Relevant research has been provided depending on the TCP-G family, for example, TCP-inverted Topp–Leone model, [33], TCP-inverse exponential model, [34], TCP-Lomax model, [35], TCP-

Weibull-G class, [36], and TCP odd Frechet-G class, [37].

[38], proposed the LBEX by assigning a weight to the exponential (E) model using the concept proposed by [39]. They investigated that the LBEX model is more flexible than the E model. The cdf and pdf of LBEX distribution are provided below

$$G(x;\ \eta) = 1 - \left(1 + \frac{x}{\eta}\right) e^{-\frac{x}{\eta}}, x > 0 \qquad (4)$$

and

$$g(x, \eta) = \frac{x}{\eta^2} e^{-\frac{x}{\eta}}, \qquad (5)$$

where $\eta > 0$ is a scale parameter. An extension of the LBEX model is proposed, and this extension is constructed utilizing the TCP-G class and LBEX model. This extension is termed the TCP-LBEX distribution.

The remainder of this article is outlined as follows. In Section 2, the cdf and pdf of the proposed distribution are presented and an expansion of the TCP-LBEX pdf is derived. The basic properties of the distribution, including the quantile function, moments, incomplete and conditional moments, and entropy, are presented in Section 3. In Section 4, the parameter estimation employing the maximum likelihood estimation (MLE) approach is discussed and Monto Carlo simulations are utilized to study the behavior of the parameters. In Section 5, the TCP-LBEX model is performed on two real-world data sets to examine its feasibility using some information criterion (INC) of the goodness of fit, like; the Akaike INC (K1), Bayesian INC (K2), consistent Akaike INC (K3), Hannan–Quinn INC (K4), Cramèr–Von Mises (K5), Anderson–Darling (K6), Kolmogorov–Smirnov (K7) statistics and p-value (K8). Lastly, Section 6 presents the conclusions.

2 The New TCP-LBEX Model

By substituting (4) in (2), the TCP-LBEX cdf of random variable X is obtained as

$$F(x;\ \mu, \eta) = \frac{4}{\pi} arctan \left[1 - \left(1 + \frac{x}{\eta}\right) e^{-\frac{x}{\eta}}\right]^{\mu}. \quad (6)$$

The corresponding pdf is

$$f\ (x;\ \mu, \eta) = \frac{4\ \mu x e^{-\frac{x}{\eta}} \left[1 - \left(1 + \frac{x}{\eta}\right) e^{-\frac{x}{\eta}}\right]^{\mu - 1}}{\pi\, \eta^2 \left[1 + \left[1 - \left(1 + \frac{x}{\eta}\right) e^{-\frac{x}{\eta}}\right]^{2\,\mu}\right]}. \qquad (7)$$

The reliability function (sf) of the TCP-LBEX distribution is provided below

$$\overline{F}(x;\ \mu, \eta) = 1 - \frac{4}{\pi} arctan \left[1 - \left(1 + \frac{x}{\eta}\right) e^{-\frac{x}{\eta}}\right]^{\mu}.$$

For the TCP-LBEX distribution, the hazard rate function (hrf) is expressed as follows:

$$h(x;\ \mu, \eta) = \frac{4\ \mu x e^{-\frac{x}{\eta}} \left[1 - \left(1 + \frac{x}{\eta}\right) e^{-\frac{x}{\eta}}\right]^{\mu - 1}}{\pi\, \eta^2 \left[1 + \left[1 - \left(1 + \frac{x}{\eta}\right) e^{-\frac{x}{\eta}}\right]^{2\,\mu}\right] \left[1 - \frac{4}{\pi} arctan \left[1 - \left(1 + \frac{x}{\eta}\right) e^{-\frac{x}{\eta}}\right]^{\mu}\right]}.$$

The reversed hazard rate (RHR) function of the TCP-LBEX distribution is provided below

$$\tau(x;\ \mu, \eta) = \frac{\mu x e^{-\frac{x}{\eta}} \left[1 - \left(1 + \frac{x}{\eta}\right) e^{-\frac{x}{\eta}}\right]^{\mu - 1}}{\eta^2 \left[1 + \left[1 - \left(1 + \frac{x}{\eta}\right) e^{-\frac{x}{\eta}}\right]^{2\,\mu}\right] \left\{arctan \left[1 - \left(1 + \frac{x}{\eta}\right) e^{-\frac{x}{\eta}}\right]^{\mu}\right\}}.$$

The cumulative hazard rate (CHR) function of the TCP-LBEX distribution is provided below

$$H(x;\ \mu, \eta) = -\ln \left[1 - \frac{4}{\pi} arctan \left[1 - \left(1 + \frac{x}{\eta}\right) e^{-\frac{x}{\eta}}\right]^{\mu}\right].$$

Figure 1 presents the pdf of the TCP-LBEX model.

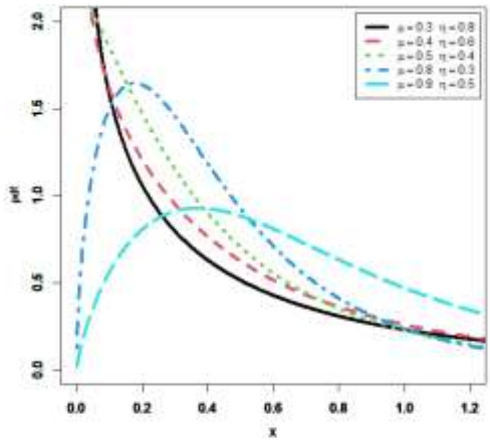

Fig. 1: cdf of the TCP-LBEX model.

Figure 2 presents the cdf of the TCP-LBEX model.

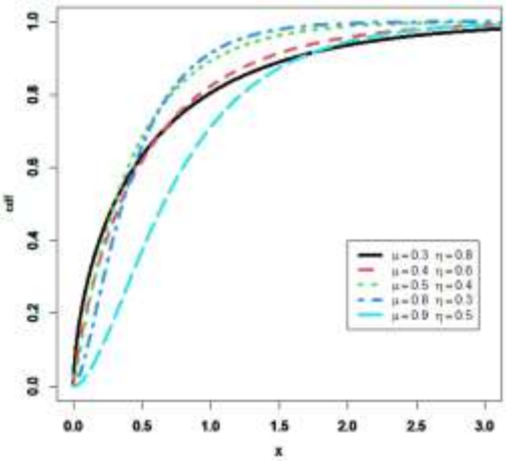

Fig. 2: cdf of the TCP-LBEX model.

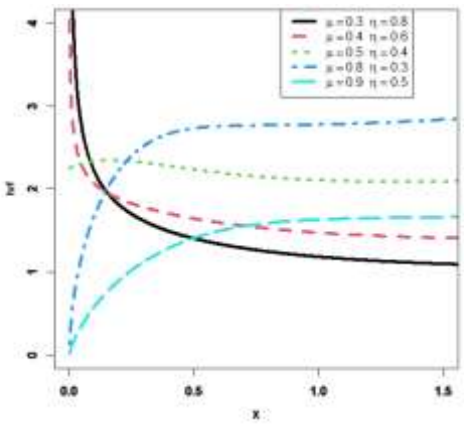

Fig. 3: hrf of the TCP-LBEX model.

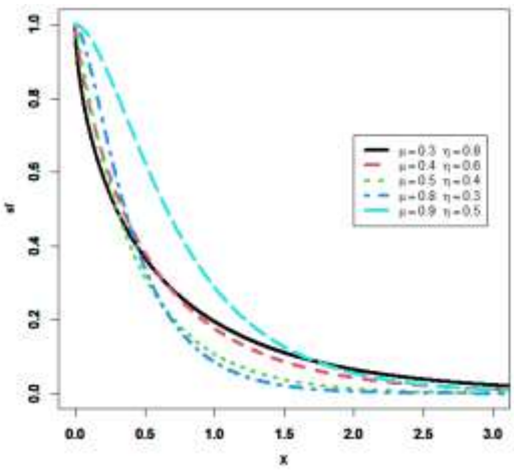

Fig. 4: sf of the TCP-LBEX model.

3 Statistical Properties of the TCP-LBEX Model

In this section, the statistical properties of the TCP-LBEX model, such as the useful expansion, quantiles, moments, generating functions, incomplete moments, and entropy, are discussed.

3.1 Useful expansion

In this subsection, the expansion of the TCP-LBEX pdf is established. Using the binomial series expansion,

$$(1+x)^{-d} = \sum_{i=0}^{\infty} (-1)^i \binom{d+i-1}{i} x^i, \qquad (8)$$

which holds for $|x| < 1$, and d is a positive real non-integer. Further, the following relation is used:

$$(1-x)^{o-1} = \sum_{j=0}^{\infty} (-1)^j \binom{o-1}{j} x^j, \quad (9)$$

where o is any positive real noninteger. By substituting (8) in (7), the TCP-LBEX pdf becomes

$$f\,(x;\, \mu, \eta) = \frac{4\,\mu x e^{-\frac{x}{\eta}}}{\pi\,\eta^2} \sum_{i=0}^{\infty} (-1)^i \left[1 - \left(1+\frac{x}{\eta}\right) e^{-\frac{x}{\eta}}\right]^{\mu(2i+1)-1}. \quad (10)$$

Further, by substituting (9) in (10) and performing some algebraic manipulations, the TCP-LBEX pdf can be written as

$$f\,(x;\, \mu, \eta) = \sum_{k=0}^{\infty} \varpi_k\, x^{k+1} e^{-(j+1)\frac{x}{\eta}}, \qquad (11)$$

where
$\varpi_k = \frac{4\,\mu}{\pi\,\eta^{2+k}} \sum_{i,j=0}^{\infty} (-1)^{i+j} \binom{\mu(2i+1)-}{j} \binom{j}{k}.$

3.2 Quantiles

Quantiles are fundamental for estimating and simulating the distribution parameters. The quantile function of X can be obtained by inverting (6) as

$$\left(1+\frac{x}{\eta}\right)e^{-\frac{x}{\eta}} + \left(tan\frac{u\pi}{4}\right)^{\frac{1}{\mu}} - 1 = 0, 1 < u < 0. \qquad (12)$$

We cannot have closed form for Equation (12) but we can solve it numerically.

3.3 Moments

For X with a pdf (11), the r_{th} moment of X is

$$\mu_r'(x) = \int_0^{\infty} x^r f(x)dx = \sum_{k=0}^{\infty} \varpi_k \int_0^{\infty} x^{r+k+1} e^{-(j+1)\frac{x}{\eta}} dx.$$

By substituting $y = (j+1)\frac{x}{\eta}$ and some algebraic manipulations, the r_{th} moment achieves the following form:

$$\mu_r'(x) = \sum_{k=0}^{\infty} \varpi_k \left(\frac{\eta}{j+1}\right)^{r+k+2} \Gamma(r+k+2). \quad (13)$$

The mg function $M_X(t)$ of X can be derived from (11) as follows:

$$M_X(t) = E(e^{tX}) = \int_0^{\infty} e^{tx} f\,(x)dx = \sum_{k=0}^{\infty} \varpi_k \int_0^{\infty} x^{k+1} e^{-\left[\frac{j+1}{\eta}-t\right]x} dx$$
$$= \sum_{k=0}^{\infty} \varpi_k \frac{\Gamma(k+2)}{\left(\frac{j+1}{\eta}-t\right)^{k+2}}. \qquad (14)$$

The numerical values of specific parameters of the first four ordinary moments variance (σ^2), skewness (S), kurtosis (K), and coefficient of variation (CV) of the TCP-LBEX model are mentioned in Table 1.

From the data presented in Table 1, as the value of μ increases, the values of moments decrease for constant η. In contrast, the values of σ^2, S, K, and CV decrease.

3.4 Conditional Moments

The s_{th} upper incomplete moment of the TCP-LBEX distribution can be given by

$$\Phi_s(t) = \int_t^{\infty} x^s f(x)dx = \sum_{k=0}^{\infty} \varpi_k \int_t^{\infty} x^{s+k+1} e^{-(j+1)\frac{x}{\eta}} dx$$
$$= \sum_{k=0}^{\infty} \varpi_k \left(\frac{\eta}{j+1}\right)^{s+k+2} \Gamma\left(s+k+2, (j+1)\frac{t}{\eta}\right), \qquad (15)$$

where $\Gamma(s,t) = \int_t^{\infty} x^{s-1} e^{-x} dx$ is the upper incomplete gamma function. Similarly, the s_{th} lower incomplete moment of the distribution is expressed as

$$\Lambda_s(t) = \int_0^{t} x^s f\,(x)dx = \sum_{k=0}^{\infty} \varpi_k \int_0^{t} x^{s+k+1} e^{-(j+1)\frac{x}{\eta}} dx$$
$$= \sum_{k=0}^{\infty} \varpi_k \left(\frac{\eta}{j+1}\right)^{s+k+2} \gamma\left(s+k+2, (j+1)\frac{t}{\eta}\right), \qquad (16)$$

where $\gamma(s,t) = \int_0^{t} x^{s-1} e^{-x} dx$ is the lower incomplete gamma function.

Table 1. Some numerical results of moments for the TCP-LBEX model at η = 0.5

μ	μ_1'	μ_2'	μ_3'	μ_4'	$\boldsymbol{\sigma^2}$	S	K	CV
0.5	1.032	9.481	2.106	0.327	2.579	1.106	0.633	0.554
1.0	0.739	7.197	1.646	0.409	5.101	2.143	1.159	0.866
1.5	0.617	6.53	1.479	0.441	7.545	3.104	1.599	1.076
2.0	0.548	6.228	1.393	0.457	9.908	3.997	1.978	1.233
2.5	0.502	6.061	1.342	0.466	12.195	4.832	2.312	1.359
3.	0.4	5.9	1.3	0.4	14.	5.6	2.6	1.4

0	69	59	09	7	408	17	1	63
3.5	0.443	5.891	1.286	0.473	16.555	6.359	2.881	1.552
4.0	0.423	5.843	1.269	0.475	18.64	7.063	3.13	1.629
4.5	0.406	5.809	1.256	0.476	20.668	7.734	3.359	1.698
5.0	0.392	5.783	1.246	0.477	22.642	8.375	3.573	1.76
5.5	0.38	5.764	1.238	0.477	24.568	8.99	3.773	1.816
6.0	0.37	5.749	1.232	0.477	26.447	9.581	3.962	1.867
6.5	0.361	5.737	1.226	0.477	28.282	10.15	4.14	1.914
7.0	0.353	5.727	1.222	0.476	30.078	10.7	4.308	1.958
7.5	0.345	5.72	1.219	0.476	31.836	11.231	4.469	1.998
8.0	0.339	5.714	1.216	0.476	33.558	11.746	4.622	2.036
8.5	0.333	5.709	1.213	0.475	35.246	12.245	4.768	2.072
9.0	0.327	5.705	1.211	0.475	36.902	12.729	4.908	2.106
9.5	0.322	5.702	1.209	0.474	38.528	13.201	5.043	2.137
10.0	0.317	5.7	1.207	0.474	40.126	13.66	5.173	2.168
10.5	0.313	5.698	1.206	0.473	41.696	14.107	5.298	2.196
11.0	0.309	5.696	1.205	0.473	43.24	14.543	5.418	2.224
11.5	0.305	5.695	1.204	0.472	44.759	14.969	5.535	2.25
12.0	0.302	5.695	1.203	0.472	46.254	15.385	5.648	2.275
12.5	0.299	5.694	1.202	0.471	47.727	15.792	5.757	2.299
13.0	0.295	5.694	1.202	0.471	49.178	16.19	5.863	2.322
13.5	0.293	5.694	1.201	0.47	50.608	16.58	5.967	2.344
14.0	0.29	5.694	1.2	0.47	52.019	16.962	6.067	2.366

3.5 Entropy

The Rényi entropy is defined using ($\nu > 0, \nu \neq 1$):

$$I_R(\nu) = \frac{1}{1-\nu} log\left[\int_0^\infty f^{\,\nu}(x)dx\right].$$

Using (2.2), the following expression is obtained:

$$\int_0^\infty f^{\,\nu}(x)dx$$

$$= \left(\frac{4\,\mu}{\pi\,\eta^2}\right)^\nu \int_0^\infty x^\nu\, e^{-\nu\frac{x}{\eta}} \frac{\left[1-\left(1+\frac{x}{\eta}\right)e^{-\frac{x}{\eta}}\right]^{\nu(\,\mu-1)}}{\left[1+\left[1-\left(1+\frac{x}{\eta}\right)e^{-\frac{x}{\eta}}\right]^{2\,\mu}\right]^\nu} dx.$$

Using the same procedure employed for the useful expansion (8) and by performing some simplifications, the following expression is obtained:

$$\int_0^\infty f^{\,\nu}(x)dx = \sum_{k=0}^\infty \varpi_k^* \int_0^\infty x^{\nu+k} e^{-(\nu+j)\frac{x}{\eta}} dx$$

$$= \sum_{k=0}^\infty \varpi_k^*\, \Gamma(\nu+k+1),$$

Where

$$\varpi_k^* = \left(\frac{4\,\mu}{\pi\,\eta}\right)^\nu \sum_{i,j=0}^\infty (-1)^{i+j} \binom{-\nu}{i} \binom{\mu(2i+\nu)-\nu}{j} \binom{j}{k} \left(\frac{1}{\nu+j}\right)^{\nu+k}.$$

Thus,

$$I_R(\nu) = \frac{1}{1-\nu} log\left\{\sum_{k=0}^\infty \varpi_k^*\Gamma(\nu+k+1)\right\}. \quad (17)$$

Some numerical results of $I_R(\nu)$ for the TCP-LBEX distribution for some choices of parameter η and μ are listed in Table 2.

Table 2. Some numerical results of Rényi entropy for the TCP-LBEX model.

η	μ	$\nu = 0.5$	$\nu = 1.5$	$\nu = 3.0$
0.5	0.5	0.028	0.09	0.354
	1.0	0.19	0.266	0.451
	1.5	0.238	0.311	0.485
	2.0	0.258	0.33	0.501
	2.5	0.269	0.34	0.51
	3.0	0.275	0.346	0.516
	3.5	0.279	0.35	0.52
2.0	0.5	0.574	0.692	0.956
	1.0	0.792	0.868	1.054
	1.5	0.84	0.913	1.087
	2.0	0.86	0.932	1.103
	2.5	0.871	0.942	1.112
	3.0	0.877	0.948	1.118
	3.5	0.881	0.952	1.122

The numerical values in Table 2 belong to [0.028, 1.122]; this indicates that η and μ have an important impact on the amount of information.

4 Parameter Estimation

The MLE technique is utilized in this section to estimate the unknown parameters of the TCP-LBEX distribution. Suppose that $x_1,\ldots,x_n$ be an n-th random sample from the specified distribution (7). The TCP-LBEX distribution's log-likelihood function (LLF) is supplied by

$$L_n = n\,log\left(\frac{4}{\pi}\right) + n\,log(\,\mu) - 2n\,log(\,\eta) + \sum_{i=1}^{n} x_i - \sum_{i=1}^{n}\frac{x_i}{\eta}$$

$$+(\,\mu-1)\sum_{i=1}^{n} log\left[1-\left(1+\frac{x}{\eta}\right)e^{-\frac{x}{\eta}}\right] + \sum_{i=1}^{n} log\left[1 + \left[1 - \left(1+\frac{x}{\eta}\right)e^{-\frac{x}{\eta}}\right]^{2\,\mu}\right]. \qquad (18)$$

The LLF in equation (18) can be computed by differentiating (18) regards to μ and η:

$$\frac{\partial L_n}{\partial\,\mu} = \frac{n}{\mu} + \sum_{i=1}^{n} log\left[1-\left(1+\frac{x_i}{\eta}\right)e^{-\frac{x_i}{\eta}}\right]$$

$$-2\,\mu\sum_{i=1}^{n}\frac{\left[1-\left(1+\frac{x_i}{\eta}\right)e^{-\frac{x_i}{\eta}}\right]^{2\,\mu} log\left[1-\left(1+\frac{x_i}{\eta}\right)e^{-\frac{x_i}{\eta}}\right]}{1+\left[1-\left(1+\frac{x_i}{\eta}\right)e^{-\frac{x_i}{\eta}}\right]^{2\,\mu}} = 0, \qquad (19)$$

and

$$\frac{\partial L_n}{\partial\,\eta} = \frac{-2n}{\eta} + \sum_{i=1}^{n}\frac{x_i}{\eta^2} + (\,\mu - 1)\sum_{i=1}^{n}\frac{\frac{x_i}{\eta}(2+\frac{x_i}{\eta})e^{-\frac{x_i}{\eta}}}{1-\left(1+\frac{x_i}{\eta}\right)e^{-\frac{x_i}{\eta}}}$$

$$-2\,\mu\sum_{i=1}^{n}\frac{\left[1-\left(1+\frac{x_i}{\eta}\right)e^{-\frac{x_i}{\eta}}\right]^{2\,\mu-1}\frac{x_i}{\eta}\left(2+\frac{x_i}{\eta}\right)e^{-\frac{x}{\eta}}}{1+\left[1-\left(1+\frac{x_i}{\eta}\right)e^{-\frac{x_i}{\eta}}\right]^{2\,\mu}} = 0, \qquad (20)$$

respectively. The MLEs of the parameters μ and η are denoted by $\widehat{\mu}$ and $\widehat{\eta}$ and are obtained by solving the above last system of equations (19) and (20). It is commonly more convenient to employ nonlinear optimization techniques, like the quasi-Newton approach, to numerically optimize the sample likelihood function.

4.1 Simulation Study

In this section, a simulation study has been conducted to illustrate the MLEs of η and μ for the TCP-LBEX model. The estimates are assessed and compared based on the root mean square errors (RMSErs). For this purpose, the following algorithm is adopted.

Step 1: A random sample of size n = 50, 100, 200, 300, 500, 700, and 1000 are generated from the TCP-LBEX distribution.

Step 2: The parameter values are considered as $set\ 1: (\eta = 0.5, \mu = 0.5)$, $set\ 2: (\eta = 0.5, \mu = 1.5)$, $set\ 3: (\eta = 0.5, \mu = 2.0)$, and $set\ 4: (\eta = 1.5, \mu = 1.5)$.

Step 3: For the selected values of parameters and each sample of size n, the MLEs are calculated.

Step 4: Steps 1–3 are repeated, and N = 10000 times, representing various samples.

Step 5: The outcomes of the simulation study are presented in Table 3 and Table 4.

Table 3. MLEs and RMSEr of the TCP-LBEX distribution for set 1 and set 2

n	set 1		set 2	
	MLE	RMSEr	MLE	RMSEr
50	0.519876	0.010675	0.549776	0.015231
	0.583619	0.096384	1.763210	0.652452
100	0.519020	0.006504	0.526846	0.006895
	0.580228	0.050699	1.731120	0.467523
200	0.507657	0.002476	0.512182	0.002960
	0.532262	0.013490	1.621920	0.109372
300	0.506596	0.001206	0.501286	0.001231
	0.504893	0.006243	1.561980	0.063621
500	0.503032	0.000785	0.501934	0.000863
	0.510978	0.003014	1.488500	0.028265
700	0.503911	0.000366	0.504906	0.000489
	0.505968	0.001960	1.527870	0.024482
1000	0.502257	0.000234	0.500685	0.000212
	0.499607	0.001162	1.505420	0.007254

Table 4. MLEs and RMSEr of the TCP-LBEX distribution for set 3 and set 4

n	set 3		set 4	
	MLE	RMSEr	MLE	RMSEr
50	0.536645	0.013190	1.504310	0.032153
	2.312980	1.042390	1.507250	0.037610
100	0.514007	0.004442	1.518710	0.010780
	2.160900	0.424585	1.521170	0.013241
200	0.506586	0.003029	1.505840	0.003223
	2.009150	0.119023	1.505360	0.003831
300	0.502757	0.001286	1.501350	0.001183
	2.000800	0.090311	1.500930	0.001369
500	0.505382	0.000974	1.503320	0.000541
	2.071520	0.064444	1.503850	0.000738
700	0.502108	0.000509	1.502030	0.000213
	2.036330	0.042179	1.501890	0.000286
1000	0.500067	0.000247	1.498890	0.000053
	2.019420	0.016566	1.498660	0.000069

From the above Table 3 and Table 4, the RMSErs of the TCP-LBEX model decrease when *n* increase.

5 Applications

In this section, we used two real-world data sets to compare the TCP-LBEX model to several other known competing models like; the LBEX model, Burr X-EX (BrXEX) model, Marshall–Olkin EX (MOEX) model, Kumaraswamy Marshall–Olkin EX (KMOEX) model, Kumaraswamy EX (KEX) model, beta EX (BEX) model, generalized Marshall–Olkin EX (GMOEX) model, EX model and Marshall–Olkin Kumaraswamy EX (MOKEX) model) in this part to highlight its usefulness in data modeling. To estimate the parameters of the competing models, the MLE approach is employed. To choose the optimal model, the K1, K2, K3, K4, K5, K6, K7, and K8model selection criteria and goodness of fit tests are utilized.

The first Data set: This data set contains the survival periods (in days) of 72 guinea pigs infected with virulent tubercle bacilli, [40]. For the first data set, MLEs and standard errors (SErs) are computed. Table 5 shows the numerical results of MLEs and the SErs for all competitive models for the first data set. Based on the numerical numbers in Table 6 and the information in Figure 3, we can conclude that the TCP-LBEX model provides the best fit for the first data set because the TCP-LBEX model has the lowest numerical value in K1, K2, K3, K4, K5, K6, K7 but has the largest value in K8.

Table 5. MLEs and SErs for the first data set.

Models	MLEs				SErs			
TCP-LBEX (η,μ)	0.781	1.665			0.092	0.301		
LBEX (η)	0.9252				0.0768			
BrXEX (η,μ)	0.475	0.2055			0.060	0.012		
MOEX (η,μ)	8.778	1.379			3.555	0.193		
KMOEX **(**η**, δ, γ,** μ**)**	0.3731	3.4782	3.3063	0.2990	0.1358	0.862	0.781	1.113
KEX (δ, γ, μ**)**	3.3041	1.1002	1.0371		1.1061	0.7642	0.6141	
BEX (δ, γ, β)	0.8073	3.4612	1.3311		0.6961	1.0032	0.8551	
GMOEX (λ, α, μ**)**	0.1789	47.6350	4.4652		0.0702	44.9011	1.3270	
EX (η)	0.540				0.063			
MOKEX **(**η**, δ, γ,** μ**)**	0.0081	2.7162	1.9861	0.0992	0.0021	1.3158	0.7839	0.0481

Table 6. Numerical values of K1, K2, K3, K4, K5, K6, K7, and K8 for the first data set.

Models	K1	K2	K3	K4	K5	K6	K7	K8
TCP-LBEX	191.60	191.31	191.77	193.41	0.08	0.48	0.09	(0.635)
LBEX	210.40	212.68	210.45	211.30	0.25	1.52	0.14	(0.130)
BrXEX	235.30	239.90	235.50	237.10	0.52	2.90	0.22	(0.002)
MOEX	210.36	214.92	210.53	212.16	0.17	1.18	0.10	(0.430)
KMOEX	207.82	216.94	208.42	211.42	0.11	0.61	0.09	(0.530)
KEX	209.42	216.24	209.77	212.12	0.11	0.74	0.09	(0.500)
BEX	207.38	214.22	207.73	210.08	0.15	0.98	0.11	(0.340)
GMOEX	210.54	217.38	210.89	213.24	0.16	1.02	0.09	(0.510)
EX	234.63	236.91	234.68	235.54	1.25	6.53	0.27	(0.060)
MOKEX	209.44	218.56	210.04	213.04	0.12	0.79	0.10	(0.440)

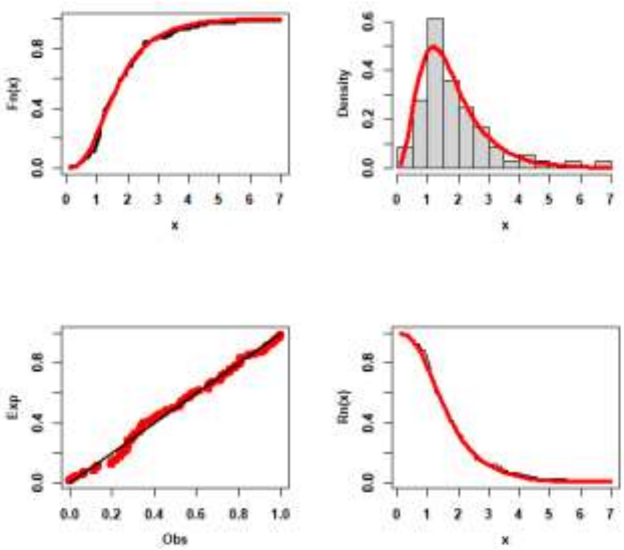

Fig. 3: Fitted cdf, pdf, and pp plots for the first data set

The second data set: This data collection contains information from 20 individuals and consists of histories pertaining to relief periods (in minutes) for patients who have taken an analgesic, [41]. Table 7 shows the numerical results of MLEs and the SErs for all competitive models for the second data set. Based on the numerical numbers in Table 8 and the information in Figure 4, we can conclude that the TCP-LBEX provides the best fit for the second data set because the TCP-LBEX model has the lowest numerical value in K1, K2, K3, K4, K5, K6, K7 but has the largest value in K8.

Table 7. MLEs and SErs for the second data set.

Models	MLEs				SErs			
TCP-LBEX (η,μ)	0.413	11.714			0.077	6.939		
LBEX (η)	0.9502				0.1501			
BrXEX (η,μ)	1.1635	0.3207			0.330	0.030		
MOEX (η,μ)	54.474	2.316			35.582	0.374		
KMOEX **(η, δ, γ, μ)**	8.8679	34.8258	0.2989	4.8988	9.1459	22.3119	0.2387	3.1757
KEX (δ, γ, μ)	83.7558	0.5679	3.3329		42.3612	0.3261	1.1880	
BEX (δ, γ, β)	81.6333	0.5421	3.5142		120.410	0.3272	1.4101	
GMOEX (λ, α, μ)	0.5192	89.4623	3.1691		0.2561	66.2782	0.7721	
EX (η)	0.526				0.117			
MOKEX (η, δ, γ, μ)	0.1333	33.2322	0.5711	1.6691	0.3320	57.8371	0.7211	1.8141

Table 8. Numerical values of K1, K2, K3, K4, K5, K6, K7, and K8 for the second data set.

Models	K1	K2	K3	K4	K5	K6	K7	K8
TCP-LBEX	36.43	35.03	37.14	36.82	0.06	0.35	0.13	(0.90)
LBEX	54.32	55.31	54.54	54.50	0.53	2.76	0.32	(0.07)
BrXEX	48.10	50.10	48.80	48.50	0.24	1.39	0.25	(0.17)
MOEX	43.51	45.51	44.22	43.90	0.14	0.80	0.18	(0.55)
KMOEX	42.80	46.84	45.55	43.60	0.19	0.08	0.15	(0.86)
KEX	41.78	44.75	43.28,	42.32	0.07	0.45	0.14	(0.86)
BEX	43.48	46.45	44.98	44.02	0.12	0.70	0.16	(0.80)
GMOEX	42.75	45.74	44.25	43.34	0.08	0.51	0.15	(0.78)
EX	67.67	68.67	67.89	67.87	0.96	4.60	0.44	(0.004)
MOKEX	41.58	45.54	44.25	42.30	0.11	0.60	0.14	(0.87)

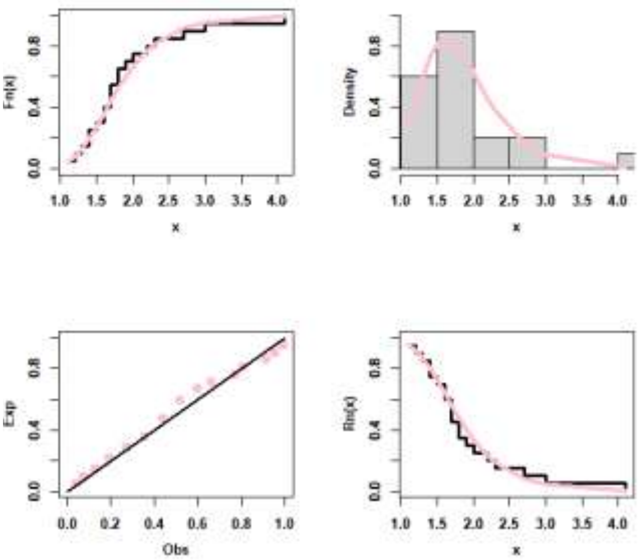

Fig. 4: Fitted cdf, pdf, and pp plots for the second data set

6 Conclusion

A new two-parameter model named the TCP-LBEX model was proposed. Some of the statistical properties of the TCP-LBEX model were investigated. The maximum likelihood estimator of the TCP-LBEX model was derived. Monto Carlo simulations are used to assess the behavior of parameters. Using two real data sets, the proposed model achieved better goodness of fit than some of the other competitive models. The limitation of our work is that we only used the complete samples and maximum likelihood method to estimate the parameters of the suggested model. For future directions, the other authors can estimate the parameters of the suggested model using different methods of estimation and utilizing different censored schemes.

References:

[1] Alzaatreh, A., Lee, C., and Famoye, F., A new method for generating families of continuous distributions. *Metron*, 71, 2013, P 63–79.

[2] Haq, M., and Elgarhy, M., The odd Fréchet- G family of probability distributions. *J. Stat. Appl. Probab.* 7, 2018, P 189–203.

[3] Cordeiro, G. M., and Castro, M., A new family of generalized distributions. *Journal of Statistical Computation and Simulation*, 81(7), 2011, P 883–898.

[4] Afify, A.Z. and Alizadeh, M., The odd Dagum family of distributions: properties and applications. *J. Appl. Probab. Stat.*, 15, 2020, P 45-72.

[5] Nadia, H. A., and Lamyaa, K. H., Weighted exponential-G family of probability distributions. *Saudi J. Eng. Technol*., 3(2), 2018, P 51–59.

[6] Mahdavi, A., Kundu, D., A new method for generating distributions with an application to exponential distribution. *Commun. Stat.-Theory Methods*, 46, 2017, P 6543–6557.

[7] Bakouch, H. S., Chesneau, C., and Enany, M. G., A new weighted exponential distribution as an alternative to the Weibull distribution and its fit to reliability data. *International Journal of Data Science*, 6(3), 2021, P 223–240.

[8] Nassar, M., Kumar, D., Dey, S., Cordeiro, G.M. and Afify, A.Z., The Marshall-Olkin alpha power family of distributions with applications. *Journal of Computational and Applied Mathematics*, 351, 2019, P 41-53.

[9] Bantan, R.A., Jamal, F., Chesneau, C., Elgarhy, M., Truncated inverted Kumaraswamy generated family of distributions with applications. *Entropy*, 21, 2019, 1089.

[10] Afify, A.Z., Alizadeh, M., Yousof, H.M., Aryal, G., Ahmad, M., The transmuted geometric-G family of distributions: Theory and applications. *Pak. J. Statist*, 32, 2016, P 139–160.

[11] Alizadeh, M., Yousof, H.M., Afify, A.Z., Cordeiro, G.M., Mansoor, M., The complementary generalized transmuted Poisson-G family of distributions. *Austrian J. Stat.* 47, 2018, P 60–80.

[12] Alizadeh, M., Tahmasebi, S., Haghbin, H., The exponentiated odd log-logistic family of distributions: properties and applications. *J Stat Model Theory Appl*, 1, 2018, P 1–24.

[13] Hassan, A.S., Elgarhy, M., and Shakil, M., Type II half Logistic family of distributions with applications. *Pak. J. Stat. Oper. Res.* 13, 2017, P 245–264.

[14] Al-Shomrani, A., Arif, O., Shawky, K., Hanif, S., Shahbaz, M.Q. Topp-Leone family of distributions: some properties and application. *Pak. J. Stat. Oper.* Res. 12, 2016, P 443–451.

[15] Alotaibi, N., Elbatal, I., Almetwally, E.M., Alyami, S.A., Al-Moisheer, A.S., Elgarhy, M., Truncated Cauchy Power Weibull-G Class of Distributions: Bayesian and Non-Bayesian Inference Modelling for COVID-19 and Carbon Fiber Data. *Mathematics.* 10, 2022, 1565.

[16] Cordeiro, G.M., Ortega, E.M.M., Popovi ́c, B.V., and Pescim, R.R., The Lomax generator of distributions: properties, minification process and regression model. *Appl. Math. Comput*. 247, 2014, P 465–486.

[17] Algarni, A., MAlmarashi, A., Elbatal, I., Hassan, A.S., Almetwally, E.M., Daghistani, A.M., and Elgarhy, M., Type I half logistic Burr XG family: Properties, Bayesian, and non-Bayesian estimation under censored samples and applications to COVID-19 data. *Math. Probl. Eng.* 2021.

[18] Hamedani, G.G., Yousof, H.M., Rasekhi, M., Alizadeh, M., Najibi, S.M., Type I general exponential class of distributions. *Pak. J. Stat. Oper. Res.* 14, 2018, P 39–55.

[19] Al-Babtain, A.A., Elbatal, I., Chesneau, C., Elgarhy, M., Sine Topp-Leone-G family of distributions: Theory and applications. *Open Phys.* 18, 2020, P 74–593.

[20] Korkmaz, M.Ç., Alizadeh, M., Yousof, H.M., Butt, N.S., The generalized odd Weibull generated family of distributions: statistical properties and applications. *Pak. J. Stat. Oper. Res.* 14, 2018, P 541–556.

[21] Bantan, R.A., Jamal, F., Chesneau, C., Elgarhy, M., A New Power Topp–Leone Generated Family of Distributions with Applications. *Entropy*, 21, 2019, 1177.

[22] Korkmaz, M.Ç., Altun, E., Yousof, H.M., Hamedani, G.G., The odd power Lindley generator of probability distributions: properties, characterizations and regression modeling. *Int. J. Stat. Probab.* 8, 2019, P 70–89.

[23] Mansour, M.M., Abd Elrazik, E.M., Afify, A.Z., Ahsanullah, M., and Altun, E., The transmuted

transmuted-G family: properties and applications. *J. Nonlinear Sci. Appl.* 12, 2019, P 217–229.

[24] Bantan, R.A., Chesneau, C., Jamal, F., Elgarhy, M., On the Analysis of New COVID-19 Cases in Pakistan Using an Exponentiated Version of the M Family of Distributions. *Mathematics*, 8, 2020, 953.

[25] Reyad, H., Jamal, F., Othman, S., and Hamedani, G.G., The transmuted Gompertz-G family of distributions: properties and applications. *Tbilisi. Math. J.* 11, 2018, P 47–67.

[26] Badr, M.M., Elbatal, I., Jamal, F., Chesneau, C., and Elgarhy, M., The transmuted odd Fréchet-G family of distributions: Theory and applications. *Mathematics*, 8, 2020, 958.

[27] Reyad, H., Jamal, F., Othman, S., Hamedani, G.G., The transmuted odd Lindley-G family of distributions. *Asian. J. Probab. Stat.* 1, 2018, P 1–2526.

[28] Reyad, H., Alizadeh, M., Jamal, F., and Othman, S., The Topp Leone odd Lindley-G family of distributions: properties and applications. *J. Stat. Manag. Syst.* 21, 2018, P 1273–1297.

[29] Brito, E., Cordeiro, G.M., Yousof, H.M., Alizadeh, M., and Silva, G.O. Topp-Leone odd log-logistic family of distributions. *J. Stat. Comput. Simul.* 87, 2017, P 3040–3058.

[30] Elbatal, I., Alotaibi, N., Almetwally, E.M., Alyami, S.A., and Elgarhy, M., On Odd Perks-G Class of Distributions: Properties, Regression Model, Discretization, Bayesian and Non-Bayesian Estimation, and Applications. *Symmetry*, 14, 2022, 883.

[31] Afify, A.Z., Cordeiro, G.M., Yousof, H.M., Alzaatreh, A., Nofal, Z.M. The Kumaraswamy transmuted-G family of distributions: Properties and applications. *J. Data Sci.* 14, 2016, 245–270.

[32] Aldahlan, M., Jamal, F., Chesneau, C., Elgarhy, M., Elbatal, I., The truncated Cauchy power family of distributions with inference and applications. *Entropy*, 22, 2020, 346.

[33] Mohamed, R.A.H., Elgarhy, M., Alabdulhadi, M.H., Almetwally, E.M., and Radwan, T., Statistical Inference of Truncated Cauchy Power-Inverted Topp–Leone Distribution under Hybrid Censored Scheme with Applications. *Axioms*, 12, 2023, 148.

[34] Chaudhary, A.K., Sapkota, L.P., and Kumar, V., Truncated Cauchy Power Inverse Exponential Distribution: Theory and Applications. *IOSR J. Math.* 16, 2020, P 12–23.

[35] Almarashi, A.M., Truncated Cauchy power Lomax model and its application to biomedical data. *Nanosci. Nanotechnol. Lett.* 12, 2020, P 16–24.

[36] Alotaibi, N., Elbatal, I., Almetwally, E.M., Alyami, S.A., Al-Moisheer, A.S., and Elgarhy, M., Truncated Cauchy Power Weibull-G Class of Distributions: Bayesian and Non-Bayesian Inference Modelling for COVID-19 and Carbon Fiber Data. *Mathematics*, 10, 2022, 1565.

[37] Shrahili, M., and Elbatal, I., Truncated Cauchy power odd Fréchet family of distributions: Theory and applications. *Complexity*, 2021, 4256945.

[38] Dara, S.T., and Ahmad. M., Recent advances in moment distribution and their hazard rates. *Lap Lambert Academic Publishing GmbH KG.* 2012.

[39] Fisher, R.A., The effects of methods of ascertainment upon the estimation of frequencies. *Annals of Eugenics*, 6(1), 1934, P 13–25.

[40] Bjerkedal, T., Acquisition of resistance in guinea pigs infected with different doses of virulent tubercle bacilli. *American Journal of Epidemiology*, 72(1), 1960, P 130–48.

[41] Gross, A.J., and Clark, V.A., Survival distributions: Reliability applications in the biomedical sciences. *John Wiley and Sons*, New York. 1975.

Contribution of Individual Authors to the Creation of a Scientific Article (Ghostwriting Policy)

Najwan Alsadat has written, reviewed, and actively participated in all the publication stages of this manuscript.

Sources of Funding for Research Presented in a Scientific Article or Scientific Article Itself

No funding was received for conducting this study.

Conflict of Interest

The author has no conflict of interest to declare that is relevant to the content of this article.